UNIVALENT α-SPIRAL FUNCTIONS

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1. Introduction. Suppose f is regular in the open unit disk Δ, $|z| < 1$, and has a simple zero at the origin and no other zeros. Špaček **(15)** essentially showed that f is univalent in Δ if and only if

$$\int_{t_1}^{t_2} \frac{f'(re^{it})}{f(re^{it})} e^{it}\, dt \neq 0 \qquad \text{for all } r,\ t_1, \text{ and } t_2$$

such that $0 < r < 1$ and $0 < t_2 - t_1 \leqslant 2\pi$. Consequently, if there exists a complex number ζ for which

$$\text{(1.1)} \qquad \operatorname{Re}[\zeta\{zf'(z)/f(z)\}] > 0, \qquad z \in \Delta,$$

then

$$\int_{t_1}^{t_2} \frac{f'(re^{it})}{f(re^{it})} e^{it}\, dt = \frac{1}{\zeta}\int_{t_1}^{t_2} \frac{\zeta f'(re^{it})}{f(re^{it})} e^{it}\, dt \neq 0;$$

therefore f is univalent in Δ. Functions satisfying Špaček's condition (1.1) have been called "spiral-like" **(8)** and have, in recent years, been the source of useful and important counter-examples in geometric function theory **(2, 5)**.

Classes of spiral-like functions are defined and their mapping properties are described below; the "β-spiral" radius is introduced and calculated for some families of univalent functions. Sharp coefficient bounds are established.

2. Preliminary remarks. Let $\mathfrak{S}$ be the family of all functions f which are regular and univalent in Δ and are normalized by the conditions

$$\text{(2.1)} \qquad f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

DEFINITION 1. *f is regular in Δ and satisfies* (2.1); *f is an α-spiral function, $f \in \mathfrak{T}_\alpha$, if and only if there is a real number α such that*

$$\text{(2.2)} \qquad \operatorname{Re}[e^{i\alpha}\{zf'(z)/f(z)\}] > 0, \qquad z \in \Delta.$$

Špaček's results, described above, show that $\mathfrak{T}_\alpha \subset \mathfrak{S}$. It is clear that $-\pi/2 \leqslant \alpha \leqslant \pi/2$ and that $|\alpha| = \pi/2$ only for the identity function $f(z) = z$; hence we exclude that case from further consideration.

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A geometric interpretation of (2.2) may be given as follows: Let f satisfy (2.2) on the circle $z = re^{i\theta}$, $0 < r < 1$, θ real, and let $C(r)$ be the image of this circle under f. Denote by $T(r, \theta)$ the tangent vector to $C(r)$ at the point $f(re^{i\theta})$ with direction determined by that of increasing θ. Then, with proper choice of arguments, (2.2) is equivalent to

$$0 < \arg[ie^{i\alpha}\{re^{i\theta}f'(re^{i\theta})/f(re^{i\theta})\}] < \pi \tag{2.3}$$

and, since

$$\arg\{ire^{i\theta}f'(re^{i\theta})\} = \arg\{T(r, \theta)\}$$

(**9**, p. 223), we may rewrite (2.3) as

$$0 < \arg\{T(r, \theta)\} - [\arg\{f(re^{i\theta})\} - \alpha] < \pi. \tag{2.4}$$

Consequently, the vector $T(r, \theta)$ is directed into the open half-plane whose boundary has the equation

$$w = f(re^{i\theta}) + t \exp i[\arg\{f(re^{i\theta})\} - \alpha], \qquad t \text{ real}. \tag{2.5}$$

This half-plane does not contain the origin in its interior when $\alpha > 0$, but does contain the origin when $\alpha < 0$. It is not difficult to show that the line given by (2.5) is parallel to the line tangent to the spiral $\rho = ke^{\phi\cot(-\alpha)}$ at the point (ρ, ϕ). (The spiral and the point are given in polar coordinates; the direction of the radius vector is identified with the direction $\arg\{f(re^{i\theta})\}$.) S. Ozaki (**10**) has discussed some of these ideas for the case when f is multivalent.

Let $\mathfrak{P}$ represent the class of normalized holomorphic functions of positive real part, i.e., $P \in \mathfrak{P}$ if and only if P is holomorphic in Δ, $\operatorname{Re}\{P(z)\} > 0$ for z in Δ and $P(0) = 1$.

If $f \in \mathfrak{T}_\alpha$, the introduction of appropriate normalizing factors enables us to write

$$\sec \alpha[e^{i\alpha}\{zf'(z)/f(z)\} - i \sin \alpha]_{z=0} = 1.$$

This leads to useful representation formulas for members of $\mathfrak{T}_\alpha$ in terms of functions in $\mathfrak{P}$. f is an α-spiral function if and only if there exists a function P in $\mathfrak{P}$ such that

$$\frac{zf'(z)}{f(z)} = \frac{\cos \alpha \cdot P(z) + i \sin \alpha}{\cos \alpha + i \sin \alpha}, \qquad z \in \Delta, \tag{2.6}$$

or, equivalently,

$$f(z) = z \exp\left\{\cos \alpha \cdot e^{-i\alpha} \int_0^z \frac{P(t) - 1}{t}\, dt\right\}, \qquad z \in \Delta. \tag{2.7}$$

Many geometric and analytic properties of α-spiral functions can be obtained from (2.6), (2.7), and known properties of the class $\mathfrak{P}$ which has been studied extensively (**12**). These results can be sharpened considerably by a suitable refinement of $\mathfrak{P}$ into subclasses.

DEFINITION 2. *P is a function with positive real part of order ρ, $P \in \mathfrak{P}_\rho$, $0 \leqslant \rho \leqslant 1$ if and only if $P \in \mathfrak{P}$ and $\mathrm{Re}\{P(z)\} \geqslant \rho$ for z in Δ.*

Evidently, $\mathfrak{P}_\rho \subset \mathfrak{P}_\sigma$ whenever $\rho \leqslant \sigma$; therefore $\mathfrak{P}_\rho \subset \mathfrak{P}_0 = \mathfrak{P}$ for all ρ. Furthermore, Q is in $\mathfrak{P}_\rho$ if $Q(z) = (1-\rho)P(z) + \rho$ for some P in $\mathfrak{P}$, and conversely. A particularly distinguished **(6, 12)** member of $\mathfrak{P}$ is the function P_0 defined by

$$P_0(z) = (1+z)/(1-z), \qquad z \in \Delta. \tag{2.8}$$

For convenience we let

$$P_\rho(z) = (1-\rho)P_0(z) + \rho = \{1 + (1-2\rho)z\}/(1-z). \tag{2.9}$$

DEFINITION 3. *$f \in \mathfrak{T}_{\alpha,\rho}$, or f is an α-spiral function of order ρ, if and only if f is defined by (2.7) with $P \in \mathfrak{P}_\rho$.*

For $\rho = 0$, we obtain the sets of functions of Definition 1 and we write $\mathfrak{T}_{\alpha,0} = \mathfrak{T}_\alpha$. On the other hand when $\alpha = 0$ we get the functions which are starlike of order ρ with respect to the origin **(6, 11, 14)** and which are frequently denoted by $\mathfrak{S}_\rho^*$, i.e., $\mathfrak{S}_\rho^* = \mathfrak{T}_{0,\rho}$. The class of all starlike functions is usually symbolized by $\mathfrak{S}^*$; in keeping with this we let $\mathfrak{T}$ represent all α-spiral functions, that is, $\mathfrak{T} = \bigcup_{\alpha,\rho} \mathfrak{T}_{\alpha,\rho}$, $0 \leqslant \rho \leqslant 1$, $-\frac{1}{2}\pi \leqslant \alpha \leqslant \frac{1}{2}\pi$.

The concepts "radius of convexity" and "radius of starlikeness" for classes of univalent functions are useful and have attracted investigators **(6, 9, 13)**. More recently Krzyż **(4)** obtained the radius of the largest disk centred at the origin whose map by every member of $\mathfrak{S}$ is close-to-convex **(3)**. A similar question can be posed for the notion given in Definition 1.

DEFINITION 4. *If f is in $\mathfrak{S}$ and $-\pi/2 < \beta < \pi/2$, then the β-spiral radius of f is*

$$\beta\text{-s.r.}\{f\} = \sup\left[r \colon \mathrm{Re}\left\{e^{i\beta}\frac{zf'(z)}{f(z)}\right\} > 0, \qquad |z| < r\right].$$

DEFINITION 5. *If $\mathfrak{A} \subset \mathfrak{S}$ and $-\pi/2 < \beta < \pi/2$, then the β-spiral radius of $\mathfrak{A}$ is*

$$\beta\text{-s.r.}\ \mathfrak{A} = \inf_{f\in\mathfrak{A}} [\beta\text{-s.r.}\{f\}].$$

The geometric interpretation of the β-spiral radius is clear from the preceding discussion. It is evident that α-s.r. $\mathfrak{T}_\alpha = 1$ and that the 0-s.r. $\mathfrak{A}$ is the radius of starlikeness of $\mathfrak{A}$. By using the principle of subordination **(6, 9, 11)** we shall find β-s.r. $\mathfrak{T}_{\alpha,\rho}$ for arbitrary α, β, and ρ, $-\pi/2 < \alpha, \beta < \pi/2$, and $0 \leqslant \rho \leqslant 1$.

3. Coefficient bounds.

THEOREM 1. *If $f \in \mathfrak{T}_{\alpha,\rho}$ and*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in \Delta, \tag{3.1}$$

then

$$|a_n| \leqslant \prod_{k=0}^{n-2} \frac{|2(1-\rho)\cos\alpha \cdot e^{-i\alpha} + k|}{k+1}, \qquad n = 2, 3, \ldots, \tag{3.2}$$

and these bounds are sharp for all admissible α and ρ and for each n.

Proof. If ω is a function holomorphic in Δ, satisfying the Schwarz lemma **(9)**, and

$$P(z) = \{1 + (1-2\rho)\omega(z)\}/\{1 - \omega(z)\}, \qquad z \in \Delta, \tag{3.3}$$

then $P \in \mathfrak{P}_\rho$, and conversely **(9)**. Consequently, the representation given in (2.6) may be written

$$e^{i\alpha}\sec\alpha \cdot \frac{zf'(z)}{f(z)} - i\tan\alpha = \frac{1 + (1-2\rho)\omega(z)}{1-\omega(z)}. \tag{3.4}$$

Hence

$$\left\{\sum_{k=1}^{\infty} [ke^{i\alpha}\sec\alpha + (1 - 2\rho - i\tan\alpha)]a_k z^k\right\}\omega(z) = \sum_{k=2}^{\infty} [ke^{i\alpha}\sec\alpha - (1 + i\tan\alpha)]a_k z^k, \tag{3.5}$$

or

$$\left\{\sum_{k=1}^{n-1} [ke^{i\alpha}\sec\alpha + (1 - 2\rho - i\tan\alpha)]a_k z^k\right\}\omega(z) = \sum_{k=2}^{n} [ke^{i\alpha}\sec\alpha - (1 + i\tan\alpha)]a_k z^k + \sum_{k=n+1}^{\infty} b_k z^k, \tag{3.6}$$

with the last sum convergent in Δ and $n = 2, 3, \ldots$.

Let $z = re^{i\theta}$, $0 < r < 1$, $0 \leqslant \theta < 2\pi$; then

$$\begin{aligned}
&\sum_{k=1}^{n-1} |ke^{i\alpha}\sec\alpha + (1 - 2\rho - i\tan\alpha)|^2\, |a_k|^2\, r^{2k} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{k=1}^{n-1} [ke^{i\alpha}\sec\alpha + (1 - 2\rho - i\tan\alpha)]a_k r^k e^{i\theta k}\right|^2 d\theta \\
&\geqslant \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{k=1}^{n-1} [ke^{i\alpha}\sec\alpha + (1 - 2\rho - i\tan\alpha)]a_k r^k e^{i\theta k}\right|^2 |\omega(re^{i\theta})|^2\, d\theta \\
&\geqslant \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{k=2}^{n} [ke^{i\alpha}\sec\alpha - (1 + i\tan\alpha)]a_k r^k e^{i\theta k} + \sum_{k=n+1}^{\infty} b_k r^k e^{i\theta k}\right|^2 d\theta \\
&\geqslant \sum_{k=2}^{n} |ke^{i\alpha}\sec\alpha - (1 + i\tan\alpha)|^2 \cdot |a_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |b_k|^2 r^{2k} \\
&\geqslant \sum_{k=2}^{n} |ke^{i\alpha}\sec\alpha - (1 + i\tan\alpha)|^2\, |a_k|^2\, r^{2k}.
\end{aligned} \tag{3.7}$$

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Letting $r \to 1$ and rewriting the preceding inequality, we obtain

$$(3.8)\qquad \sum_{k=1}^{n-1} \{|ke^{i\alpha}\sec\alpha + (1-2\rho - i\tan\alpha)|^2 - |ke^{i\alpha}\sec\alpha - (1+i\tan\alpha)|^2\}|a_k|^2 \geqslant |ne^{i\alpha}\sec\alpha - (1+i\tan\alpha)|^2\cdot|a_n|^2;$$

some simplification reduces this to

$$(3.9)\qquad 4(1-\rho)\sum_{k=1}^{n-1}(k-\rho)|a_k|^2 \geqslant (n-1)^2\sec^2\alpha\cdot|a_n|^2.$$

For $n = 2$, (3.9) reads $4(1-\rho)^2 \geqslant \sec^2\alpha\cdot|a_2|^2$
or

$$(3.10)\qquad |a_2| \leqslant 2(1-\rho)\cos\alpha$$

which is equivalent to (3.2). (3.2) is established for larger n from (3.9) by an induction argument.

Fix n, $n \geqslant 3$, and suppose that (3.2) holds for $k = 2, 3, \ldots, n-1$. Then

$$(3.11)\qquad |a_n|^2 \leqslant \frac{4(1-\rho)\cos^2\alpha}{(n-1)^2}\left\{(1-\rho) + \sum_{k=2}^{n-1}(k-\rho)\prod_{j=0}^{k-2}\frac{|2(1-\rho)\cos\alpha\cdot e^{-i\alpha}+j|^2}{(j+1)^2}\right\}.$$

We must show that the square of the right side of (3.2) is equal to the right side of (3.11); that is,

$$(3.12)\qquad \left\{\prod_{j=0}^{m-2}\frac{|2(1-\rho)\cos\alpha\cdot e^{-i\alpha}+j|}{j+1}\right\}^2 = \frac{4(1-\rho)\cos^2\alpha}{(m-1)^2} \times\left\{(1-\rho)+\sum_{k=2}^{m-1}(k-\rho)\left[\prod_{j=0}^{k-2}\frac{|2(1-\rho)\cos\alpha\cdot e^{-i\alpha}+j|}{j+1}\right]^2\right\}$$

for $m = 3, 4, \ldots$. A brief calculation verifies (3.12) for $m = 3$ and proves (3.2) for $n = 3$. Assume that (3.12) is valid for all m, $3 < m \leqslant n-1$; then (3.9) and (3.11) give

$$\begin{aligned}|a_n|^2 &\leqslant \frac{4(1-\rho)\cos^2\alpha}{(n-1)^2}\left\{(1-\rho)+\sum_{k=2}^{n-2}(k-\rho)\prod_{j=0}^{k-2}\frac{|2(1-\rho)\cos\alpha\cdot e^{-i\alpha}+j|^2}{(j+1)^2}\right.\\ &\qquad\left. + (n-1-\rho)\prod_{j=0}^{n-3}\frac{|2(1-\rho)\cos\alpha\cdot e^{-i\alpha}+j|^2}{(j+1)^2}\right\}\\ &= \prod_{j=0}^{n-3}\frac{|2(1-\rho)\cos\alpha\cdot e^{-i\alpha}+j|^2}{(j+1)^2}\cdot\left\{\frac{(n-2)^2+4(n-1-\rho)(1-\rho)\cos^2\alpha}{(n-1)^2}\right\}\\ &= \prod_{j=0}^{n-2}\frac{|2(1-\rho)\cos\alpha\cdot e^{-i\alpha}+j|^2}{(j+1)^2}.\end{aligned}$$

This concludes the proof of (3.2).

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To show the estimate is sharp we use (2.9) together with (2.7) and obtain

$$F_{\alpha,\rho}(z) = \frac{z}{(1-z)^{2(1-\rho)\cos\alpha \cdot e^{-i\alpha}}} \tag{3.13}$$

$$= z + \sum_{n=2}^{\infty} \prod_{k=0}^{n-2} \frac{[2(1-\rho)\cos\alpha \cdot e^{-i\alpha} + k]}{k+1} z^n.$$

By choosing $\rho = 0$ in Theorem 1, we get the following result, which was proved recently by the late J. Zamorski **(16)**.

COROLLARY 1. *If $f \in \mathfrak{T}_\alpha$ and has the representation* (3.1), *then*

$$|a_n| \leqslant \prod_{j=0}^{n-2} \frac{|2e^{-i\alpha}\cos\alpha + j|}{j+1}, \qquad n = 2, 3, \ldots . \tag{3.14}$$

Taking $\alpha = 0$, we obtain a theorem of Robertson **(11)** which was also obtained recently by Schild **(14)**.

COROLLARY 2. *If f is starlike of order ρ and has* (3.1) *as its Maclaurin series, then*

$$|a_n| \leqslant \frac{\prod_{k=2}^{n} (k - 2\rho)}{(n-1)!}, \qquad n = 2, 3, \ldots . \tag{3.15}$$

The technique of using (3.3) and the integration in (3.7) is due to Clunie **(1)**. The same method applied to (3.3) yields coefficient estimates for functions in $\mathfrak{P}_\rho$.

THEOREM 2. *If*

$$P(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \in \mathfrak{P}_\rho,$$

then

$$|p_k| \leqslant 2(1-\rho), \qquad k = 1, 2, \ldots ;$$

P_ρ, (2.9), *renders these bounds sharp.*

For $\rho = 0$ we obtain the classical theorem of Carathéodory **(12)**, and a proof of it by Clunie's method is given elsewhere by the author **(6**, Lemma 3.2**)**. It is easily adapted to give a proof of Theorem 2.

It is interesting to note that the method of equating coefficients in (2.6) and using Theorem 2, as can be done for $\mathfrak{S}^*$, **(9)**, or for $\mathfrak{S}_\rho^*$ **(11)**, does not yield sharp estimates for the functions of Theorem 1 or Corollary 1.

4. The β-spiral radius. If $f \in \mathfrak{T}_{\alpha,\rho}$, then by making use of Definitions 3 and 5 and (2.6) we see that the β-spiral radius of f is the largest number r such that for some P in $\mathfrak{P}_\rho$ and $|z| < r$

$$\operatorname{Re}\{e^{i(\beta-\alpha)}[\cos\alpha \cdot P(z) + i \sin\alpha]\} \geqslant 0. \tag{4.1}$$

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It is well known and indeed follows from (2.8) and (3.3) that every member of $\mathfrak{P}$ is subordinate to P_0 **(9)**; therefore every function in $\mathfrak{P}_\rho$ is subordinate to P_ρ. Consequently, if (4.1) holds for P_ρ, then it holds, a fortiori, for all P in $\mathfrak{P}_\rho$.

Let

$$B(z) = e^{i(\beta-\alpha)}[\cos\alpha \cdot P_\rho(z) + i\sin\alpha] = e^{i(\beta-\alpha)}\left\{\cos\alpha\cdot\left[\frac{1+(1-2\rho)z}{1-z}\right] + i\sin\alpha\right\}. \tag{4.2}$$

$B(z)$ maps the disk $|z| \leqslant r$ onto a disk whose centre is

$$e^{i(\beta-\alpha)}\left\{\cos\alpha\cdot\left[\frac{1+(1-2\rho)r^2}{1-r^2}\right] + i\sin\alpha\right\} \tag{4.3}$$

and whose radius is

$$\left| e^{i(\beta-\alpha)}\cos\alpha\cdot\frac{2(1-\rho)r}{1-r^2}\right|. \tag{4.4}$$

Hence, $\mathrm{Re}\{B(z)\} \geqslant 0$ if and only if

$$\mathrm{Re}\left\{e^{i(\beta-\alpha)}\left[\cos\alpha\cdot\frac{1+(1-2\rho)r^2}{1-r^2} + i\sin\alpha\right]\right\} \geqslant \cos\alpha\cdot\frac{2(1-\rho)r}{1-r^2}, \tag{4.5}$$

or

$$\cos\beta\cdot(1-r^2) + 2(1-\rho)r^2\cos(\beta-\alpha)\cdot\cos\alpha \geqslant \cos\alpha\cdot 2(1-\rho)r. \tag{4.6}$$

These results are summarized below:

THEOREM 3. *β-s.r. $\mathfrak{T}_{\alpha,\rho}$ is the smallest positive root r of the equation*

$$[2(1-\rho)\cos(\beta-\alpha)\cdot\cos\alpha - \cos\beta]r^2 - 2(1-\rho)\cos\alpha\cdot r + \cos\beta = 0. \tag{4.7}$$

$F_{\alpha,\rho}$, defined in (3.13), *shows this result is sharp.*

By fixing the parameters α, β, and ρ in the theorem we obtain some interesting special cases.

COROLLARY 3. *The radius of starlikeness of $\mathfrak{T}_\alpha$ is*

$$0\text{-s.r. } \mathfrak{T}_\alpha = 1/(\cos\alpha + |\sin\alpha|). \tag{4.8}$$

The last result was obtained recently by M. S. Robertson **(13)**.

COROLLARY 4. *β-s.r. $\mathfrak{S}_\rho^*$ is the smallest positive root r of*

$$\cos\beta\cdot(1-2\rho)r^2 - 2(1-\rho)r + \cos\beta = 0. \tag{4.11}$$

When $\rho = \frac{1}{2}$, (4.11) is linear; since $\mathfrak{K}$, the class of convex functions in $\mathfrak{S}$, is contained in $\mathfrak{S}_{\frac{1}{2}}^*$, **(7)**, we may state the following interesting conclusion.

COROLLARY 5. *β-s.r. $\mathfrak{K} = \cos\beta$.*

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