HARDY INEQUALITIES WITH MIXED NORMS

BY

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ABSTRACT. We give a necessary and sufficient condition on
weight functions u and v such that for 1 ≤ p ≤ q ≤ ∞ there exists a
constant C for which

\( \left( \int_0^\infty |u(x)|^p \left( \int_0^x f(t) dt \right)^q dx \right)^{\frac{1}{pq}} \leq C \left( \int_0^\infty |f(x)v(x)|^p dx \right)^{\frac{1}{p}}. \)

A corresponding dual result is also given. This extends a result of B.

1. Introduction. The classical Hardy inequality ([1], [2]) states that for
\( f(x) \geq 0 \) and \( p > 1 \)

\( \int_0^\infty \left[ \frac{1}{x} \int_0^x f(t) dt \right]^p dx \leq \left( \frac{p}{p-1} \right)^p \int_0^\infty f(x)^p dx. \)

Muckenhoupt, in [3], showed that the more general inequality

\( \left( \int_0^\infty |u(x)|^p \left( \int_0^x f(t) dt \right)^q dx \right)^{\frac{1}{pq}} \leq C \left( \int_0^\infty |f(x)v(x)|^p dx \right)^{\frac{1}{p}} \)

holds if and only if

\( \sup_{r>0} \left( \int_0^r |u(x)|^p dx \right)^{\frac{1}{p}} \left( \int_0^r |v(x)|^{-p'} dx \right)^{\frac{1}{p'}} = K < \infty \)

and \( K \leq C \leq K(p)^{1/p}(p')^{1/p'} \). A similar result for the dual inequality

\( \left( \int_0^\infty |u(x)|^q \left( \int_0^x f(t) dt \right)^p dx \right)^{\frac{1}{pq}} \leq C \left( \int_0^\infty |f(x)v(x)|^p dx \right)^{\frac{1}{p}} \)

was also obtained.

2. Generalized Hardy inequalities. Our results are the following:

THEOREM 1. Let \( 1 \leq p \leq q \leq \infty \). Suppose u and v are non-negative. Then

\( \left( \int_0^\infty \left[ u(x) \int_0^x f(t) dt \right]^q dx \right)^{\frac{1}{q}} \leq C \left( \int_0^\infty |f(x)v(x)|^p dx \right)^{\frac{1}{p}} \)

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holds for non-negative \( f \) if and only if
\[
\left( \int_0^\infty u(x)^q \, dx \right)^{1/q} \left( \int_0^r v(x)^{-p'} \, dx \right)^{1/p'} = K < \infty.
\]

Furthermore \( K \leq C \leq K(p)^{1/q} (p')^{1/p'} \) for \( 1 < p < q < \infty \) and \( K = C \) if \( p = 1 \) or \( q = \infty \).

The result on the constants is best possible since \( K = C = 1 \) if \( u \) is the characteristic function of \([1, 2]\) and \( v \) is 1 on \([0, 1]\) and \( \infty \) elsewhere while, if \( p = q \), \( C = K p^{1/p} (p')^{1/p} \) in the classical case.

**Proof.** To prove the theorem for \( 1 < p \leq q < \infty \) we first suppose that (2.1) holds. A reduction in the intervals of integration yields
\[
\left( \int_0^\infty u(x)^q \, dx \right)^{1/q} \left( \int_0^r f(x) \, dx \right) \leq C \left( \int_0^r \left[ f(x) v(x) \right]^p \, dx \right)^{1/p}
\]
and choosing \( f(x) = v(x) - p' \) gives (2.2) with \( K \leq C \). To prove that (2.2) implies (2.1) we define \( h(t) = \int_0^t v(s)^{-p'} \, ds \). Then by Hölder’s inequality and Minkowski’s integral inequality [4] we see that
\[
I \equiv \int_0^\infty \left[ u(x) \int_0^x f(t) \, dt \right]^q \, dx \\
\leq \int_0^\infty u(x)^q \left( \int_0^\infty \left[ f(t) v(t) h(t) \chi_{(0 \leq r < \infty)}(x, t) \right]^p \, dt \right)^{q/p} \\
\times \left( \int_0^\infty \left[ v(s) h(s) \right]^{-p'} \, ds \right)^{q/p'} \, dx \\
\leq \left\{ \int_0^\infty \left[ f(t) v(t) h(t) \right]^p \left( \int_0^\infty u(x)^q \left( \int_0^\infty \left[ v(s) h(s) \right]^{-p'} \, ds \right)^{q/p'} \, dx \right)^{p/q} \, dt \right\}^{q/p}.
\]
Performing the innermost integration yields
\[
\left( \int_0^\infty \left[ v(s) h(s) \right]^{-p'} \, ds \right)^{q/p'} = (p')^{q/p'} \left( \int_0^\infty v(u)^{-p'} \, du \right)^{1/p'},
\]
which by (2.2) is bounded by
\[
K(q/p')(p')^{q/p'} \left( \int_0^\infty u(s)^q \, ds \right)^{-1/q}.
\]
Hence
\[
I \leq (K p')^{q/p'} \left\{ \int_0^\infty \left[ f(t) v(t) h(t) \right]^p \left( \int_0^\infty u(x)^q \left( \int_0^\infty u(s)^q \, ds \right)^{-1/p'} \, dx \right)^{p/q} \, dt \right\}^{q/p}.
\]
and again evaluating the inner integral and applying (2.2) we obtain
\[
\left( \int_1^\infty (u(x))^p \left( \int_1^\infty (v(x))^q \, dx \right)^{-1/p'} \, dx \right)^{p/q} = K^p \int_1^\infty (v(x))^{-p'} \, dx \right)^{1/p'} = K^p \int_1^\infty (v(x))^{-p'} \, dx \right)^{1/p'}.
\]
Consequently
\[
I \leq K^p \left( \int_0^\infty [f(t)v(t)]^p \, dt \right)^{1/q},
\]
which proves (2.1) with \( C \leq K(p)^{1/q}(p')^{1/p'} \).

If \( p = 1 \) and/or \( q = \infty \) we show that (2.1) implies (2.2) by an argument which is essentially the same as the corresponding one used in [3] for the cases \( p = 1 \) and \( \infty \), and is hence omitted. To prove the reverse implication, if \( p = 1 \) and \( q < \infty \) we apply Minkowski’s inequality to the left side of (2.1) while, if \( 1 \leq p \leq q = \infty \) we use Hölder’s inequality. The result follows immediately.

The following dual result is obtained analogously.

**Theorem 2.** Suppose that \( 1 \leq p \leq q \leq \infty \) and that \( u \) and \( v \) are non-negative. Then
\[
\left( \int_0^\infty \left( u(x) \int_1^\infty f(t) \, dt \right)^q \, dx \right)^{1/q} \leq C \left( \int_0^\infty [f(x)v(x)]^p \, dx \right)^{1/p}
\]
if and only if
\[
\sup_{r > 0} \left( \int_0^r u(x)^p \, dx \right)^{1/q} \left( \int_0^r (u(x))^{-p'} \, dx \right)^{1/p'} = K < \infty.
\]
In addition \( K \leq C \leq K(p)^{1/q}(p')^{1/p'} \).

We single out the following specific case:

**Corollary 1.** Let \( 1 < p \leq q < \infty \). Then if \( ap > 1 \), \( bp < 1 \) and \( f \) is non-negative
\[
\left( \int_0^\infty \left( x^{-a} \int_1^x f(t) \, dt \right)^q \, dx \right)^{1/q} \leq C \left( \int_0^\infty \left( x^{-a+1/q+1/p'} f(x) \right)^p \, dx \right)^{1/p}
\]
and
\[
\left( \int_0^\infty \left( x^{-b} \int_1^x f(t) \, dt \right)^q \, dx \right)^{1/q} \leq C \left( \int_0^\infty \left( x^{-b+1/q+1/p'} f(x) \right)^p \, dx \right)^{1/p}.
\]

**References**