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Mutually Aposyndetic Decomposition of Homogeneous Continua

Dedicated to Charles L. Hagopian and James T. Rogers

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Abstract. A new decomposition, the *mutually aposyndetic decomposition* of homogeneous continua into closed, homogeneous sets is introduced. This decomposition is respected by homeomorphisms and topologically unique. Its quotient is a mutually aposyndetic homogeneous continuum, and in all known examples, as well as in some general cases, the members of the decomposition are semi-indecomposable continua. As applications, we show that hereditarily decomposable homogeneous continua and path connected homogeneous continua are mutually aposyndetic. A class of new examples of homogeneous continua is defined. The mutually aposyndetic decomposition of each of these continua is non-trivial and different from Jones' aposyndetic decomposition.

Given a category $\mathcal{C}$ of objects, and two properties $\mathcal{P}_0$ and $\mathcal{P}_1$ representing opposite extremes of the same spectrum, in various areas of mathematics we find theorems affirming that an object in $\mathcal{C}$ admits a unique (minimal or maximal) decomposition into subspaces (hyperspaces, co-sets, etc.) having property $\mathcal{P}_0$ such that the quotient object, that is, the image under quotient morphism, has the property $\mathcal{P}_1$. These theorems tend to be important but rare, because they provide fundamental structural information about the objects with respect to the particular spectrum of properties. For instance, in topology the most fundamental decomposition of this type would be the decomposition into (connected) components with totally disconnected quotient.

In this study we present a new decomposition of this type in an intriguing class of topological spaces, the class of homogeneous continua. Homogeneous continua naturally generalize the two following important classes of spaces: closed connected manifolds, and compact connected topological groups. The significance of this class was noticed as early as the 1920's. Progress in understanding these spaces and finding new examples has been slow. However, persistent efforts in this direction have been rewarded with occasional unexpected turns and spectacular breakthroughs.

Jones' aposyndetic decomposition theorem [12], a remarkable decomposition of the type discussed above, is one of these breakthroughs. It corresponds to the opposite extremes of being an aposyndetic continuum and indecomposable continuum. It asserts that each homogeneous continuum has a unique minimal decomposition into indecomposable homogeneous continua such that the quotient is an aposyndetic homogeneous continuum. Much more is now known about this decomposition,

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as it has been a subject of study for over fifty years (see [22, 23, 25] for later improvements by James Rogers). One of the most spectacular outcomes of this theorem was an early theoretical prediction that a space now called the *circle of pseudo-arcs* might exist. Shortly afterwards, this prediction was followed by actual construction of the circle of pseudo-arcs [4], adding a new surprising example of a homogeneous plane continuum. Decades later many more examples of this type were defined [14]. Jones' theorem is an excellent example of a mathematical discovery coming from the study of seemingly simple set-theoretical ideas, yet leading to deep results reaching far beyond our usual intuition.

Investigating homogeneous continua by studying their *filament* and *ample* subcontinua, I was surprised to realize there is yet another decomposition of these spaces, one of the type described in the beginning of the paper. I call it the *mutually aposyndetic decomposition*. It corresponds to the opposite extremes of being mutually aposyndetic and semi-indecomposable. These properties were introduced and studied in the late 1960's by Charles Hagopian [8]. This paper is devoted to the study of the mutually aposyndetic decomposition. First, we investigate its general properties in a larger class of Kelley continua. Next, narrowing down the class of spaces to homogeneous continua, we obtain stronger basic properties of this decomposition. As applications, we prove that hereditarily decomposable homogeneous continua and path connected homogeneous continua are mutually aposyndetic. Finally, we give nontrivial examples of the mutually aposyndetic decomposition. Some of them coincide with the aposyndetic decomposition; however we also show that many examples exist that are different from the aposyndetic decomposition. This article initiates the research of mutually aposyndetic decompositions of homogeneous continua. Though we do not know how deep this research can go, what we have learned so far seems to be very promising.

1 Preliminaries

All spaces are assumed to be metric. If K is a subset of X, a *neighborhood* of K is any subset of X containing K in its interior. A *continuum* is a compact, connected, nonempty metric space, and a *curve* is a 1-dimensional continuum. If X is a space, $C(X)$ is the hyperspace of subcontinua of X equipped with the Hausdorff metric. If $f\colon X \to X$ is a map, we define $\tilde{d}(f) = \sup\{d(x, f(x)) \mid x \in X\}$. A map $f\colon X \to Y$ is said to be *confluent* if for every continuum $K \subset Y$ and every component C of $f^{-1}(K)$ we have $f(C) = K$. It is known that open maps between continua are confluent.

The following concepts were introduced in [18] and further studied in [19–21]. If X is a space, a subcontinuum K is called a *filament continuum* (in X) provided there exists a neighborhood N of K such that the component of N containing K has empty interior. A set $S \subset X$ is called a *filament set* provided every continuum contaik

A space X is called a *Kelley space*[1] if whenever a sequence $\{x_n\} \subset X$ converges to a point $x \in X$ and x belongs to a subcontinuum K of X, there are continua K_n converging to K in $C(X)$ such that $x_n \in K_n$ for each n [13].

[1]Historically the name has varied. It was originally called a *space with property* 3.2 in [13], and later also a *space with the property of Kelley* and a *space with property* $[k]$.

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The following fact, which is fundamental to the study of filament and ample continua, was established in [18, Proposition 2.3].

Proposition 1.1 *Every subcontinuum of a Kelley continuum X is either filament or ample. In particular, if a continuum Y in X contains a subcontinuum that is ample in X, then Y is ample in X.*

Every homogeneous continuum is a Kelley space [27]. In a homogeneous continuum X the filament subcontinua form an open collection in $C(X)$.

In this paper we will employ the following set function J, which is a modification of Jones' function T. If X is a compactum and Y a subset of X, then

$$J(Y) = \{y \in X \mid \text{ every ample continuum containing } y \text{ intersects } Y\}.$$

It should be noted the function J is not identical with Jones' function T, even for closed sets. Indeed, if X is a one-point union of two nondegenerate indecomposable continua X_1 and X_2 with $X_1 \cap X_2 = \{p\}$, then for any $p' \in X_1 \setminus \{p\}$ we have $T(p') = X_1 \neq X = J(p')$. Nevertheless, using Proposition 1.1, it can easily be shown that for closed sets Y the function J coincides with T in the class of Kelley continua. We note the following.

Proposition 1.2 *Let X be a continuum and Y a closed subset of X.*

(i) *$J(Y)$ is closed in X.*
(ii) *$J^2(Y) = J(J(Y)) = J(Y)$.*
(iii) *If, additionally, X is a Kelley continuum and Y is a subcontinuum of X, then $J(Y)$ is a continuum.*

Proof (i) If $p \notin J(Y)$, then there exists an ample continuum $A \subset X - Y$ containing p, which can be enlarged to a continuum $L \subset X - Y$ having A in its interior. The set $\operatorname{Int} L$ is a neighborhood of p in $X - J(Y)$, and thus $J(Y)$ is closed.

(ii) Clearly $J(Y) \subset J^2(Y)$. Let $p \in X - J(Y)$, and A, L be as in part (i). Then $A \subset \operatorname{Int} L \subset X - J(Y)$, and thus A is an ample continuum containing p disjoint with $J(Y)$. Hence $p \in X - J^2(Y)$.

(iii) This property is known [6] for Jones' function T, which coincides with J in Kelley continua. ■

A continuum X is *aposyndetic* at x with respect to y if there exists a continuum $K \subset X \setminus \{y\}$ containing x in its interior. It is *aposyndetic* at x if it is aposyndetic at x with respect to every other point, and *aposyndetic* if it is aposyndetic everywhere [11]. If X is a homogeneous continuum, the sets $T(x)$ (or equivalently $J(x)$), for $x \in X$, form Jones' aposyndetic decomposition [12] of X. The element of Jones' decomposition containing $x \in X$ can also be equivalently defined as: (i) the intersection of the continua in X having x in their interiors; or (ii) the intersection of the ample continua in X containing x [21].

A continuum X is said to be *decomposable* if it has proper subcontinua Y and Z such that $X = Y \cup Z$. If every non-degenerate subcontinuum of X is decomposable, X is called *hereditarily decomposable*. Non-decomposable continua are called *indecomposable*. Note that a continuum X is indecomposable if and only if it is non-aposyndetic at each of its points with respect to any other point. A subcontinuum Y of a space X is called *terminal* if, for every continuum $Z \subset X$ intersecting Y, either $Y \subset Z$ or $Z \subset Y$.

The following concepts of *mutual aposyndesis* and *semi-indecomposability*[2] have been introduced by Hagopian in [8]. A continuum X is *mutually aposyndetic at* points x and y, provided there are disjoint continua K and L in X containing x and y in their corresponding interiors. It is *mutually aposyndetic* if it is mutually aposyndetic at each pair of its distinct points. If X is not mutually aposyndetic at any pair of its distinct points, then X is called a *semi-indecomposable* continuum. In Kelley spaces we can equivalently express mutual aposyndesis using ample continua. Indeed, a Kelley continuum X is mutually aposyndetic at x and y if and only if there are disjoint ample continua K and L in X containing x and y, respectively.

If X is a homogeneous compactum, then for every positive ε there is a number δ, called an *Effros number* for ε, such that for each $x, y \in X$ with $d(x, y) < \delta$, there is some homeomorphism $f \colon X \to X$ such that $f(x) = y$ and $d(z, f(z)) < \varepsilon$ for each $z \in X$. This is called the *Effros theorem*. It follows from the more general statement that for each $x \in X$, the evaluation map, $g \mapsto gx$, from the homeomorphism group onto X is open. The latter follows from [7, Theorem 2]. (See also [26, Theorem 3.1].)

2 Mutually Aposyndetic Decomposition in Kelley Continua

Let X be a continuum. If $x, y \in X$, we write $x \asymp y$ provided that every two ample continua A_x and A_y in X such that $x \in A_x$ and $y \in A_y$ have nonempty intersection. We also let $Q_x = \{y \in X \mid x \asymp y\}$.

If $x \not\asymp y$, there are disjoint ample continua A_x and A_y such that $x \in A_x$ and $y \in A_y$. Since A_x and A_y are ample, there exist continua B_x and B_y such that $A_x \subset \operatorname{Int} B_x$, $A_y \subset \operatorname{Int} B_y$, and $B_x \cap B_y = \varnothing$. Note that $B_x \times B_y$ is a neighborhood of (x, y) in $X \times X$ composed of pairs (x', y') such that $x' \not\asymp y'$. We have shown the following.

Proposition 2.1 *If X is a continuum, then the relation $\asymp$ is a closed subset of $X \times X$. In particular, the sets Q_x are closed and the function $x \mapsto Q_x$ is upper semi-continuous.*

Given a continuum X and a subset Y of X, recall that $J(Y)$ denotes the modified Jones' set function of Y, which was introduced in the previous section.

Proposition 2.2 *For every continuum X and $x \in X$,*

$$Q_x = \bigcap \{J(A) \mid A \text{ is an ample subcontinuum of } X \text{ containing } x\}.$$

Proof Let $\mathcal{A}_x$ be the collection of ample continua containing x. If $y \notin Q_x$, then $x \not\asymp y$. Thus there exist disjoint ample continua A_x and A_y containing x and y, respectively. Clearly, $A_x \in \mathcal{A}_x$ and $y \notin J(A_x)$. Thus $y \notin \bigcap\{J(A) \mid A \in \mathcal{A}_x\}$.

[2]Originally, Hagopian used the name *strictly non-mutually aposyndetic*. We changed the name to *semi-indecomposable* for brevity, and also because the continua in question manifest properties similar to the properties of indecomposable continua.

Conversely, if $y \notin \bigcap\{J(A) \mid A \in \mathcal{A}_x\}$, then $y \notin J(A_x)$ for some $A_x \in \mathcal{A}_x$. Therefore, there is an ample continuum A_y disjoint from A_x such that $y \in A_y$. Consequently, $y \notin Q_x$. ∎

Now we study the sets Q_x in Kelley continua. Note that if X is a Kelley continuum and $x, y \in X$, then $x \asymp y$ if and only if X is not mutually aposyndetic at x and y.

Proposition 2.3 *If A_1 and A_2 are disjoint ample subcontinua of a Kelley continuum X, then $J(A_1)$ and $J(A_2)$ are disjoint.*

Proof Suppose $p \in J(A_1) \cap J(A_2)$. Let P_t for $t \in [0,1]$ be an order arc of continua such that $P_0 = \{p\}$, $P_1 = X$, and $P_t \subset P_{t'}$ for $t < t'$. There is a smallest $\alpha \in [0,1]$ such that $P_\alpha \cap (A_1 \cup A_2) \neq \varnothing$. For some $i \in \{1,2\}$ we have $P_\alpha \cap A_i \neq \varnothing$. Let j be such that $\{j\} = \{1,2\} - \{i\}$. Since A_i is ample, there exists a continuum B_i such that $A_i \subset \operatorname{Int} B_i \subset B_i \subset X - A_j$. Thus, for a number $\beta \in [0,\alpha)$ sufficiently near to α, we have $P_\beta \cap B_i \neq \varnothing$. Consequently, the continua $P_\beta \cup B_i$ and A_j are disjoint, and $P_\beta \cup B_i$ is not filament. Thus $P_\beta \cup B_i$ is ample by Proposition 1.1. Since $p \in P_\beta \cup B_i$, it follows $p \notin J(A_j)$, a contradiction. ∎

Proposition 2.4 *If X is a Kelley continuum, then $\asymp$ is an equivalence relation in X.*

Proof It suffices to show that $\asymp$ is transitive. Suppose $x \asymp y$ and $y \asymp z$ but $x \not\asymp z$. There are disjoint ample continua A_x and A_z such that $x \in A_x$ and $z \in A_z$. By Proposition 2.3 the continua $J(A_x)$ and $J(A_z)$ are disjoint. If $y \in J(A_x)$, then $J(A_x)$ and A_z are disjoint ample continua containing y and z, respectively. So $y \not\asymp z$, a contradiction. If $y \notin J(A_x)$, there is an ample continuum A_y such that $y \in A_y$ and $A_x \cap A_y = \varnothing$. Hence $x \not\asymp y$, a contradiction. ∎

By Propositions 2.1 and 2.4 we have the following.

Corollary 2.5 *For every Kelley continuum X the collection $\mathcal{Q} = \{Q_x \mid x \in X\}$ is an upper semi-continuous decomposition of X into closed sets.*

The decomposition $\mathcal{Q}$ of X, also denoted by $\mathcal{Q}(X)$, will be called the *mutually aposyndetic decomposition of X*.

Proposition 2.6 *Let X be a Kelley continuum, $\mathcal{Q} = \{Q_x \mid x \in X\}$, and $q\colon X \to X/\mathcal{Q}$ the quotient map. Then for every ample continuum A in X the continuum $J(A)$ is saturated with respect to q, that is, $q^{-1}(q(J(A))) = J(A)$.*

Proof If $x \in J(A)$, then $q^{-1}(q(x)) = Q_x \subset J(J(A))$ by Proposition 2.2. We also have $J(J(A)) = J(A)$ by Proposition 1.2. The conclusion follows. ∎

Proposition 2.7 *If X is a Kelley continuum, $\mathcal{Q} = \{Q_x \mid x \in X\}$, and $q\colon X \to X/\mathcal{Q}$ the quotient map, then for every ample continuum A in X the continuum $q(J(A))$ is ample in $X/\mathcal{Q}$.*

Proof Since A is ample, so is the continuum $J(A)$. For every $\varepsilon > 0$ there is a continuum B in X such that $J(A_x) \subset \operatorname{Int} B \subset B \subset N_\varepsilon(J(A_x))$. Since $J(A)$ is saturated with respect to q (see Proposition 2.6), the continuum B is a neighborhood of every set $q^{-1}(q(p))$, where $p \in J(A)$. Thus $q(p) \in \operatorname{Int}(q(B))$ for every such B and p. Hence $q(J(A)) \subset \operatorname{Int}(q(B))$, and thus $q(J(A))$ is ample in $X/\mathcal{Q}$. ∎

Proposition 2.8 *If X is a Kelley continuum and $\mathcal{Q} = \{Q_x \mid x \in X\}$, then the continuum $X/\mathcal{Q}$ is mutually aposyndetic.*

Proof Let $q\colon X \to X/\mathcal{Q}$ be the quotient map and $x, y \in X$ two points such that $Q_x \neq Q_y$. Since $x \not\asymp y$, there are ample, disjoint continua A_x and A_y such that $x \in A_x$ and $y \in A_y$. By Proposition 2.3 the continua $J(A_x)$ and $J(A_y)$ are disjoint. They are also saturated with respect to q by Proposition 2.6. Therefore,

$$q(J(A_x)) \cap q(J(A_y)) = \varnothing.$$

Applying Proposition 2.7 we see that $q(J(A_x))$ and $q(J(A_y))$ are disjoint ample continua in $X/\mathcal{Q}$ containing $q(x)$ and $q(y)$, respectively. ■

Question 1 Let X be a Kelley continuum. Are the members of the mutually aposyndetic decomposition of X connected?

This question remains open also for homogeneous continua. Homogeneous spaces are the focus of the rest of the paper.

3 Mutually Aposyndetic Decomposition in Homogeneous Continua

The results of the previous section apply to all homogeneous continua because each such continuum is Kelley [27]. In this section we further study the mutually aposyndetic decomposition $\mathcal{Q} = \{Q_x \mid x \in X\}$ in the case when X is a homogeneous continuum. Note that the function $x \mapsto Q_x$ is respected by self-homeomorphisms of X, that is, for every homeomorphism $h\colon X \to X$ and $x \in X$ if $h(x) = y$, then $h(Q_x) = Q_y$ (*cf.* [20, §4]). A decomposition of a homogeneous continuum respected by homeomorphisms leads to a particularly regular structure of the space. For the partition $\mathcal{Q}$, by Corollary 2.5, Proposition 2.8, and [20, Proposition 4.1], we have the following.

Theorem 3.1 *If X is a homogeneous continuum, then the relation $x \asymp y$, meaning X is not mutually aposyndetic at x and y, is an equivalence relation in X. The collection of the equivalence classes of this relation, $\mathcal{Q}(X) = \{Q_x \mid x \in X\}$, is a continuous decomposition of X into closed sets, which is respected by self-homeomorphisms of X. Moreover, the sets Q_x are mutually homeomorphic and homogeneous. The quotient space $X/\mathcal{Q}$ is a homogeneous mutually aposyndetic continuum.*

In addition, the decomposition $\mathcal{Q}(X)$ in Theorem 3.1 is completely regular in the sense of [22]. This is the case because X is homogeneous, and the self-homeomorphisms of X respect $\mathcal{Q}(X)$.

Remark 3.2 If X is a homogeneous continuum and $\mathcal{D}_a(X)$ its aposyndetic decomposition, then by definition each member of $\mathcal{D}_a(X)$ is contained in some member of the decomposition $\mathcal{Q}(X)$. In other words, $\mathcal{Q}(X)$ is coarser than $\mathcal{D}_a(X)$. As the study of Section 5 below shows, we have many examples X for which $\mathcal{Q}(X) = \mathcal{D}_a(X)$, and many with $\mathcal{Q}(X)$ essentially coarser than $\mathcal{D}_a(X)$. Whether there exists X with

$\mathcal{Q}(X) \neq \mathcal{D}_a(X)$ having non-degenerate members of both $Q(X)$ and $\mathcal{D}_a(X)$, is an open problem whose solution depends on a possible counterexample to Question 2 from Section 5.

To further strengthen Theorem 3.1, we devote the rest of this section to the study of the following conjecture.

Conjecture *If X is a homogeneous continuum, then the fibers of the mutually aposyndetic decomposition of X are semi-indecomposable continua.*

We begin with the following lemma.

Lemma 3.3 *Let X be a homogeneous continuum. If for every $\varepsilon > 0$ there are two disjoint ample subcontinua A and B of X such that $A = J(A)$ and $B = J(B)$ and two continua $Y, Z \subset X/\mathcal{Q}$ such that*

(i) $Y \cap Z \neq \varnothing$,
(ii) $Y \cap q(A) \neq \varnothing \neq Z \cap q(B)$,
(iii) $Y \cap q(B) = \varnothing = Z \cap q(A)$, *and*
(iv) $\operatorname{diam} Y, \operatorname{diam} Z < \varepsilon$,

then each set Q_x in X is connected.

Proof Suppose the contrary. By homogeneity each Q_x is not connected. Let A_n, B_n, Y_n, and Z_n be continua as in the hypothesis for $\varepsilon = 1/n$. We choose a sequence p_n such that $p_n \in q(A_n) \cap Y_n$. Replacing all defined sequences by their subsequences, without loss of generality we assume $\{p_n\}$ converges to some point $p \in X/\mathcal{Q}$. Since $X/\mathcal{Q}$ is a homogeneous continuum, and the decomposition $\mathcal{Q}$ is respected by the homeomorphisms, using the Effros theorem we modify these sets, by a sequence of homeomorphisms sending p_n to p, so that $p = p_n$ for each n. By the assumption $q^{-1}(p)$ is not connected. Let $x, y \in X$ be such that $q(x) = q(y) = p$ and $q^{-1}(p) = Q_x = Q_y$ is not connected between x and y. Let $S \subset Q_x$ and $T \subset Q_x$ be disjoint, open-closed sets relative to Q_x such that $x \in S$, $y \in T$, and $S \cup T = Q_x$. Let U, V be disjoint neighborhoods in X of S and T, respectively. The sets $Y_n \cup Z_n$ converge to the singleton $\{p\}$, and thus, by the continuity of q, we have $q^{-1}(Y_n \cup Z_n) \subset U \cup V$ for almost all n. We fix such an n. Let K_x and K_y be the components of $q^{-1}(Y_n \cup Z_n)$ such that $x \in K_x$ and $y \in K_y$. Clearly $K_x \subset U$ and $K_y \subset V$. Since q is open, and thus confluent, $q(K_x) = q(K_y) = Y_n \cup Z_n$. Let L_x, L_y be the components of $q^{-1}(Y_n)$ containing x and y, respectively. We have $L_x \subset K_x \subset U$ and $L_y \subset K_y \subset V$, and $q(L_x) = q(L_y) = Y_n$ by the confluence of q. Let $r \in Y_n \cap Z_n$. Fix $x_r \in L_x$ and $y_r \in L_y$ such that $q(x_r) = q(y_r) = r$. Let M_x and M_y be the components of $q^{-1}(Z_n)$ containing x_r and y_r, respectively. Clearly, $M_x \subset K_x \subset U$ and $M_y \subset K_y \subset V$, and $q(M_x) = q(M_y) = Z_n$ by the confluence of q. In particular $M_y \cap A_n = \varnothing \neq M_y \cap B_n$. Since the sets $A_n \cup L_x$ and $B_n \cup M_y$ contain ample continua, they are ample by Proposition 1.1. Therefore, they are disjoint ample continua containing x_r and y_r, respectively. By the definition of the quotient map q, this is impossible, because $x_r, y_r \in q^{-1}(r)$. ■

Lemma 3.4 *Let X be a homogeneous continuum and $\mathcal{D}$ a monotone, continuous decomposition of X, with the quotient map $q\colon X \to X/\mathcal{D}$, such that the group $\mathcal{H}_{\mathcal{D}}$*

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of self-homeomorphisms of X that respect $\mathcal{D}$ acts transitively on X. Then for every $D \in \mathcal{D}$, an ample subcontinuum A of D, and $\varepsilon > 0$, there exists a closed set L such that $A \subset \operatorname{Int} L \subset L \subset N_\varepsilon(A) \subset X$ and the map $q|L : L \to q(L) \subset X/\mathcal{D}$ is monotone.

Proof Note that $\mathcal{H}_\mathcal{D}$ is a closed subset of the group $H(X)$ of self-homeomorphisms of X. Thus the Effros theorem applies to the homeomorphisms from $\mathcal{H}_\mathcal{D}$ [5]. Let d_H be the Hausdorff distance for closed subsets of X, and, for $x, y \in X/\mathcal{D}$, let $d(x, y) = d_H(q^{-1}(x), q^{-1}(y))$. Define $C_\mathcal{D}(X) = \{C \in C(X) \mid C \subset D \in \mathcal{D} \text{ for some } D \in \mathcal{D}\}$, and note that $C_\mathcal{D}(X)$ is a closed subcollection of $C(X)$.

Let $\varepsilon > 0$. Let $D \in \mathcal{D}$ and A be an ample subcontinuum of D. For every $\xi > 0$ define $\mathcal{H}_{\mathcal{D},\xi} = \{h \in \mathcal{H}_\mathcal{D} \mid \tilde{d}(h) < \xi\}$. Since A is ample in D, there exists a continuum $A_0 \subset D \cap N_{\varepsilon/3}(A)$ such that A is in the interior of A_0 relative to D.

Claim There exists a $\xi > 0$ such that for every $h \in \mathcal{H}_{\mathcal{D},\xi}$ and $B \in C_\mathcal{D}(X)$ with $d_H(B, A_0) < \xi$, if $h(A_0)$ and B are subsets of the same member of $\mathcal{D}$, then $h(A_0) \cap B \neq \varnothing$.

Suppose there is no such ξ. There are two sequences

$$\{h_n\} \subset \mathcal{H}_{\mathcal{D},\xi} \quad \text{and} \quad \{B_n\} \subset C_\mathcal{D}(X)$$

such that h_n converges to the identity, B_n converges to A_0 in the sense of the Hausdorff distance, and $h_n(A_0)$ and B_n are in the same member of $\mathcal{D}$ but disjoint. Thus $A_0 = h_n^{-1}(h_n(A_0))$ and $h_n^{-1}(B_n)$ are in D, and they are disjoint for each n. Since $h_n^{-1}(B_n)$ converges to A_0 and the interior of A_0 relative to D is non-empty, $h_n^{-1}(B_n) \cap A_0 \neq \varnothing$ for almost all n, which is a contradiction.

Let $\delta = \min\{\varepsilon/3, \xi/2\}$, where ξ is as in the Claim. Define

$$U = \bigcup\{h(A_0) \mid h \in \mathcal{H}_{\mathcal{D},\delta}\}.$$

We have $A_0 \subset \operatorname{Int} U$ by the Effros theorem applied to the homeomorphisms from $\mathcal{H}_\mathcal{D}$. Thus there exists a closed set L_0 such that $A_0 \subset \operatorname{Int} L_0 \subset L_0 \subset U$. Let

$$\mathcal{L}_0 = \{h(A_0) \mid h \in \mathcal{H}_{\mathcal{D},\delta} \text{ and } q(h(A)) \subset q(L_0)\}$$

and $\mathcal{L}$ be the closure of $\mathcal{L}_0$ in $C(X)$. Since $\mathcal{L}_0 \subset C_\mathcal{D}(X)$, and $C_\mathcal{D}(X)$ is closed, we have $\mathcal{L} \subset C_\mathcal{D}(X)$. Since L_0 is closed, $q(L_0)$ is also closed. Therefore $q(S)$ is a singleton in $q(L_0)$ for each $S \in \mathcal{L}$ and $\bigcup\{q(S) \mid S \in \mathcal{L}\} = q(L_0)$. By definition, for each $S \in \mathcal{L}$ there exists a member $h(A_0) \in \mathcal{L}_0$ such that $q(S) = q(h(A_0))$, and thus S and $h(A_0)$ are contained in the same member of $\mathcal{D}$. Moreover, $d(A_0, S) \leq \delta < \xi$ and $\tilde{d}(h) < \delta < \xi$. By the Claim, $h(A_0) \cap S \neq \varnothing$ for each such pair $h(A_0)$ and S. This implies that the union L_x of the collection of the members of $\mathcal{L}$ contained in a single set $q^{-1}(x) \in \mathcal{D}$, $x \in q(L_0)$, is connected. Define $L = \bigcup \mathcal{L} = \bigcup\{L_x \mid x \in q(L_0)\}$. Since $\mathcal{L}$ is a closed collection and the decomposition $\mathcal{D}$ is continuous, the monotone decomposition $\{L_x \mid x \in q(L_0)\}$ of L is upper semi-continuous. We also have

$$A_0 \subset \operatorname{Int} L_0 \subset \operatorname{Int} L \subset L \subset \operatorname{cl}(N_\delta(A_0)) \subset \operatorname{cl}(N_{\varepsilon/3}(N_{\varepsilon/3}(A))) \subset N_\varepsilon(A). \qquad \blacksquare$$

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Lemma 3.5 *Let X be a homogeneous continuum. If for every $\varepsilon > 0$ there are two disjoint ample subcontinua A and B of X such that $A = J(A)$ and $B = J(B)$ and two continua $Y, Z \subset X/\mathcal{Q}$ such that:*

(i) $Y \cap Z \neq \varnothing$,
(ii) $Y \cap q(A) \neq \varnothing \neq Z \cap q(B)$,
(iii) $Y \cap q(B) = \varnothing = Z \cap q(A)$, *and*
(iv) $\operatorname{diam} Y, \operatorname{diam} Z < \varepsilon$,

then each set Q_x in X is a semi-indecomposable continuum.

Proof By Lemma 3.3 the members of $\mathcal{Q}$ are connected. Suppose they are not semi-indecomposable. As in the proof of Lemma 3.3 we choose continua A_n, B_n, Y_n, and Z_n satisfying the hypothesis for $\varepsilon = 1/n$, with a point $p \in q(A_n) \cap Y_n$ for each n. By the assumption, the continuum $q^{-1}(p)$ is not semi-indecomposable, which implies there are two disjoint ample subcontinua C_1 and C_2 of $q^{-1}(p)$. Let $\varepsilon > 0$ be such that $2\varepsilon < \min\{d(a_1, a_2) \mid a_1 \in C_1 \text{ and } a_2 \in C_2\}$. Let L_1 and L_2 be closed sets guaranteed by Lemma 3.4 for the decomposition $\mathcal{Q}$ with C_1 and C_2, respectively, and the number ε. Note that $p \in \operatorname{Int}(q(L_1)) \cap \operatorname{Int}(q(L_2))$ by the openness of q, and $L_1 \cap L_2 = \varnothing$ by the assumption on ε. The continua $Y_n \cup Z_n$ converge to p, and thus $Y_n \cup Z_n \subset q(L_1) \cap q(L_2)$ for almost all n. Fix such an n and an $r \in Y_n \cap Z_n$. Let $q_1 = q|L_1 \colon L_1 \to q(L_1)$ and $q_2 = q|L_2 : L_2 \to q(L_2)$, and note that q_1 and q_2 are monotone by Lemma 3.4. The continua $A_n \cup q_1^{-1}(Y_n)$ and $B_n \cup q_2^{-1}(Z_n)$ are disjoint ample continua both intersecting $q^{-1}(r)$, which is impossible by the definition of q. ■

Given a continuum X, let $\Phi_n(X)$ be the collection of continua K in X such that $K = K_1 \cup \cdots \cup K_m$ for some continua $K_1, \dots, K_m$, each having diameter less than $1/n$. Let $\Phi(X)$ be the collection of continua L in X such that there is a sequence $\{L_n\}$, with $L_n \in \Phi_n(X)$, converging to L in $C(X)$. Clearly $\Phi(X)$ is a closed subcollection of $C(X)$ containing all singletons.

Theorem 3.6 *Let X be a homogeneous continuum and $\mathcal{Q}$ its mutually aposyndetic decomposition. Suppose at least one of the three following conditions holds:*

(i) *there is a hereditarily decomposable continuum in X intersecting two different members of $\mathcal{Q}$;*
(ii) *the quotient space $X/\mathcal{Q}$ contains a non-degenerate, hereditarily decomposable subcontinuum;*
(iii) *the quotient space $X/\mathcal{Q}$ contains a non-degenerate member of $\Phi(X/\mathcal{Q})$.*

Then each member of $\mathcal{Q}$ is a semi-indecomposable continuum.

Proof It suffices to show that the conditions of Lemma 3.5 hold. Let $\varepsilon > 0$.

Case 1. There is a hereditarily decomposable continuum M in X intersecting two different members of $\mathcal{Q}$.

Let M_0 be a subcontinuum of M maximal with respect to the property that M_0 is contained in a single set Q_x. We can slightly enlarge M_0 to a continuum M_1 such that $0 < \operatorname{diam} q(M_1) < \varepsilon$. Since $q(M_1)$ is non-degenerate, M_1 intersects two different sets Q_a and Q_b for some $a, b \in M_1$. Consequently, there are disjoint ample continua A'

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and B' in X such that $a \in A'$ and $b \in B'$. Let $A = J(A')$ and $B = J(B')$. By Proposition 1.2, the continua A and B satisfy $J(A) = A$ and $J(B) = B$, and, by Proposition 2.3, are disjoint. They are saturated with respect to q by Proposition 2.6, and thus $Q_a \subset A$ and $Q_b \subset B$. Let M_2 be a subcontinuum of M_1 irreducibly intersecting A and B. Since M_2 is decomposable, there are proper subcontinua K and L of M_2 such that $K \cup L = M_2$, $K \cap A \neq \varnothing \neq L \cap B$, and $K \cap B = \varnothing = L \cap A$. Letting $Y = q(K)$ and $Z = q(L)$, we have $Y \cap q(A) \neq \varnothing \neq Z \cap q(B)$. Since A and B are saturated with respect to q, it follows $Y \cap q(B) = \varnothing = Z \cap q(A)$. Clearly $Y \cap Z \neq \varnothing$ and $\operatorname{diam} Y, \operatorname{diam} Z \leq \operatorname{diam} q(M_2) \leq \operatorname{diam} q(M_1) < \varepsilon$. The conditions of Lemma 3.5 hold.

Case 2. The quotient space $X/\mathcal{Q}$ contains a non-degenerate hereditarily decomposable continuum.

Let P be a hereditarily decomposable non-degenerate subcontinuum of $X/\mathcal{Q}$ of diameter less than ε, and p_1, p_2 two different points of P. Let $a \in q^{-1}(p_1)$ and $b \in q^{-1}(p_2)$. By the definition of q there are disjoint ample continua A' and B' containing a and b, respectively. By Propositions 2.3, 2.6, and 2.7 the continua $A = J(A')$ and $B = J(B')$ are ample and have disjoint images under q. The continuum P contains a subcontinuum P' irreducible with respect to intersecting $q(A)$ and $q(B)$. Since P is hereditarily decomposable, there are continua Y and Z in P' such that $Y \cup Z = P'$ and the conditions (i)–(iii) of Lemma 3.5 hold. Condition (iv) is satisfied as well.

Case 3. The quotient space contains a non-degenerate member P^* of $\Phi(X/\mathcal{Q})$.

By definition, P^* is the limit, in the sense of the Hausdorff distance, of a sequence continua $P_n^* \in \Phi_n(X/\mathcal{Q})$. For sufficiently large n we can choose subcontinua P_n of P_n^* such that $\varepsilon/2 < \operatorname{diam} P_n < 2\varepsilon/3$ and $P_n \in \Phi_n(X/\mathcal{Q})$. The limit P of a convergent subsequence of P_n is a non-degenerate member of $\Phi(X/\mathcal{Q})$ having diameter less than ε.

Let p_1, p_2 be two different points of P. Let $a \in q^{-1}(p_1)$ and $b \in q^{-1}(p_2)$. There are disjoint ample continua A' and B' containing a and b, respectively. Since A' and B' are ample, there are disjoint continua A'' and B'' in X containing a and b in their corresponding interiors. By Propositions 2.3, 2.6, and 2.7 the continua $A = J(A'')$ and $B = J(B'')$ are ample and have disjoint images under q. Since $a \in \operatorname{Int} A$ and $b \in \operatorname{Int} B$, and the map q is open, $P_n \cap q(A) \neq \varnothing \neq P_n \cap q(B)$. Fix an n so large that $3/n < \max\{d(u,v) \mid u \in q(A) \text{ and } v \in q(B)\}$. The continuum P_n, being the finite union of subcontinua of diameter less than $1/n$, contains continua $K_1, \dots, K_m$, $m \geq 3$, such that (i) $K_i \cap K_j \neq \varnothing$ if and only if $|i - j| \leq 1$, (ii) $K_i \cap q(A) \neq \varnothing$ if and only if $i = 1$, and (iii) $K_i \cap q(B) \neq \varnothing$ if and only if $i = m$. Letting $Y = K_1$ and $Z = K_2 \cup \dots \cup K_m$, we see that the conditions (i)–(iv) of Lemma 3.5 hold. ∎

Remark 3.7 In Theorem 3.6 the conjecture stated in the beginning of this section has been confirmed in three important cases. Note that it also holds in the trivial cases when the sets Q_x are either singletons or each Q_x is the whole space. Moreover, the quotient space $X/\mathcal{Q}$, being mutually aposyndetic, does not have proper, non-degenerate, terminal subcontinua. Therefore, decomposable continua are everywhere in $X/\mathcal{Q}$, which makes the conditions of Lemma 3.5 seem likely and the conjecture is probably true in general. Solving the conjecture would be a major breakthrough in the study of the mutually aposyndetic decomposition of homogeneous continua.

4 Applications

In this section we present some applications of the mutually aposyndetic decomposition, which lead to new results on homogeneous continua that (i) are hereditarily decomposable, (ii) have dense path components, or (iii) are path connected.

We begin with a theorem on semi-indecomposable homogeneous continua. Though it plays an auxiliary role here, it may be of interest in its own right. Indeed, in view of the results from the previous section, semi-indecomposable continua seem to deserve special attention and, perhaps, a separate study.

Theorem 4.1 *If X is a semi-indecomposable, homogeneous continuum, then every minimal ample subcontinuum of X is either indecomposable or the union of two indecomposable continua.*

Proof Suppose a minimal ample continuum A in X cannot be represented as the union of at most two indecomposable continua. Then A, being decomposable, is the union of two of its proper subcontinua, B and C. Using Zorn's lemma, we can assume the union $B \cup C$ is irreducible in the sense that $B' \cup C' \neq A$ for every subcontinua $B' \subset B$ and $C' \subset C$, at least one of which is proper. At least one of the continua B and C is decomposable, say C. We have $C = C_1 \cup C_2$ for some proper subcontinua C_1 and C_2 of C. Consequently, A is the union of its three proper subcontinua B, C_1, and C_2, no two of which have their union equal to A. We note that the closed sets $G_1 = B \cup C_1$, $G_2 = C_1 \cup C_2$, $G_3 = B \cup C_2$ are filament sets at least two of which, say G_1 and G_2, are connected. The third, G_3, has at most two components.

Case 1. Only G_1 and G_2 are connected.

Let U_1 and U_2 be open filament neighborhoods of G_1 and G_2, respectively [18, Theorem 3.2]. Define

$$\begin{aligned}
\varepsilon_1 &= \min\{d(x,y) \mid x \in G_1,\ y \in X - U_1\},\\
\varepsilon_2 &= \min\{d(x,y) \mid x \in G_2,\ y \in X - U_2\},\\
\varepsilon_3 &= \min\{d(x,y) \mid x \in B,\ y \in C_2\},\\
\varepsilon &= \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}/3.
\end{aligned}$$

Since G_1 is a filament continuum, by the Effros theorem there exists a homeomorphism $h_1 \colon X \to X$ with $\tilde{d}(h_1) < \varepsilon$ sending G_1 to the component of U_1 different from the component of U_1 containing G_1. Thus $G_1 \cap h_1(G_1) = \varnothing$. Let

$$\varepsilon_4 = \min\{d(x,y) \mid x \in G_1,\ y \in h_1(G_1)\} \quad \text{and} \quad \varepsilon_5 = \min\{\varepsilon, \varepsilon_4\}.$$

Since $h_1(G_2)$ is a filament continuum, by the Effros theorem there exists a homeomorphism $h_2\colon X \to X$ with $\tilde{d}(h_2) < \varepsilon_5$ sending $h_1(G_2)$ to the component of U_2 different from the component of U_2 containing G_2. Letting $h = h_2 \circ h_1$, we note that $h(G_1) \cap G_1 = \varnothing = h(G_2) \cap G_2$ and $h(B) \cap C_2 = \varnothing = h(C_2) \cap B$. Hence $A \cap h(A) = \varnothing$, and X contains two disjoint ample continua A and $h(A)$. This is impossible because X is semi-indecomposable.

Case 2. All three sets G_1, G_2, and G_3 are connected.

This case is similar to the previous one. In this case we also use a third filament neighborhood U_3 of G_3, the number $\varepsilon_3' = \min\{d(x, y) \mid x \in G_3,\ y \in X - U_3\}$, and a third homeomorphism $h_3\colon X \to X$ sending $h_2(h_1(G_3))$ to a component of U_3 different from the one containing G_3. By choosing h_3 with sufficiently small $\tilde{d}(h_3)$, we also have $h_3(h_2(h_1(G_1))) \cap G_1 = \varnothing = h_3(h_2(h_1(G_2))) \cap G_2$. The homeomorphism h is defined as the composition $h_3 \circ h_2 \circ h_1$. The sets $h_3(h_2(h_1(G_i)))$ and G_i are disjoint for $i \in \{1, 2, 3\}$. Hence $h(A) \cap A = \varnothing$, an impossibility. The details are left to the reader. ■

The problem whether a circle is the only homogeneous, non-degenerate, hereditarily decomposable continuum, by Józef Krasinkiewicz and Piotr Minc, is still open. Some recent partial results can be found in [19]. Here we add the following.

Theorem 4.2 *If X is a hereditarily decomposable homogeneous continuum, then X is mutually aposyndetic.*

Proof The map $q\colon X \to X/\mathcal{Q}$ is open, and thus $X/\mathcal{Q}$ is hereditarily decomposable (see [15, (9.2), p. 76 and Table II, p. 28]). By Theorems 3.1 and 3.6 the sets Q_x are homogeneous semi-indecomposable continua. If non-degenerate, clearly the Q_x's are non-locally connected. Non-locally connected homogeneous continua have non-degenerate minimal ample subcontinua, and thus the Q_x's contain non-degenerate continua that are the unions of at most two indecomposable continua by Theorem 4.1. Thus X contains a non-degenerate indecomposable subcontinuum, which is a contradiction. Hence the Q_x's are singletons and X is mutually aposyndetic. ■

Theorem 4.3 *If X is a homogeneous continuum containing an arc that intersects two different members of $\mathcal{Q}(X)$, then X is mutually aposyndetic.*

Proof Let P be an arc with end points x and y such that the sets Q_x and Q_y are distinct. The image $q(P)$ is a non-degenerate locally connected subcontinuum of $q(X)$, which contains an arc. By Theorem 3.6 the sets Q_p, for $p \in X$, are semi-indecomposable continua.

Suppose the sets Q_p are non-degenerate. Since the sets Q_x and Q_y are distinct, there are disjoint ample continua A' and B' containing x and y, respectively, such that $J(A') = A'$ and $J(B') = B'$ (see Proposition 2.3). Let ab be an arc with end points $a \in A'$ and $b \in B'$ irreducible with respect to the property $ab \cap A' \neq \varnothing \neq ab \cap B'$. Let $z \in P - (A' \cup B')$ and note that $Q_z \cap A' = \varnothing = Q_z \cap B'$ by Proposition 2.6. We enlarge A' and B' slightly to some disjoint continua A and B, respectively, such that $A \cap Q_z = \varnothing = B \cap Q_z$, $A' \subset \operatorname{Int} A$, $B' \subset \operatorname{Int} B$ and $az \cap B = \varnothing = zb \cap A$, where az and zb are arcs in ab with the corresponding end points.

Claim. $Q_z \cap ab = \{z\}$.

Indeed, otherwise there would be two different points $z_a, z_b \in Q_z \cap ab$, the first and the last, respectively, in the sense of the ordering in ab from a to b. Let az_a and $z_b b$ be arcs in ab with the corresponding end points. The continua $A' \cup az_a$ and $B' \cup z_b b$ are disjoint ample continua containing, respectively, two different points z_a and z_b of Q_z, an impossibility.

By the Effros theorem we can choose a homeomorphism $h\colon X \to X$, satisfying $h(z) \in Q_z - \{z\}$, so near to the identity that $h(a) \in \operatorname{Int} A$, $h(b) \in \operatorname{Int} B$, and $h(az) \cap B = \varnothing = h(zb) \cap A$.

Case 1. $h(ab) \cap zb = \varnothing$.

In this case we notice that $A \cup h(az)$ and $B \cup zb$ are disjoint ample continua in X containing, correspondingly, two different points, z and $h(z)$, of Q_z. This is impossible by the definition of Q_z.

Case 2. $h(ab) \cap zb \neq \varnothing$.

Let r be the first point in zb, with respect to the order from z to b, that belongs to $h(ab)$, and $u = h^{-1}(r)$. Observe that $r \neq z$ because $z \neq h(z) \in h(ab) \cap Q_z$, and $h(ab)$ intersects Q_z at exactly one point for the same reason as ab does (see the Claim above). Again by the Claim, $r \notin Q_z$. Therefore $u \neq z$. Let $zr \subset zb$ be the arc from z to r.

Suppose $u \in az - \{z\}$. Let $au \subset az$ be the arc from a to u. The disjoint ample continua $A \cup h(au) \cup zr$ and $B \cup h(zb)$ contain, respectively, the two different points z and $h(z)$ of Q_z. This is impossible by the definition of Q_z.

Suppose $u \in zb - \{z\}$. Let $ub \subset zb$ be the arcs from u to b. The disjoint ample continua $A \cup h(az)$ and $B \cup h(ub) \cup zr$ contain, respectively, the two different points $h(z)$ and z of Q_z. This is impossible by the definition of Q_z. Hence the sets Q_p, for $p \in X$, are singletons, and thus X is mutually aposyndetic. ■

Corollary 4.4 *If X is a homogeneous continuum with dense path components, then X is either mutually aposyndetic or semi-indecomposable.*

Compact, connected topological groups are significant examples of homogeneous continua with dense path components [9, Theorem 9.60, p. 500]. By the previous result we have the following.

Corollary 4.5 *Each compact, connected topological group is either mutually aposyndetic or semi-indecomposable.*

From the view point of Corollary 4.5, the products of solenoids are particularly interesting. First, solenoids are examples of semi-indecomposable compact topological groups. In fact, they are even indecomposable. Second, Hagopian [8] has shown the product of three non-degenerate continua is always mutually aposyndetic. Thus the product of at least three solenoids is mutually aposyndetic. The case of the product of two solenoids is intriguing. Alejandro Illanes [10] proved that if S_p and S_q are the p-adic and q-adic solenoids, respectively, with p and q relatively prime, then

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$S_p \times S_q$ is mutually aposyndetic. In a recent paper [17] the author characterized pairs of solenoids having semi-terminal products. From that characterization the product $S \times S$ is semi-indecomposable for every solenoid S. Thus some products of pairs of solenoids are semi-indecomposable, compact, connected topological groups that are decomposable.

Another class of special interest is that of path connected homogeneous continua. It may be considered fundamental in the sense that it connects with other areas of study in topology and beyond topology. This class is much larger than the class of homogeneous, locally connected continua. The first example of homogeneous path connected, non-locally connected continuum was defined in [16]. Some general properties of homogeneous path connected continua can be found in [1, 2]. Using Jones' decomposition, it is easy to see that such spaces are aposyndetic. Here we show that they are also mutually aposyndetic.

Theorem 4.6 *Each path connected homogeneous continuum is mutually aposyndetic.*

Proof Suppose a path connected homogeneous continuum X is not mutually aposyndetic. According to Theorem 4.3, it is semi-indecomposable. Let $x \in X$ and $\mathrm{Fcs}(x)$ be the filament composant of X determined by x, that is, the union of the filament continua containing x. The set $\mathrm{Fcs}(x)$ of x is a first category subset of X [18, Proposition 1.8], and thus there is a $y \in X - \mathrm{Fcs}(x)$. Let A be an arc containing x and y. Since A is not filament, A is an ample subcontinuum of X by Proposition 1.1. It contains a minimal ample subcontinuum A_0. Since X is not locally connected, A_0 is nondegenerate. Therefore A_0 is an arc, whereas, by Theorem 4.1, A_0 is either indecomposable or the union of two indecomposable continua. This is impossible, and hence the proof is complete. ■

Remark 4.7 Theorem 4.3 yields a comparison of the mutually aposyndetic decomposition $\mathcal{Q}(X)$ of a homogeneous continuum X to a decomposition determined by the partition of X into path components. Using the Effros theorem, it is easy to see that the closures of the path components of X form a decomposition of X into mutually disjoint sets, denoted here by $\mathcal{P}(X)$. Each member of $\mathcal{P}(X)$ is a homogeneous continuum having dense path components, $\mathcal{P}(X)$ is respected by homeomorphisms, and it is completely regular in the sense of [22]. By Theorem 4.3 it is clear that either $\mathcal{Q}(X)$ is coarser than $\mathcal{P}(X)$, or $\mathcal{P}(X)$ is coarser than $\mathcal{Q}(X)$. Moreover, if $\mathcal{P}(X)$ is coarser than and different from $\mathcal{Q}(X)$, then the members of $\mathcal{Q}(X)$ are singletons. Since the members of the aposyndetic decomposition $\mathcal{D}_a(X)$ of X are terminal, the same holds true for $\mathcal{D}_a(X)$ in relation to $\mathcal{P}(X)$. As pointed out in Remark 3.2, $\mathcal{Q}(X)$ is coarser than $\mathcal{D}_a(X)$, and hence for each two of the decompositions $\mathcal{Q}(X)$, $\mathcal{D}_a(X)$, and $\mathcal{P}(X)$, one is coarser than the other.

5 Examples of Mutually Aposyndetic Decompositions

In this section we discuss examples of mutually aposyndetic decompositions of homogeneous continua. It turns out that many known aposyndetic decompositions are also mutually aposyndetic. Moreover, we also show that there exist aposyndetic homogeneous continua with nontrivial mutually aposyndetic decompositions.

First, we notice that for any mutually aposyndetic homogeneous continuum X the sets Q_x are singletons. To have non-trivial members of the decomposition $\mathcal{Q}$, consider the circle of pseudo-arcs $\widehat{S}$ defined in [4]. The mutually aposyndetic decomposition of $\widehat{S}$ is the same as the aposyndetic one. The members of this decomposition are the maximal (terminal) pseudo-arcs in $\widehat{S}$. Similarly, according to the results of [14], for every mutually aposyndetic homogeneous curve M there exists a homogeneous curve $\widehat{M}$ having a unique continuous monotone decomposition into maximal terminal pseudo-arcs with the quotient homeomorphic to M. As in the case of the circle of pseudo-arcs, this decomposition is both aposyndetic and mutually aposyndetic. Thus we have a large class of examples of mutually aposyndetic decompositions with non-degenerate fibers. This leads to the question whether the aposyndetic and mutually aposyndetic decompositions coincide in every homogeneous continuum. This question was answered in the negative by Hagopian [8], using the following example.

Example 5.1 **(Hagopian)** Let P be the pseudo-arc and $X = P \times P$. Note that X is homogeneous because P is homogeneous. As the product of two non-degenerate continua, X is aposyndetic. Therefore the fibers of its aposyndetic decomposition are singletons. Hagopian proved [8] that X is semi-indecomposable. Thus the mutually aposyndetic decomposition of X has only one member, the set X itself.

It is not known whether there exists a similar example in dimension one. The following open question is of particular interest from the view point of this study.

Question 2 Is every homogeneous aposyndetic curve mutually aposyndetic?

One can ask whether there exist homogeneous continua having the mutually aposyndetic decomposition $\mathcal{Q}$ different from the aposyndetic one, with proper subcontinua for the members of $\mathcal{Q}$. Below, we answer this question in the affirmative. In our argument we will use the following result by David Bellamy and Janusz Łysko [3].

Theorem 5.2 **(Bellamy–Łysko)** *A subcontinuum K of the product $P \times P$ of the pseudo-arc P with itself is ample if and only if the projections of K to the first and second coordinates both equal P.*

Let M be a (homogeneous) continuum of dimension at most 1 and $\widehat{M}$ a continuous curve of terminal pseudo-arcs, as defined in [14], with M as the quotient space and the quotient map $g\colon \widehat{M} \to M$. Let $g \times g\colon \widehat{M} \times \widehat{M} \to M \times M$ be the product map, and $\Delta = \{(x, x) \mid x \in M\}$ the diagonal of the product $M \times M$. Define $\widehat{M}^2_\Delta = (g \times g)^{-1}(\Delta)$ and $q_\Delta = (g \times g)|\widehat{M}^2_\Delta\colon \widehat{M}^2_\Delta \to \Delta$. The map q_Δ yields a continuous decomposition of $\widehat{M}^2_\Delta$ into products of two pseudo-arcs, with the copy Δ of M as the quotient space. Let $\pi_1, \pi_2\colon \widehat{M}^2_\Delta \to \widehat{M}$ be the projections onto the first and second coordinates, respectively. Finally, let $\mathcal{D}_a$ be the aposyndetic decomposition of M with the quotient map $q_a\colon M \to M/\mathcal{D}_a$. The defined spaces and maps are shown in the following commuting diagram.

$$
\begin{tikzcd}
& \widehat{M}^2_\Delta \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, "q_\Delta"] & \\
\widehat{M} \arrow[dr, "g"'] & & \widehat{M} \arrow[dl, "g"] \\
& M \arrow[d, "q_a"] & \\
& M/\mathcal{D}_a &
\end{tikzcd}
$$

Proposition 5.3 *If M is a homogeneous curve, then the group of self-homeomorphisms of $\widehat{M} \times \widehat{M}$ that keep $\widehat{M}^2_\Delta$ invariant and respect the decomposition $\{q_\Delta^{-1}(x) \mid x \in M\}$ of $\widehat{M}^2_\Delta$ acts transitively on $\widehat{M}^2_\Delta$. In particular, if M is homogeneous, then so is $\widehat{M}^2_\Delta$.*

Proof Let $\mathcal{H}$ be the group of homeomorphisms $h\colon \widehat{M} \times \widehat{M} \to \widehat{M} \times \widehat{M}$ such that $h(\widehat{M}^2_\Delta) = \widehat{M}^2_\Delta$ and h respects the decomposition $\{q_\Delta^{-1}(x) \mid x \in M\}$. Given a point in $\widehat{M}^2_\Delta$, it suffices to show that (i) its orbit with respect to $\mathcal{H}$ intersects all sets $q_\Delta^{-1}(x)$ for $x \in M$, and (ii) if it intersects one such set, then it contains it.

If $x, y \in M$, by the homogeneity of M there is a homeomorphism $f\colon M \to M$ with $f(x) = y$. This homeomorphism can be lifted to the homeomorphism $\widehat{f}\colon \widehat{M} \to \widehat{M}$ such that $g \circ \widehat{f} = f \circ g$ by [14, Theorem 3, p. 95]. Note that the homeomorphism $\widehat{f} \times \widehat{f}\colon \widehat{M} \times \widehat{M} \to \widehat{M} \times \widehat{M}$ belongs to $\mathcal{H}$ and maps $q_\Delta^{-1}(x)$ onto $q_\Delta^{-1}(y)$, which shows condition (i).

If $(p_1, p_2), (p_1', p_2') \in q_\Delta^{-1}(x)$ for some $x \in M$, by [14, Theorem 5, p. 98] there are homeomorphisms $\widehat{h}_1, \widehat{h}_2\colon \widehat{M} \to \widehat{M}$ such that $\widehat{h}_1(p_1) = p_1'$, $\widehat{h}_2(p_2) = p_2'$ and $g \circ \widehat{h}_1 = g \circ \widehat{h}_2 = g$. Observe that $\widehat{h}_1 \times \widehat{h}_2\colon \widehat{M} \times \widehat{M} \to \widehat{M} \times \widehat{M}$ belongs to $\mathcal{H}$ and maps (p_1, p_2) to (p_1', p_2'). Hence condition (ii) holds. The proof is complete. ■

Proposition 5.4 *If M is a homogeneous curve and K is a subcontinuum of $q_\Delta^{-1}(x)$ for some $x \in M$, then the two following statements are equivalent:*

(i) *For every $\varepsilon > 0$ there is a continuum K' such that $K \subset K' \subset N_\varepsilon(K) \subset \widehat{M}^2_\Delta$ and $K' - q_\Delta^{-1}(x) \neq \varnothing$;*

(ii) *K is an ample subcontinuum of $q_\Delta^{-1}(x)$.*

In particular, if a continuum Y in $\widehat{M}^2_\Delta$ intersects both $q_\Delta^{-1}(x)$ and its complement, then Y contains an ample subcontinuum of $q_\Delta^{-1}(x)$.

Proof Suppose K is not ample in $q_\Delta^{-1}(x)$. Since $q_\Delta^{-1}(x)$ is the product of two pseudo-arcs with the projections

$$\pi_1|q_\Delta^{-1}(x)\colon q_\Delta^{-1}(x) \to g^{-1}(x) \quad \text{and} \quad \pi_2|q_\Delta^{-1}(x)\colon q_\Delta^{-1}(x) \to g^{-1}(x),$$

by Theorem 5.2 at least one of the continua $\pi_1(K)$ and $\pi_2(K)$ is properly contained in $g^{-1}(x)$. Say $\pi_1(K) \neq g^{-1}(x)$ (the other case is similar). The continuum $g^{-1}(x)$ is terminal in $\widehat{M}$, and thus $\pi_1(K') \subset g^{-1}(x)$ for any continuum K' that is sufficiently near to and intersects K. Therefore $K' \subset \pi_1^{-1}(g^{-1}(x)) = q_\Delta^{-1}(x)$ for such K', and hence (i) does not hold.

Suppose K is ample in $q_\Delta^{-1}(x)$. We apply Lemma 3.4 to the space $X = \widehat{M}^2_\Delta$ and decomposition $\mathcal{D} = \{q_\Delta^{-1}(x) \mid x \in M\}$. By Proposition 5.3 the conditions of Lemma 3.4 are satisfied. Let L be the closed set guaranteed by that lemma. Since $K \subset \operatorname{Int} L$, it follows that $x \in \operatorname{Int} q_\Delta(L)$. Let $Y \subset \operatorname{Int} q_\Delta(L)$ be a nondegenerate continuum containing x. Then $K' = (q_\Delta|L)^{-1}(Y)$ contains K. Moreover, K' is a continuum because $q_\Delta|L$ is monotone. We also have $K' - q_\Delta^{-1}(x) = (q_\Delta|L)^{-1}(Y - \{x\}) \neq \varnothing$. Since $K \subset K' \subset L$, and L can be chosen as near to K as we wish, the continuum K' can be defined arbitrarily near to K. Hence (i) holds. The proof is complete. ■

In the next theorem we assume that the aposyndetic and mutually aposyndetic decompositions of a homogeneous curve M are the same. Since the quotients of non-trivial aposyndetic decompositions are at most one-dimensional [25], we do not know whether this assumption is essential (*cf.* Question 2).

Theorem 5.5 *Let M be a homogeneous continuum of dimension at most one, whose aposyndetic decomposition is mutually aposyndetic. Then $\widehat{M}^2_\Delta$ is a homogeneous continuum, the mutual aposyndetic decomposition of $\widehat{M}^2_\Delta$ is the collection*

$$\{q_\Delta^{-1}(q_a^{-1}(u)) \mid u \in M/\mathcal{D}_a\},$$

and each member of this collection is a semi-indecomposable continuum.

Proof The homogeneity of $\widehat{M}^2_\Delta$ is shown in Proposition 5.3. We show that the sets $q_\Delta^{-1}(q_a^{-1}(u))$ are the members of the mutually aposyndetic decomposition of $\widehat{M}^2_\Delta$.

If $x \in q_\Delta^{-1}(q_a^{-1}(u))$ and $y \in q_\Delta^{-1}(q_a^{-1}(v))$, where $u \neq v$, by the mutual aposyndesis of $M/\mathcal{D}_a$ there are disjoint continua A and B in $M/\mathcal{D}_a$ having u and v in their corresponding interiors. The map $q_a \circ q_\Delta$ is monotone, and thus $q_\Delta^{-1}(q_a^{-1}(A))$ and $q_\Delta^{-1}(q_a^{-1}(B))$ are disjoint continua in $\widehat{M}^2_\Delta$ containing x and y in their corresponding interiors. Hence $\widehat{M}^2_\Delta$ is mutually aposyndetic at x and y.

Let $x, y \in q_\Delta^{-1}(q_a^{-1}(u))$ for some $u \in M/\mathcal{D}_a$, and A_x and A_y be continua in $\widehat{M}^2_\Delta$ containing x and y in their interiors, respectively. If M is a singleton, then $\widehat{M}^2_\Delta$ is the product of the pseudo-arcs with itself. In this case $A_x \cap A_y \neq \varnothing$ by Example 5.1.

Suppose M is non-degenerate. The map q_Δ is open, so each of the continua $q_\Delta(A_x)$ and $q_\Delta(A_y)$ has some point of $q_a^{-1}(u)$ in its interior. Since $q_a^{-1}(u)$ is a terminal continuum M, it follows that $q_a^{-1}(u) \subset q_\Delta(A_x) \cap q_\Delta(A_y)$. Let $p \in q_a^{-1}(u)$. Then $A_x \cap q_\Delta^{-1}(p) \neq \varnothing \neq A_y \cap q_\Delta^{-1}(p)$. Since M is non-degenerate, the interior of

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$q_\Delta^{-1}(p)$ is empty. Therefore both A_x and A_y must intersect the complement of $q_\Delta^{-1}(p)$. By Proposition 5.4 there are continua $B_x \subset A_x \cap q_\Delta^{-1}(p)$ and $B_y \subset A_y \cap q_\Delta^{-1}(p)$ such that B_x and B_y are ample in $q_\Delta^{-1}(p)$. These continua have non-empty intersection by Example 5.1. Therefore $A_x \cap A_y \neq \varnothing$, and hence $\widehat{M}^2_\Delta$ is not mutually aposyndetic at x and y. We have shown $\mathcal{Q} = \{q_\Delta^{-1}(q_a^{-1}(u)) \mid u \in X/\mathcal{D}_a\}$ is the mutually aposyndetic decomposition of $\widehat{M}^2_\Delta$.

The maps q_Δ and q_a are monotone, and thus the members of $\mathcal{Q}$ are connected. It remains to show that they are semi-indecomposable. If q_a is a homeomorphism, the members $q_\Delta^{-1}(q_a^{-1}(u))$ of $\mathcal{Q}$ are products of two pseudo-arcs, which are semi-indecomposable by Example 5.1.

Suppose q_a is not a homeomorphism. Given $u \in M/\mathcal{D}_a$, let K and L be subcontinua of $q_\Delta^{-1}(q_a^{-1}(u))$ with non-empty interiors relative to $q_\Delta^{-1}(q_a^{-1}(u))$. The maps π_1 and π_2 are open and $q_\Delta^{-1}(q_a^{-1}(u))$ is saturated with respect to them. Therefore the maps $\pi_i|q_\Delta^{-1}(q_a^{-1}(u))\colon q_\Delta^{-1}(q_a^{-1}(u)) \to g^{-1}(q_a^{-1}(u))$ for $i \in \{1,2\}$ are open. Consequently the sets $\pi_1(K)$, $\pi_1(L)$, $\pi_2(K)$, and $\pi_2(L)$ have non-empty interiors relative to $g^{-1}(q_a^{-1}(u))$. Since $q_a^{-1}(u)$ is a member of the aposyndetic decomposition of M, it is indecomposable [24, Theorem 1, p. 277]. The map g is open and has terminal point inverses (such maps are also called *atomic*), which implies that $g^{-1}(q_a^{-1}(u))$ is indecomposable. Thus $\pi_1(K) = \pi_1(L) = \pi_2(K) = \pi_2(L) = g^{-1}(q_a^{-1}(u))$. Since q_a is not a homeomorphism, there are two different points $p, p' \in q_a^{-1}(u)$. Note that $K \cap q_\Delta^{-1}(p) \neq \varnothing \neq L \cap q_\Delta^{-1}(p)$ and $K \cap q_\Delta^{-1}(p') \neq \varnothing \neq L \cap q_\Delta^{-1}(p')$ by the last statement. By Proposition 5.4, the continua K and L contain, correspondingly, ample subcontinua A_K and A_L of $q_\Delta^{-1}(p)$. The set $q_\Delta^{-1}(p)$ is the product of two pseudo-arcs, so $A_K \cap A_L \neq \varnothing$ by Example 5.1. Hence $K \cap L \neq \varnothing$, which completes the proof. ■

By Proposition 5.3, for every homogeneous curve M the set $\widehat{M}^2_\Delta$ is a homogeneous continuum. Since $\widehat{M}$ is 1-dimensional, from the definition it follows that $\widehat{M}^2_\Delta$ is a two-dimensional continuum. Moreover, $\widehat{M}^2_\Delta$ has no proper, non-degenerate, terminal subcontinuum. Indeed, such a subcontinuum of $\widehat{M}^2_\Delta$ would either be properly contained in some set $q_\Delta^{-1}(x)$, or be the union of such sets. The former case is impossible because $q_\Delta^{-1}(x)$ is the product of two pseudo-arcs, which has no proper terminal subcontinua. The latter case is impossible because the continuum $\{(x,x) \mid x \in \widehat{M}\}$ intersects all the set $q_\Delta^{-1}(x)$, but contains none of them.

The fibers of the aposyndetic decomposition are terminal, and thus they must be singletons in $\widehat{M}^2_\Delta$. Consequently, $\widehat{M}^2_\Delta$ is aposyndetic. Since the sets $q_\Delta^{-1}(q_a^{-1}(u))$ are non-degenerate, $\widehat{M}^2_\Delta$ is not mutually aposyndetic.

By Theorem 5.5 we conclude the following:

(i) If M is indecomposable, then $M/\mathcal{D}_a$ is a singleton. Consequently, the mutually aposyndetic decomposition of $\widehat{M}^2_\Delta$ is trivial and $\widehat{M}^2_\Delta$ is semi-indecomposable.

(ii) If M is mutually aposyndetic, then the sets $q_\Delta^{-1}(x)$, which are the products of two pseudo-arcs, are the fibers of the mutually aposyndetic decomposition of $\widehat{M}^2_\Delta$.

(iii) If the members of the aposyndetic decomposition of M are homeomorphic to an indecomposable curve Y and $M/\mathcal{D}_a$ is mutually aposyndetic, then the members of the mutually aposyndetic decomposition of $\widehat{M}^2_\Delta$ are homeomorphic to $\widehat{Y}^2_\Delta$.

If the answer to Question 2 is yes, than for each homogeneous curve M at least one of the cases (i), (ii), or (iii) holds. The spaces $\widehat{M}^2_\Delta$ form a large collection of homogeneous continua having the mutually aposyndetic decomposition different from the aposyndetic decomposition.

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