

# ON POSITIVITY OF SEVERAL COMPONENTS OF SOLUTION VECTOR FOR SYSTEMS OF LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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**Abstract.** In the classical theorems about lower and upper vector functions for systems of linear differential equations very heavy restrictions on the signs of coefficients are assumed. These restrictions in many cases become necessary if we wish to compare *all* the components of a solution vector. The formulas of the integral representation of the general solution explain that these theorems claim actually the positivity of *all* elements of the Green's matrix. In this paper we define a principle of partial monotonicity (comparison of only several components of the solution vector), which assumes only the positivity of elements in a corresponding row of the Green's matrix. The main theorem of the paper claims the equivalence of positivity of all elements in the  $n$ th row of the Green's matrices of the initial and two other problems, non-oscillation of the  $n$ th component of the solution vector and a corresponding assertion about differential inequality of the de La Vallée Poussin type. Necessary and sufficient conditions of the partial monotonicity are obtained. It is demonstrated that our sufficient tests of positivity of the elements in the  $n$ th row of the Cauchy matrix are exact in corresponding cases. The main idea in our approach is a construction of an equation for the  $n$ th component of the solution vector. In this sense we can say that an analog of the classical Gauss method for solving systems of functional differential equations is proposed in the paper.

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**1. Introduction.** Consider the system

$$(M_i x)(t) \equiv x'_i(t) + \sum_{j=1}^n (B_{ij} x_j)(t) = f_i(t), \quad t \in [0, \omega], \quad i = 1, \dots, n, \quad (1.1)$$

where  $x = \text{col}(x_1, \dots, x_n)$ ,  $B_{ij}: C_{[0, \omega]} \rightarrow L_{[0, \omega]}$ ,  $i, j = 1, \dots, n$ , are linear continuous operators and  $C_{[0, \omega]}$  and  $L_{[0, \omega]}$  are the spaces of continuous and measurable essentially

bounded functions  $y : [0, \omega] \rightarrow R^1$  respectively. We will also consider the case  $\omega = \infty$ , which is important, for example, in problems of stability.

Let  $l : C^n_{[0,\omega]} \rightarrow R^n$  be a linear bounded functional. If the homogeneous boundary value problem  $(M_i x)(t) = 0, t \in [0, \omega], i = 1, \dots, n, lx = 0$ , has only the trivial solution, then the boundary value problem

$$(M_i x)(t) = f_i(t), \quad t \in [0, \omega], \quad i = 1, \dots, n, \quad lx = \alpha, \tag{1.2}$$

has for each  $f_i \in L_{[0,\omega]}$ ,  $\alpha \in R^n$  a unique solution, which has the following representation [2]:

$$x(t) = \int_0^\omega G(t, s)f(s) ds + X(t)\alpha, \quad t \in [0, \omega], \tag{1.3}$$

where the  $n \times n$  matrix  $G(t, s)$  is called the Green’s matrix of problem (1.2),  $X(t)$  is an  $n \times n$  fundamental matrix of the system  $(M_i x)(t) = 0, i = 1, \dots, n$ , such that  $lX = E$  ( $E$  is the unit  $n \times n$  matrix),  $f = \text{col}(f_1, \dots, f_n)$ . It is clear from the solution representation (1.3) that the matrices  $G(t, s)$  and  $X(t)$  determine all properties of solutions.

If the Green’s matrix  $G(t, s)$  is positive then from the conditions

$$(M_i x)(t) \geq (M_i y)(t), \quad t \in [0, \omega], \quad i = 1, \dots, n, \quad lx = ly, \tag{1.4}$$

it follows that

$$x_i(t) \geq y_i(t), \quad t \in [0, \omega], \quad i = 1, \dots, n. \tag{1.5}$$

The great importance of the property (1.4)  $\rightarrow$  (1.5) in the approximate integration was noted by S.A. Tchaplygin [25]. Series of papers, starting with the paper by N.N. Luzin [24], were devoted to the various aspects of Tchaplygin’s approximate method of integration. Note in this connection the well-known monograph by V. Lakshmikantham and S. Leela [21] and the recent monograph by I. Kiguradze and B. Puza [19].

As a particular case of system (1.1) let us consider the following delay system:

$$x'_i(t) + \sum_{j=1}^n p_{ij}(t)x_j(h_{ij}(t)) = f_i(t), \quad i = 1, \dots, n, \quad t \in [0, \omega],$$

$$x(\theta) = 0 \text{ for } \theta < 0, \tag{1.6}$$

where  $p_{ij}$  are measurable essentially bounded functions and  $h_{ij}$  are measurable functions such that  $h_{ij}(t) \leq t$  for  $i, j = 1, \dots, n, t \in [0, \omega]$ . Its general solution has the representation

$$x(t) = \int_0^t C(t, s)f(s) ds + C(t, 0)x(0), \quad t \in [0, \omega], \tag{1.7}$$

where  $C(t, s) = \{C_{ij}(t, s)\}_{i,j=1,\dots,n}$  is called the Cauchy matrix of system (1.6). Note that for each fixed  $s$  the matrix  $C(t, s)$  is the fundamental matrix of the system

$$x'_i(t) + \sum_{j=1}^n p_{ij}(t)x_j(h_{ij}(t)) = 0, \quad i = 1, \dots, n, \quad t \in [0, \omega],$$

$$x(\theta) = 0 \text{ for } \theta < s,$$

such that  $C(s, s)$  is the unit  $n \times n$  matrix [2].

The classical Wazewskii's theorem claims [27] that the condition

$$p_{ij} \leq 0 \text{ for } j \neq i, \quad i, j = 1, \dots, n, \tag{1.8}$$

is necessary and sufficient for non-negativity of all elements  $C_{ij}(t, s)$  of the Cauchy matrix and consequently of the property (1.4), (1.5) for the system of ordinary differential equations

$$x'_i(t) + \sum_{j=1}^n p_{ij}(t)x_j(t) = f_i(t), \quad i = 1, \dots, n, \quad t \in [0, \omega]. \tag{1.9}$$

In Section 3 we obtain an extension of this result on boundary value problem (1.2). Note that results of this type for various boundary problems can be found in [18, 19].

The assertions of Section 3 are auxiliary for our main results, which will be obtained in Section 4. We focus our attention upon the problem of comparison for only one of the components of solution vector.

Let  $k_i$  be either 1 or 2. In Section 4 we consider the following problem: when from the conditions

$$(-1)^{k_i}[(M_i x)(t) - (M_i y)(t)] \geq 0, \quad t \in [0, \omega], \quad lx = ly, \quad i = 1, \dots, n, \tag{1.10}$$

it does follow that for a corresponding fixed component  $x_r$  of the solution vector the inequality

$$x_r(t) \geq y_r(t), \quad t \in [0, \omega], \tag{1.11}$$

is satisfied. This property is a weakening of the property (1.4)  $\rightarrow$  (1.5) and, as we will obtain below, leads to essentially less hard limitations on the given system. From the formula of solution's representation it follows that this property is reduced to sign-constancy of all elements standing only in the  $r$ th row of the Green's matrix.

It should be noted here that the classical monotone technique in the theory of non-linear differential system is based on monotonicity of corresponding operators acting on spaces of solutions vector of systems. These operators are obtained as a result of regularisation procedures reducing the boundary value problems to equivalent integral equations [2, 17, 21]. The monotonicity of such operators, obtained on the basis of positivity of corresponding Green's matrices of boundary value problems, leads, as it was noted above in the case of the initial problem, to very heavy restrictions on given systems. In this paper we propose new ideas of such a regularisation, leading to a problem of so-called partial monotonicity of these operators, which can be described in a linear case as their positivity in the space of vector functions consisting of only *several* and *not all* the components of solution vectors of a given system. It will be clear that in this case positivity of several components of solution vector is achieved on the basis of positivity of elements standing in several corresponding rows of Green's matrices.

Our technique in proofs of main assertions of the paper is based on a construction of a corresponding scalar functional differential equation for  $n$ th component of a solution vector. In this sense it is similar to the idea of the classical Gauss method for solving systems of algebraic equations. In Section 2 the auxiliary results for first-order scalar functional differential equations are included. These assertions develop the known non-oscillation results of [9, 13].

The problem of the asymptotic stability of delay differential systems is one of the most important applications of results on positivity of the Cauchy matrix  $C(t, s)$ . The technique of the use of positivity of  $C(t, s)$  in the exponential stability was proposed in [7], where necessary and sufficient conditions of the exponential stability for a system possessing positivity of the Cauchy matrix were obtained. In other terminology, this approach can be found in [11]. The same idea of a regularisation, leading to an analysis of vector integral equations with positive operators, was proposed in [3, 15]. Important development of this approach can be found in [12].

It should also be noted that our approach can be applied to differential equations with unbounded delay, which have been intensively studied over the past 10 years. The foundations of the theory of such equations were developed in [14, 22]. Non-oscillation properties of such equations were considered, for example, in [9] (note also a recent paper [5]). Integro-differential equations can be considered as a class of equations with unbounded memory. Various applications of integro-differential and functional differential equations are presented, for example, in [4, 8, 10].

In this paper we consider the boundary value problems with boundary conditions of the following form:

$$l_i x_i = 0, \quad i = 1, \dots, n, \quad (1.12)$$

where  $l_i: C_{[0,b]} \rightarrow \mathbb{R}^1$ ,  $i = 1, \dots, n$ , is a linear boundary functional. Note that each of the following types of boundary conditions

$$x_i(0) = 0, \quad i = 1, \dots, n, \quad (1.13)$$

$$x_i(\omega) = 0, \quad i = 1, \dots, n, \quad (1.14)$$

$$x_i(0) = x_i(\omega), \quad i = 1, \dots, n, \quad (1.15)$$

$$x_i(0) = 0, \quad x_j(\omega) = 0, \quad i = 1, \dots, k, \quad j = k + 1, \dots, n \quad (1.16)$$

is a particular case of condition (1.12).

**2. First-order scalar functional differential equation – auxiliary results.** In this Section we formulate auxiliary results on first-order scalar equations. Let us consider boundary value problems described by the following scalar equation:

$$(My)(t) \equiv x'(t) + (Bx)(t) = f(t), \quad t \in [0, +\infty), \quad (2.1)$$

and one of the following boundary conditions:

$$x(0) = 0, \quad (2.2)$$

$$x(\omega) = 0, \quad (2.3)$$

$$x(0) = x(\omega), \quad (2.4)$$

where  $B: C \rightarrow L$  is a bounded linear Volterra operator acting from the space of continuous functions to the space of measurable essentially bounded functions determined on  $[0, \infty)$ . Denote by  $C_{[0,\omega]}$  the space of continuous functions and  $L_{[0,\omega]}$  – the space of measurable essentially bounded functions  $y: [0, \omega] \rightarrow \mathbb{R}^1$ . The fact that  $B: C \rightarrow L$  is a Volterra operator allows us to consider also the operator  $B_{[0,\omega]}: C_{[0,\omega]} \rightarrow L_{[0,\omega]}$ , where  $(B_{[0,\omega]}x) = (Bx)(t)$  for  $t \in [0, \omega]$  and each continuous function  $x \in C$ . Below we write  $B$  instead of  $B_{[0,\omega]}$ .

If the periodic boundary value problem (2.1), (2.4) has a unique solution, then it has the following representation:

$$x(t) = \int_0^\omega P(t, s)f(s) ds, \tag{2.5}$$

where  $P(t, s)$  is called the Green’s function of the periodic problem (2.1), (2.4).

If the boundary value problem (2.1), (2.3) has a unique solution, it has the following representation:

$$x(t) = \int_0^\omega G(t, s)f(s) ds, \tag{2.6}$$

where  $G(t, s)$  is called the Green’s function of the problem (2.1), (2.3).

Define an operator  $N : C_{[0,\omega]} \rightarrow C_{[0,\omega]}$  as follows:

$$(Nx)(t) = \int_t^\omega (Bx)(s) ds. \tag{2.7}$$

For (2.1), the following assertion was proven.

**THEOREM 2.1 [1].** *Let  $B : C_{[0,\omega]} \rightarrow L_{[0,\omega]}$  be a positive non-zero Volterra operator, then the following assertions are equivalent:*

- (1) *There exists a non–negative absolutely continuous function  $v$  such that  $v' \in L_{[0,\omega]}$ ,*

$$Mv(t) \leq 0, \quad v(\omega) - \int_t^\omega (Mv)(s) ds > 0, \quad t \in [0, \omega]. \tag{2.8}$$

- (2) *The spectral radius of the operator  $N$  is less than 1.*
- (3) *The problem (2.1), (2.3) is uniquely solvable and its Green’s function  $G(t, s)$  is negative for  $0 \leq t < s \leq \omega$  and non-positive for  $0 \leq s \leq t \leq \omega$ .*
- (4) *A non-trivial solution of the homogeneous equation  $(My)(t) = 0, t \in [0, \omega]$  has no zeros on  $[0, \omega]$ .*
- (5) *The Cauchy function of (2.1) is positive for  $0 \leq s \leq t \leq \omega$ .*
- (6) *The periodic problem (2.1), (2.4) is uniquely solvable and its Green’s function  $P(t, s)$  is positive for  $t, s \in [0, \omega]$ .*
- (7) *There exists a positive continuous function  $v$  such that  $v(t) > Nv(t), t \in [0, \omega]$ .*
- (8) *There exists a positive essentially bounded function  $u$  such that*

$$B e^{\int_s^t u(\xi) d\xi}(t) \leq u(t), \quad t \in [0, \omega].$$

**REMARK 2.1.** If we consider (2.1) on the semi-axis, the equivalence of the corresponding assertions (1a) – (8a) is also fulfilled. Let us describe these assertions:

- (1a) *there exists a positive locally absolutely continuous function  $v$  such that  $v' \in L$  and  $Mv(t) < 0$  for  $t \in [0, +\infty)$ .*

If one sets  $\omega = +\infty$  in the assertions (4), (5) and (8), we obtain the assertions (4a), (5a) and (8a) respectively.

If we require that each of the assertions (2), (3), (6) and (7) is fulfilled for each  $\omega \in (0, +\infty)$ , the assertions (2a), (3a), (6a) and (7a) will be obtained respectively.

**REMARK 2.2.** The assertion (1)  $\rightarrow$  (4) is an analog for the first-order functional differential equations of the classical de La Vallée Poussin theorem about the

differential inequality obtained in [26] for ordinary second-order equations. Assertions (4) → (3) and (4) → (5) are analogs of the corresponding assertions connecting non-oscillation and positivity of Green’s functions for the  $n$ -th-order ordinary differential equations [23]. Note that for the delayed differential equation

$$x'(t) + p(t)x(t - g(t)) = f(t), \quad t \in [0, +\infty),$$

the inequality in assertion (8a) is of the following form,

$$p(t) e^{\int_{t-g(t)}^t u(s) ds} (t) \leq u(t), \quad t \in [0, +\infty),$$

and the equivalence of assertions (4a) and (8a) is the well-known result (see [9], p. 29).

Consider the equation

$$(My)(t) \equiv x'(t) + (Bx)(t) - (Ax)(t) = f(t), \quad t \in [0, +\infty), \tag{2.9}$$

where  $A : C \rightarrow L$  is a bounded linear Volterra operator acting from the space of continuous functions on the space of essentially bounded functions determined on  $[0, \infty)$ .

**THEOREM 2.2.** *If  $B$  and  $A$  are positive operators, and the Cauchy function  $C^+(t, s)$  of (2.1) is positive for  $0 \leq s \leq t \leq +\infty$ , then the Cauchy function  $C(t, s)$  and the Green’s matrix  $P(t, s)$  of the periodic problem for (2.9) satisfy the inequalities  $C(t, s) \geq C^+(t, s)$  for  $0 \leq s \leq t \leq +\infty$  and if  $(A1)(t) < (B1)(t)$ , then  $P(t, s) \geq P^+(t, s)$  for  $t, s \in [0, \omega]$  for each  $\omega \in (0, +\infty)$  (here  $P^+(t, s)$  is the Green’s function of periodic problem (2.1), (2.4)).*

*Proof.* The formula of the representation of the solutions (1.7) and (2.5) allows us to write the initial and periodic problems ((2.1), (2.2) and (2.1), (2.4) respectively) in the following equivalent integral forms:

$$x(t) = \int_0^t C^+(t, s)(Ax)(s) ds + \int_0^t C^+(t, s)f(s) ds + C^+(t, 0)x(0) \tag{2.10}$$

and

$$x(t) = \int_0^t P^+(t, s)(Ax)(s) ds + \int_0^t P^+(t, s)f(s) ds. \tag{2.11}$$

Denote by  $T_C$  and  $T_P$  the operators

$$(T_Cx)(t) = \int_0^t C^+(t, s)(Ax)(s) ds$$

and

$$(T_Px)(t) = \int_0^t P^+(t, s)(Ax)(s) ds$$

respectively. It is clear that the operators  $T_C$  and  $T_P$  are positive. Equations (2.10) and (2.11) can be written as follows:

$$\begin{aligned} x(t) - (T_Cx)(t) &= \varphi_1(t), \\ x(t) - (T_Px)(t) &= \varphi_2(t), \end{aligned}$$

where  $\varphi_1(t) = \int_0^t C^+(t, s)f(s) ds + C^+(t, 0)x(0)$  and  $\varphi_2(t) = \int_0^t P^+(t, s)f(s) ds$ . If  $f \geq 0$ ,  $x(0) \geq 0$ , then  $\varphi_1 \geq 0$  and

$$x(t) = ((1 - T)^{-1}\varphi_1)(t) = \varphi_1 + T\varphi_1 + T^2\varphi_1 + T^3\varphi_1 + \dots \geq \varphi_1,$$

and consequently

$$\int_0^t C(t, s)f(s) ds \geq \int_0^t C^+(t, s)f(s) ds,$$

for each  $f \geq 0$ , and this implies that  $C(t, s) \geq C^+(t, s)$ . Analogously it can be proven that  $P(t, s) \geq P^+(t, s)$ . □

**DEFINITION 2.1.** Let us determine the function  $h: [0, +\infty) \rightarrow [0, +\infty)$  as the maximal possible value for which the equality  $y_1(s) = y_2(s)$  for  $s \in [h(t), +\infty)$  implies the equality  $(By_1)(s) = (By_2)(s)$  for  $s \in [t, +\infty)$  for each two continuous functions  $y_1$  and  $y_2: [0, +\infty) \rightarrow (-\infty, +\infty)$ .

If we set  $v = \exp[-e \int_0^t (B1)(s) ds]$  in assertion (1) of Theorem 2.1, then the following result is obtained.

**THEOREM 2.3 [1].** Let  $B: C_{[0,+\infty)} \rightarrow L_{[0,+\infty)}$  be a positive linear Volterra operator and

$$\int_{h(t)}^t (B1)(s) ds \leq \frac{1}{e}, \quad t \in (0, +\infty), \tag{2.12}$$

then  $C(t, s) > 0$  for  $0 \leq s \leq t < +\infty$ , and each of the assertions (2)–(6) is satisfied for each  $\omega \in (0, +\infty)$ .

**3. Positivity of Green’s matrices of systems of functional differential equations.** In this section we consider the system

$$(M_i x)(t) \equiv x'_i(t) + \sum_{j=1}^n (B_{ij}x_j)(t) = f_i(t), \quad i = 1, \dots, n, \quad t \in [0, \omega], \tag{3.1}$$

with boundary conditions of the form

$$l_i x_i = 0, \quad i = 1, \dots, n, \tag{3.2}$$

where  $B_{ij}: C_{[0,\omega]} \rightarrow L_{[0,\omega]}$  are linear bounded Volterra operators and  $l_i: C_{[0,\omega]}^0 \rightarrow R^1$  are linear bounded functionals ( $i, j = 1, 2, 3, \dots, n$ ).

Let us define operator  $R: C_{[0,\omega]}^n \rightarrow C_{[0,\omega]}^n$  acting in the space of  $n$ th-dimensional vector functions with continuous elements  $x_i: [0, \omega] \rightarrow R^1$ ,  $\|x\|_{C_{[0,\omega]}} = \max_{1 \leq i \leq n} \max_{t \in [0, \omega]} |x_i(t)|$  as follows:

$$(Rx)(t) = \text{col} \left( - \int_0^\omega g_i(t, s) \sum_{j=1, j \neq i}^n (B_{ij}x_j)(s) ds \right)_{i=1}^n, \quad t \in [0, \omega]. \tag{3.3}$$

**THEOREM 3.1.** *Let the following conditions be fulfilled:*

(1) *The Green's functions  $g_i(t, s)$  ( $i = 1, \dots, n$ ) of  $n$  scalar boundary value problems*

$$(M_i x)(t) \equiv x'_i(t) + (B_{ii} x_i)(t) = f_i(t), \quad t \in [0, \omega], \tag{3.4}$$

$$l_i x_i = 0, \tag{3.5}$$

*exist and preserve their signs such that*

$$\int_0^\omega |g_i(t, s)| \varphi(s) ds > 0, \quad t \in [0, \omega] \tag{3.6}$$

*for each positive measurable essentially bounded function  $\varphi$ .*

(2) *The non-diagonal operators  $B_{ij}$  ( $j \neq i$ ) are positive or negative such that the operator  $R: C^n_{[0,\omega]} \rightarrow C^n_{[0,\omega]}$  determined by the formula (3.3) is positive.*

*Then the following assertions are equivalent:*

(a) *There exists a vector function  $v \in C^n_{[0,\omega]}$  with positive absolutely continuous components  $v_i: [0, \omega] \rightarrow [0, +\infty)$  such that*

$$\int_0^\omega g_i(t, s)(M_i v)(s) ds > 0, \quad t \in [0, \omega].$$

(b) *Boundary value problem (3.1), (3.2) is uniquely solvable for each right-hand side  $f = \text{col}(f_1, \dots, f_n)$  such that  $f_i \in L_{[0,\omega]}$ ,  $i = 1, \dots, n$ , and elements of its Green's matrix preserve sign and satisfy the inequality*

$$g_i(t, s)G_{ij}(t, s) \geq 0, \quad t, s \in [0, \omega], \tag{3.7}$$

*while*

$$|G_{ii}(t, s)| \geq |g_i(t, s)|, \quad t, s \in [0, \omega], \tag{3.8}$$

*for  $i, j = 1, \dots, n$ .*

(c) *The spectral radius of the operator  $R: C^n_{[0,\omega]} \rightarrow C^n_{[0,\omega]}$  is less than 1.*

*Proof.* (a)→(c). The function  $v$  satisfies the boundary value problem

$$(M_i x)(t) = \varphi_i(t), \quad l_i x_i = 0, \quad i = 1, \dots, n, \quad t \in [0, \omega], \tag{3.9}$$

where  $\varphi_i(t) = (M_i v)(t)$ ,  $t \in [0, \omega]$ . It is clear that this function  $v$  also satisfies the integral equation  $x(t) - (Rx)(t) = \psi(t)$ ,  $t \in [0, \omega]$ , where

$$\psi(t) = \text{col} \left( \int_0^\omega g_i(t, s)\varphi_i(s) ds \right)_{i=1}^n, \quad t \in [0, \omega]. \tag{3.10}$$

The condition (a) implies that all components  $\psi_i(t)$ ,  $i = 1, \dots, n$ , of the vector  $\psi(t)$  are positive for  $t \in [0, \omega]$ . By the known result of M.A. Krasnosel'skii [20, p. 86], the spectral radius of the operator  $R: C^n_{[0,\omega]} \rightarrow C^n_{[0,\omega]}$  is less than 1.

(c)→(b). If the spectral radius of the operator  $R: C^n_{[0,\omega]} \rightarrow C^n_{[0,\omega]}$  is less than 1, then the sequence  $\{x^m\}$  of vectors  $x = (x_1, \dots, x_n)$ , where  $x^m = Rx^{m-1} + \psi$ ,  $x^0 = \psi$ ,  $\psi \in C^n_{[0,\omega]}$ , converges to the solution of the equation  $x = Rx + \psi$ , which is equivalent to boundary value problem (3.1), (3.2). This means that boundary value problem (3.1),



(3.2) is uniquely solvable, while for  $\psi$  with non-negative components  $\psi_i, i = 1, \dots, n$ , we obtain  $x_i \geq \psi_i \geq 0, i = 1, \dots, n$ .

If  $f_i$  preserves its sign for  $i = 1, \dots, n$  such that

$$\int_0^\omega g_i(t, s)f_i(s) ds \geq 0, t \in [0, \omega], \tag{3.11}$$

then

$$\psi_i(t) = \int_0^\omega g_i(t, s)f_i(s) ds \geq 0, t \in [0, \omega], \tag{3.12}$$

and consequently  $x_i(t) \geq \psi_i(t), i = 1, \dots, n$ . The inequality (3.7) has been proven.

In order to prove inequality (3.8) we set  $f_j = 0$  for  $j \neq i, j = 1, \dots, n$ . In this case we obtain

$$x_i(t) - \psi_i(t) = \int_0^\omega [G_{ii}(t, s) - g_i(t, s)]f_i(s) ds, t \in [0, \omega], i = 1, \dots, n. \tag{3.13}$$

The inequality  $x_i(t) \geq \psi_i(t)$  implies the inequality (3.8).

(b)  $\rightarrow$  (a). In order to prove this implication we can set  $v(t) = \int_0^\omega G(t, s)E ds$ , where  $E = \text{col}(e_1, \dots, e_n)$  and  $e_i$  is equal to 1 or  $-1$  such that  $e_i g_i(t, s) \geq 0, t, s \in [0, \omega]$ .

In case of the Cauchy problem the spectral radius of the Volterra integral operator is equal to zero. Moreover, we can set  $\omega = +\infty$  and get the following assertion.

**THEOREM 3.2.** *Let the following conditions be fulfilled:*

- (1) *The Cauchy functions  $c_i(t, s) (i = 1, \dots, n)$  of  $n$  scalar equations*

$$x'_i(t) + (B_{ii}x_i)(t) = f_i(t), t \in [0, +\infty) \tag{3.14}$$

*are positive for  $0 \leq s \leq t < +\infty, i = 1, \dots, n$ .*

- (2) *All non-diagonal operators  $B_{ij} (j \neq i, i, j = 1, \dots, n)$  are negative.*

*Then all elements  $C_{ij}(t, s)$  of the Cauchy matrix  $C(t, s)$  of the system (3.1) are non-negative, while  $C_{ii}(t, s)$  are positive for  $0 \leq s \leq t \leq \omega, i, j = 1, \dots, n$ .*

**REMARK 3.1.** In the case of a diagonal system (the operators  $B_{ij}$  are zero operators ( $j \neq i, i, j = 1, \dots, n$ )) the elements  $C_{ii}(t, s)$  coincide with  $c_i(t, s), C_{ij}(t, s) = 0$  for  $j \neq i, i, j = 1, \dots, n$ , and condition (1) becomes necessary and sufficient for the non-negativity of all elements of the Cauchy matrix.

**REMARK 3.2.** It will be demonstrated below that condition (2) is not necessary for non-negativity of all elements of the Cauchy matrix.

**DEFINITION 3.1.** Let us determine the function  $h_{ij^*} : [0, +\infty) \rightarrow [0, +\infty)$  as the maximal possible value for which the equality  $y_1(s) = y_2(s)$  for  $s \in [h_{ij^*}(t), +\infty)$  implies the equality  $(B_{ij}y_1)(s) = (B_{ij}y_2)(s)$  for  $s \in [t, +\infty)$  for each two continuous functions  $y_1$  and  $y_2 : [0, +\infty) \rightarrow (-\infty, +\infty)$ .

It follows from a known result [16] that each bounded operator  $B_{ij} : C_{[0, +\infty)} \rightarrow L_{[0, +\infty)}$  can be written as  $B_{ij} = B_{ij}^+ - B_{ij}^-$ , where  $B_{ij}^+$  and  $B_{ij}^-$  are positive operators.

**THEOREM 3.3.** *Let  $B_{ij} : C_{[0,+\infty)}^0 \rightarrow L_{[0,+\infty)}$  be a linear Volterra operator, all non-diagonal operators  $B_{ij}$  ( $j \neq i$ ) be negative and*

$$\int_{h_{i*}(t)}^t (B_{ii}^+ 1)(s) ds \leq \frac{1}{e}, \quad t \in (0, +\infty), \quad i = 1, \dots, n. \tag{3.15}$$

*Then all elements  $C_{ij}(t, s)$  of the Cauchy matrix  $C(t, s)$  of the system (3.1) are non-negative while  $C_{ii}(t, s)$  are positive for  $0 \leq s \leq t < +\infty$ ,  $i, j = 1, \dots, n$ .*

*Proof.* Consider the following  $n$  scalar equations

$$x'_i(t) + (B_{ii}^+ x_i)(t) = f_i(t), \quad t \in [0, +\infty), \quad i = 1, \dots, n. \tag{3.16}$$

Let us set  $v_i(t) = \exp[-e \int_0^t (B_{ii} 1)(s) ds]$  in assertion (1) of Theorem 2.1. By virtue of Theorem 2.1 the Cauchy functions  $c_i^+(t, s)$  of (3.16) are positive for  $0 \leq s \leq t < +\infty$ ,  $i = 1, \dots, n$ . By virtue of Theorem 2.1 the Cauchy functions  $c_i(t, s)$  of (3.16) are positive for  $0 \leq s \leq t < +\infty$ ,  $i = 1, \dots, n$ . Now reference to Theorem 3.2 completes the proof. □

Consider the following particular case of system (3.1):

$$x'_i(t) + \sum_{k=1}^n p_{ik}(t)x_k(h_{ik}(t)) = f_i(t), \quad t \in [0, +\infty), \quad i = 1, \dots, n, \tag{3.17}$$

where  $x_i(\theta) = 0$  for  $\theta < 0$ . In this case, the function  $h_{i*}(t)$  introduced in Definition 2.1 can be determined as follows:  $h_{i*}(t) = \min_{s \geq t} h_{ii}(s)$ . Introduce  $p_{ii}^+$  and  $p_{ii}^-$  as follows:  $p_{ii} = p_{ii}^+ - p_{ii}^-$ ,  $p_{ii}^+ \geq 0$ ,  $p_{ii}^- \geq 0$ . Theorem 3.2 for system (3.17) can be formulated in the following form.

**COROLLARY 3.1.** *Assume that  $h_{ik}(t) \leq t$ ,  $p_{ik}(t) \leq 0$  for  $t \in [0, +\infty)$ ,  $i \neq k$ ,  $i, j = 1, \dots, n$ , and the following inequalities*

$$\int_{h_{ii*}(t)}^t p_{ii}^+(s) ds \leq \frac{1}{e}, \quad t \in (0, +\infty), \quad i = 1, \dots, n, \tag{3.18}$$

*are fulfilled. Then all elements  $C_{ij}(t, s)$  of the Cauchy matrix  $C(t, s)$  of the system (3.17) are non-negative, while  $C_{ii}(t, s)$  are positive for  $0 \leq s \leq t < +\infty$ ,  $i, j = 1, \dots, n$ .*

**REMARK 3.3.** Inequality (3.18) cannot be improved. Actually for the system

$$x'_i(t) + p_{ii}x_i(t - \tau_{ii}) = 0, \quad t \in [0, +\infty), \quad i = 1, \dots, n, \tag{3.19}$$

where  $p_{ii}$  and  $\tau_{ii}$  are positive constants, an opposite inequality, for example,  $p_{11} \tau_{11} > 1/e$  implies [9, 13] the oscillation of all solutions of the equation

$$x'_1(t) + p_{11}x_1(t - \tau_{11}) = 0, \quad t \in [0, +\infty), \tag{3.20}$$

and consequently by virtue of Theorem 2.1 the Cauchy function  $c_1(t, s)$  of this equation, coinciding in the case of diagonal system (3.19) with  $C_{11}(t, s)$ , changes its sign for  $0 \leq s \leq t < +\infty$ .

REMARK 3.4. In contrast with the classical Wazewski’s theorem the condition (1.8) is not necessary for non-negativity of all elements of the Cauchy matrix  $C(t, s)$  as the following example demonstrates. Consider the system

$$\begin{aligned} x'_1(t) + p_{12}x_2(t - 1) + p_{13}x_3(t - 1) &= 0, \\ x'_2(t) + p_{21}x_1(t - 1) + p_{23}x_3(t - 1) &= 0, \quad t \in [0, 3], \\ x'_3(t) + p_{31}x_1(t - 2) + p_{32}x_2(t) &= 0, \end{aligned}$$

where  $x(\theta) = 0$  for  $\theta < 0$ . Assume that all coefficients are constant,  $p_{31} > 0$  and other coefficients  $p_{ij}$  are non-positive. For each fixed  $s$  the Cauchy matrix  $C(t, s)$  is the fundamental matrix of this system on  $[s, 3]$  with initial vector function  $x(\theta) = 0$  for  $\theta < s$ , satisfying the condition  $C(s, s) = E$ , where  $E$  is the unit  $3 \times 3$  matrix [2]. Constructing the first column of the Cauchy matrix  $C(t, s)$  step by step as the solution satisfying the initial condition  $x_1(s) = 1, x_2(s) = 0, x_3(s) = 0$ , we obtain

$$\begin{aligned} C_{11}(t, s) &= 1, \quad C_{21}(t, s) = 0, \quad C_{31}(t, s) = 0 \text{ for } t \in [s, s + 1), \\ C_{11}(t, s) &= 1, \quad C_{21}(t, s) = -p_{21}(t - s - 1), \\ C_{31}(t, s) &= \frac{1}{2}p_{32}p_{21}(t - s - 1)^2 \text{ for } t \in [s + 1, s + 2), \\ C_{11}(t, s) &= -p_{12}(t - s - 2)^2 - \frac{1}{6}p_{13}p_{32}p_{21}(t - s - 2)^3 + 1, \\ C_{21}(t, s) &= -p_{21}(t - s - 2) - \frac{1}{6}p_{23}p_{32}p_{21}(t - s - 2)^3 - p_{21}, \\ C_{31}(t, s) &= \frac{1}{24}p_{32}^2p_{23}p_{21}(t - s - 2)^4 + \frac{1}{2}p_{32}p_{21}(t - s - 2)^2 + (p_{32}p_{21} - p_{31}) \\ &\quad \times (t - s - 2) + \frac{1}{2}p_{32}p_{21} \text{ for } t \in [s + 2, s + 3]. \end{aligned}$$

Analogously, the other elements of the Cauchy matrix can be constructed. All elements  $C_{ij}(t, s)$  of the Cauchy matrix  $C(t, s)$ , except  $C_{31}(t, s)$ , are non-negative for each positive  $p_{31}$  and non-positive for other coefficients  $p_{ij}$  in the zone  $0 \leq s \leq t \leq 3$ . If  $p_{31} \leq p_{32}p_{21}$ , then the element  $C_{31}(t, s)$  is also non-negative in this zone. It proves that the condition (1.18) is not necessary for non-negativity of the Cauchy matrix  $C(t, s)$  in contrast with the case of ordinary differential systems.

Let us consider the following system with unbounded delay

$$\begin{aligned} (M_i x)(t) &\equiv x'_i(t) + \sum_{k=1}^{\infty} p_{ik}(t)x_i(t - k\tau_i) + \sum_{j=1, j \neq i}^n a_{ij}(t)x_j(t - \tau_{ij}(t)) = f_i(t), \\ i &= 1, \dots, n, \quad t \in [0, +\infty). \end{aligned} \tag{3.21}$$

Assume that there exists  $\beta$  such that

$$\beta < \frac{1}{e}, \quad p_{i1} \geq \beta p_{i2}, \quad p_{i1} \geq \beta^2 p_{i3}, \dots, p_{i1} \geq \beta^{k-1} p_{ik}, \dots, \quad i = 1, \dots, n. \tag{3.22}$$

Let us substitute  $v_i(t) = e^{-\alpha_i t}$ ,  $i = 1, \dots, n$ , into assertion (1) of Theorem 2.1. The following inequality is desirable:

$$(M_i v)(t) \equiv e^{-\alpha_i t} \left\{ -\alpha_i + \sum_{k=1}^{\infty} p_k(t) e^{\alpha_i k \tau_i} \right\} \leq 0, \quad t \in [0, +\infty). \tag{3.23}$$

Using condition (3.22), we can see that this inequality is satisfied if

$$\frac{p_{i1}}{1 - \beta e^{\alpha_i \tau_i}} \leq \alpha_i e^{-\alpha_i \tau_i}. \tag{3.24}$$

The function  $g_i(\alpha_i) = \alpha_i e^{-\alpha_i \tau_i}$  in the right-hand side of this inequality is maximal at  $\alpha_i = 1/\tau_i$ . If we substitute  $\alpha_i$  into the inequality (3.24), the following inequalities are obtained:

$$\frac{p_{i1} \tau_i}{1 - \beta e} \leq \frac{1}{e}, \tag{3.25}$$

or

$$p_{i1} \tau_i + \beta \leq \frac{1}{e}. \tag{3.26}$$

**THEOREM 3.4.** *Let the conditions (3.22) and (3.26) be satisfied for (3.21) and  $a_{ij} \leq 0$  for  $j \neq i$ , then  $C_{ij}(t, s) > 0$  for  $0 \leq s \leq t < +\infty$ ,  $i, j = 1, \dots, n$ .*

*Proof.* The proof follows from Theorem 3.2. □

Let us consider the following system:

$$(Mx)(t) \equiv x'_i(t) + \int_0^{t-\tau_i} K_i(t, s)x_i(s) ds + \sum_{j=1, j \neq i}^n a_{ij}(t)x_j(t - \tau_{ij}(t)) = f(t), \tag{3.27}$$

$i = 1, \dots, n, t \in [0, +\infty),$

where  $K(t, s)$  is a positive continuous function satisfying the inequality

$$K_i(t, s) \leq b_i e^{-\gamma_i(t-s)}, \quad 0 \leq s \leq t < +\infty, \quad \gamma_i, b_i > 0. \tag{3.28}$$

Substituting  $v_i(t) = e^{-\alpha_i t}$ , where  $\gamma_i > \alpha_i$  into assertion (1) of Theorem 2.1 for each  $i = 1, \dots, n$ , we obtain

$$(M_i e^{-\alpha_i t})(t) \leq -\alpha_i + \frac{b_i}{\gamma_i - \alpha_i} e^{-(\gamma_i - \alpha_i)\tau_i}, \quad 0 \leq t < +\infty. \tag{3.29}$$

The right-hand side of (3.29) is non-positive if

$$b_i e^{-(\gamma_i - \alpha_i)\tau_i} \leq \alpha_i(\gamma_i - \alpha_i). \tag{3.30}$$

Choosing  $\alpha_i = \gamma_i/2$ , we obtain the following inequality implying non-positivity of  $(M_i e^{-\alpha_i t})(t)$  for  $0 \leq t < +\infty$ ,  $i = 1, \dots, n$ :

$$b_i \leq \frac{\gamma_i^2}{4} e^{\frac{\gamma_i}{2}\tau_i}, \quad i = 1, \dots, n. \tag{3.31}$$

**THEOREM 3.5.** *Let inequalities (3.28) and (3.31) be satisfied and  $a_{ij} \leq 0$  for  $j \neq i$ , then all elements  $C_{ij}(t, s)$  of the Cauchy matrix of (3.27) are positive for  $0 \leq s \leq t < +\infty$ ,  $i, j = 1, \dots, n$ .*

*Proof.* The proof follows from Theorem 3.2. □

**REMARK 3.5.** For the scalar equation

$$x'_1(t) + \int_0^{t-\tau_1} K_1(t, s)x_1(s) ds = f_1(t), \quad t \in [0, +\infty), \tag{3.32}$$

where  $K_1(t, s) = b_1 e^{-\gamma_1(t-s)}$  for  $0 \leq s \leq t < +\infty$ ,  $\gamma_1, b_1 > 0$  and  $\tau_1 = 0$ , the inequality (3.31) becomes of the form

$$b_1 \leq \frac{\gamma_1^2}{4}, \tag{3.33}$$

which is a necessary and sufficient condition of non-oscillation of the solutions of (3.32) [6].

**4. Positivity of the elements in the fixed  $r$ th row of Green's matrix.** In this section we consider the equation

$$(M_i x)(t) \equiv x'_i(t) + \sum_{j=1}^n (B_{ij}x_j)(t) = f_i(t), \quad t \in [0, \omega], \quad i = 1, \dots, n, \tag{4.1}$$

where  $B_{ij} : C_{[0, \omega]} \rightarrow L_{[0, \omega]}$  are linear bounded Volterra operators for  $i, j = 1, \dots, n$ .

Together with system (4.1) let us consider the following auxiliary system of the order  $n - 1$

$$x'_i(t) + \sum_{j=1}^{n-1} (B_{ij}x_j)(t) = f_i(t), \quad t \in [0, \omega], \quad i = 1, \dots, n - 1, \tag{4.2}$$

and denote by  $K(t, s) = \{K_{ij}(t, s)\}_{i,j=1, \dots, n-1}$  its Cauchy matrix. Denote by  $G(t, s) = \{G_{ij}(t, s)\}_{i,j=1, \dots, n}$  and  $P(t, s) = \{P_{ij}(t, s)\}_{i,j=1, \dots, n}$  the Green's matrices of the problems consisting of (4.1) and one of the boundary conditions

$$x_i(0) = 0, \quad i = 1, \dots, n - 1, \quad x_n(\omega) = 0 \tag{4.3}$$

or

$$x_i(0) = 0, \quad i = 1, \dots, n - 1, \quad x_n(0) = x_n(\omega) \tag{4.4}$$

respectively.

Let us start with the following assertion, explaining how the scalar functional differential equation for one of the components of the solution vector can be constructed.

**LEMMA 4.1.** *The component  $x_n$  of the solution vector of the problem (4.1) satisfies the following scalar functional differential equation*

$$(M_{x_n})(t) \equiv x'_n(t) + (B_{x_n})(t) = f^*(t), \quad t \in [0, \omega], \tag{4.5}$$

where

$$(Bx_n)(t) \equiv - \sum_{i=1}^{n-1} B_{ni} \left\{ \int_0^t \sum_{j=1}^{n-1} K_{ij}(t, s)(B_{jn}x_n)(s) ds \right\}(t) + (B_{nn}x_n)(t), \quad t \in [0, \omega], \quad (4.6)$$

and

$$f^*(t) = f_n(t) - \sum_{i=1}^{n-1} B_{ni} \left\{ \int_0^t \sum_{j=1}^{n-1} K_{ij}(t, s)f_j(s) ds \right\}(t) - \sum_{i=1}^{n-1} B_{ni} \left\{ \sum_{j=1}^{n-1} K_{ij}(t, 0)x_j(0) \right\}(t). \quad (4.7)$$

*Proof.* Using the Cauchy matrix  $K(t, s) = \{K_{ij}(t, s)\}_{i,j=1,\dots,n-1}$  of the system (4.2), we obtain

$$x_i(t) = - \int_0^t \sum_{j=1}^{n-1} K_{ij}(t, s)(B_{jn}x_n)(s) ds + \int_0^t \sum_{j=1}^{n-1} K_{ij}(t, s)f_j(s) ds + \sum_{j=1}^{n-1} K_{ij}(t, 0)x_j(0) \quad (4.8)$$

for each  $i$ . Substitution of these representations in the  $n$ th equation of the system (4.1) leads to (4.5), where the operator  $B$  and the function  $f^*$  are described by the formulas (4.6) and (4.7) respectively.  $\square$

**THEOREM 4.1.** *Let all elements of the  $(n - 1) \times (n - 1)$  Cauchy matrix  $K(t, s)$  of system (4.2) be non-negative, each of the operators  $B_{jn}$  and  $B_{ij}$  be positive or negative and the product  $-B_{nj}B_{jn}$  be a positive operator for  $j = 1, \dots, n - 1$ .*

*If  $B_{ni}$  for  $i = 1, \dots, n - 1$  are negative operators, then the following five assertions are equivalent:*

- (1) *There exists an absolutely continuous vector function  $v$  such that  $v' \in L_{[0,\omega]}$ ,  $v_n(t) > 0$ ,  $v_i(0) \leq 0$  for  $i = 1, \dots, n - 1$ ,  $(M_i v)(t) \leq 0$  for  $i = 1, \dots, n$ ,  $t \in [0, \omega]$ ;*
- (2)  *$C_{nn}(t, s) > 0$ ,  $C_{nj}(t, s) \geq 0$  for  $j = 1, \dots, n - 1$ ,  $0 \leq s \leq t \leq \omega$ ;*
- (3) *The boundary value problem (4.1), (4.3) is uniquely solvable and its Green's matrix satisfies the inequalities  $G_{nj}(t, s) \leq 0$  for  $j = 1, \dots, n$ ,  $t, s \in [0, \omega]$ , while  $G_{nn}(t, s) < 0$  for  $0 \leq t < s \leq \omega$ ;*
- (4) *If in addition the operator  $B$ , determined by equality (4.6), is a non-zero operator, the boundary value problem (4.1), (4.4) is uniquely solvable and its Green's matrix satisfies the inequalities  $P_{nj}(t, s) \geq 0$  for  $j = 1, \dots, n$ , while  $P_{nn}(t, s) > 0$  for  $t, s \in [0, \omega]$ ;*
- (5) *The  $n$ th component of the solution vector  $x$  of the homogeneous system  $M_i x = 0$ ,  $i = 1, \dots, n$ , such that  $x_i(0) \geq 0$ ,  $i = 1, \dots, n - 1$ ,  $x_n(0) > 0$ , is positive for  $t \in [0, \omega]$ .*

*If  $B_{ni}$  for  $i = 1, \dots, n - 1$  are positive operators, then the following five assertions are equivalent:*

- (1\*) *There exists an absolutely continuous vector function  $v$  such that  $v' \in L_{[0,\omega]}$ ,  $v_n(t) > 0$ ,  $v_i(0) \geq 0$ ,  $(M_i v)(t) \geq 0$  for  $i = 1, \dots, n - 1$ ,  $(M_n v)(t) \leq 0$  for  $t \in [0, \omega]$ ;*
- (2\*)  *$C_{nn}(t, s) > 0$ ,  $C_{nj}(t, s) \leq 0$  for  $j = 1, \dots, n - 1$ ,  $0 \leq s \leq t \leq \omega$ ;*
- (3\*) *The boundary value problem (4.1), (4.3) is uniquely solvable and its Green's matrix satisfies the inequalities  $G_{nj}(t, s) \geq 0$  for  $j = 1, \dots, n - 1$ ,  $G_{nn}(t, s) \leq 0$  for  $t, s \in [0, \omega]$ , while  $G_{nn}(t, s) < 0$  for  $0 \leq t < s \leq \omega$ ;*

- (4\*) If in addition the operator  $B$ , determined by equality (4.6), is a non-zero operator, the boundary value problem (4.1), (4.4) is uniquely solvable and its Green's matrix satisfies the inequalities  $P_{nj}(t, s) \leq 0$  for  $j = 1, \dots, n$ ,  $P_{nn}(t, s) > 0$  for  $t, s \in [0, \omega]$ ;
- (5\*) The  $n$ th component of the solution vector  $x$  of the homogeneous system  $M_i x = 0$ ,  $i = 1, \dots, n$ , such that  $x_i(0) \leq 0$ ,  $i = 1, \dots, n - 1$ ,  $x_n(0) > 0$ , is positive for  $t \in [0, \omega]$ .

*Proof.* Let us start with the implications (1)  $\rightarrow$  (2) and (1\*)  $\rightarrow$  (2\*). By virtue of Lemma 4.1 the component  $x_n$  of the solution vector of system (4.1) satisfies (4.5). It follows from the condition of positivity of the operator  $-B_{nj}B_{jn}$  that  $B$  is a positive operator. Each of the conditions (1) and (1\*) implies that  $(Mv_n)(t) \leq 0$  for  $t \in [0, \omega]$ . By virtue of Theorem 2.1 the Cauchy function  $R(t, s)$  of the equation  $Mv_n = 0$  is positive for  $0 \leq s \leq t \leq \omega$ .

From the formula of representation of solutions and Lemma 4.1 it follows that

$$x_n(t) = \int_0^t \sum_{j=1}^n C_{nj}(t, s)f_j(s) ds = \int_0^t R(t, s)f^*(s) ds, \quad t \in [0, \omega]. \tag{4.9}$$

If  $B_{nj}$  is a negative operator for each  $j = 1, \dots, n - 1$ , and  $f_i \geq 0$  for  $i = 1, \dots, n$ , then  $f^* \geq 0$ . The positivity of  $R(t, s)$  implies that  $x_n$  is non-negative and consequently  $C_{nj}(t, s) \geq 0$  for  $0 \leq s \leq t \leq \omega$  and  $j = 1, \dots, n$ .

If we set  $f_j = 0$  and  $x_j(0) = 0$  for  $j = 1, \dots, n - 1$ , then

$$x_n(t) = \int_0^t C_{nn}(t, s)f_n(s) ds = \int_0^t R(t, s)f_n(s) ds, \quad t \in [0, \omega], \tag{4.10}$$

and it is clear that  $C_{nn}(t, s) = R(t, s)$ . It implies that  $C_{nn}(t, s) > 0$  for  $0 \leq s \leq t \leq \omega$ .

Let us prove the implication (1) $\rightarrow$ (3). By virtue of Lemma 4.1 the component  $x_n$  of the solution vector of system (4.1) satisfies (4.5). Condition (1) by virtue of Theorem 2.1 implies that the Green's function  $G_M(t, s)$  of the boundary value problem

$$(Mx_n)(t) \equiv x'_n(t) + (Bx_n)(t) = f^*(t), \quad t \in [0, \omega], \quad x(\omega) = 0, \tag{4.11}$$

exists and satisfies the inequalities  $G_M(t, s) < 0$  for  $0 \leq t \leq s \leq \omega$  and  $G_M(t, s) \leq 0$  for  $0 \leq s \leq t \leq \omega$ . Lemma 4.1, the representations of solutions of boundary value problem (4.1), (4.3) and the scalar one-point problem (4.11) imply the equality

$$x_n(t) = \int_0^\omega \sum_{j=1}^n G_{nj}(t, s)f_j(s) ds = \int_0^\omega G_M(t, s)f^*(s) ds, \quad t \in [0, \omega]. \tag{4.12}$$

If  $B_{nj}$  is a negative operator for each  $j = 1, \dots, n - 1$ , and  $f_i \geq 0$  for  $i = 1, \dots, n$ , then  $f^* \geq 0$ . The non-positivity of  $G_M(t, s)$  implies that  $x_n$  is non-negative and consequently  $G_{nj}(t, s) \leq 0$  for  $t, s \in [0, \omega]$  and  $j = 1, \dots, n$ .

If we set  $f_j = 0$  and  $x_j(0) = 0$  for  $j = 1, \dots, n - 1$ , then

$$x_n(t) = \int_0^\omega G_{nn}(t, s)f_n(s) ds = \int_0^\omega G_M(t, s)f(s) ds, \quad t \in [0, \omega], \tag{4.13}$$

and it is clear that  $C_{nn}(t, s) = G_M(t, s)$ . This implies that  $G_{nn}(t, s) < 0$  for  $0 \leq t < s \leq \omega$ .

The proof of the implications (1\*) → (3\*), (1) → (4) and (1\*) → (4\*) are analogous.

In order to prove (3) → (1) we set  $v(t) = y(t)$ , where  $y$  is the solution of the boundary value problems

$$(M_i x)(t) = -1, \quad i = 1, \dots, n, \quad t \in [0, \omega], \tag{4.14}$$

$$x_i(0) = 0, \quad i = 1, \dots, n - 1, \quad x_n(\omega) = 0. \tag{4.15}$$

In this case  $v_i(0) = 0, \quad i = 1, \dots, n - 1$ , and

$$v_n(t) = - \int_0^\omega \sum_{j=1}^n G_{nj}(t, s) ds > 0, \quad t \in [0, \omega]. \tag{4.16}$$

In order to prove (3\*) → (1\*) we set  $v(t) = y(t)$ , where  $y$  is the solution of the boundary value problem consisting of the equations

$$(M_i x)(t) = 1, \quad i = 1, \dots, n - 1, \quad (M_n x)(t) = -1, \quad t \in [0, \omega], \tag{4.17}$$

and boundary conditions (4.15).

In order to prove the implications (2) → (1) and (2\*) → (1\*) we set  $v_i(t) = C_{ni}(t, 0)$  for  $i = 1, \dots, n, \quad t \in [0, \omega]$ .

(4) → (1) and (4\*) → (1\*). It was explained above that the element  $P_{mn}(t, s)$  coincides with the Green's function  $Q(t, s)$  of the periodic problem for scalar equation (4.5), which has the representation

$$Q(t, s) = R(t, s) + \frac{R(t, 0)R(\omega, s)}{1 - R(\omega, 0)}, \tag{4.18}$$

where  $R(t, s)$  is the Cauchy function of (4.5) and  $R(t, s) = 0$  if  $0 \leq t < s \leq \omega$ . Thus from the positivity of the Green's matrix of problem (4.4) follows the positivity of  $Q(t, s)$  and by virtue of Theorem 2.1 the positivity of  $R(t, s)$ . We can set  $v_n(t) = R(t, 0), \quad v_i(t) = 0, \quad t \in [0, \omega], \quad i = 1, \dots, n$ .

(5) → (1) and (5\*) → (1\*). Consider the solution vector of the problem  $M_i x = 0, \quad i = 1, \dots, n, \quad x_i(0) = 0, \quad i = 1, \dots, n - 1, \quad x_n(0) = \gamma$ , with  $\gamma > 0$ . By virtue of assertion (5), the component  $x_n(t) > 0$  for  $t \in [0, \omega]$ . Now we can set  $v_n(t) = x_n(t), \quad v_i(0) = 0, \quad i = 1, \dots, n - 1$ .

(2) → (5) and (2\*) → (5\*). From the properties of the Cauchy matrix it follows that the vector  $\text{col}\{C_{1n}(t, 0), \dots, C_{nn}(t, 0)\}$  is the solution of the initial problem  $M_i x = 0, \quad i = 1, \dots, n, \quad x_i(0) = 0, \quad i = 1, \dots, n - 1, \quad x_n(0) = 1$ . By virtue of Lemma 4.1 the component  $x_n$  satisfies (4.5), where the operator  $B$  is determined by the formula (4.6) and

$$f^*(t) = - \sum_{i=1}^{n-1} B_{ni} \left\{ \sum_{j=1}^{n-1} K_{ij}(t, 0)x_j(0) \right\}(t).$$

From the formula (4.10) it is clear that the Cauchy function  $R(t, s)$  of the first-order scalar equation (4.5) coincides with the element  $C_{nn}(t, s)$  of the Cauchy matrix  $C(t, s)$  of the system (4.1). The general solution of (4.5) can be written as follows

$$x(t) = \int_0^t C_{nn}(t, s)f^*(s) ds + C_{nn}(t, 0)x_n(0).$$



The conditions  $x_i(0) \geq 0$  and the negativity of the operators  $B_{ni}$  for  $i = 1, \dots, n - 1$ , in assertion (5) (the conditions  $x_i(0) \leq 0$  and the positivity of the operators  $B_{ni}$  for  $i = 1, \dots, n - 1$ , in assertion (5\*)) imply that  $f^*(t) \geq 0$  for  $t \in [0, \omega]$ . Positivity of  $C_{nn}(t, s)$  now implies that  $x_n(t) \geq C_{nn}(t, 0)x_n(0) > 0$ .  $\square$

REMARK 4.1. The assertions (1)  $\rightarrow$  (5) and (1\*)  $\rightarrow$  (5\*) are analogs for the  $n$ th component of the solution vector of  $n$ th-order functional differential systems of the classical de La Vallée Poussin theorem about the differential inequality obtained in [26] for ordinary second-order equations. Assertions (5)  $\rightarrow$  (2), (5\*)  $\rightarrow$  (2\*), (5)  $\rightarrow$  (3) and (5\*)  $\rightarrow$  (3\*) are analogs of the corresponding assertions connecting non-oscillation and positivity of Green's functions for the  $n$ th-order ordinary differential equations [23].

Let us write system (1.6) in the following form

$$x'_i(t) + \sum_{j=1}^n p_{ij}(t)x_j(t - \tau_{ij}(t)) = f_i(t), \quad i = 1, \dots, n, \quad t \in [0, +\infty), \quad (4.19)$$

where the delay  $\tau_{ij} \geq 0$  for  $i, j = 1, \dots, n$ .

Let us introduce the following denotations:  $p_{ij}^* = \text{ess sup } p_{ij}(t)$ ,  $p_{ij*} = \text{ess inf } p_{ij}(t)$ ,  $\tau_{ij}^* = \text{ess sup } \tau_{ij}(t)$ ,  $\tau_{ij*} = \text{ess inf } \tau_{ij}(t)$ ,  $p_{ij}^+(t) = \max\{0, p_{ij}(t)\}$ .

THEOREM 4.2. *Let the following conditions be fulfilled:*

- (1)  $p_{ij} \leq 0$  for  $i \neq j$ ,  $i, j = 1, \dots, n - 1$ ;
- (2)  $p_{jn} \geq 0$ ,  $p_{nj} \leq 0$  for  $j = 1, \dots, n - 1$ ;
- (3)  $\tau_{ii}^*(p_{ii}^+)^* \leq \frac{1}{e}$ , for  $i = 1, \dots, n - 1$ ,
- (4) *There exists a positive  $\alpha$  such that  $\tau_{ij}^* \alpha \leq 1/e$  for  $i = 1, \dots, n$ , and*

$$p_{nn}^+(t) e^{\alpha \tau_{nn}(t)} - \sum_{j=1}^{n-1} p_{nj}(t) e^{\alpha \tau_{nj}(t)} \leq \alpha \leq \min_{1 \leq i \leq n-1} \left\{ p_{ii}(t) e^{\alpha \tau_{ii}(t)} + \sum_{j=1, i \neq j}^n p_{ij}(t) e^{\alpha \tau_{ij}(t)} \right\}, \quad t \in [0, +\infty). \quad (4.20)$$

Then the elements of the  $n$ th row of the Cauchy matrix of system (4.19) satisfy the inequalities  $C_{nn}(t, s) > 0$ ,  $C_{nj}(t, s) \geq 0$  for  $j = 1, \dots, n - 1$ ,  $0 \leq s \leq t < +\infty$ .

*Proof.* By virtue of Theorem 3.3 all the elements of the  $(n - 1) \times (n - 1)$  Cauchy matrix of the system

$$x'_i(t) + \sum_{j=1}^{n-1} p_{ij}(t)x_j(t - \tau_{ij}(t)) = f_i(t), \quad i = 1, \dots, n - 1, \quad t \in [0, +\infty),$$

of the order  $n - 1$  are non-negative.

Let us continue the coefficients  $p_{ij}$  to the interval  $[-\tau^*, 0)$ , where  $\tau^* = \max_{i,j=1,\dots,n} \tau_{ij}^*$ , as follows:  $p_{ij}(t) = 0$  for  $i \neq j$  and  $p_{ii} = \alpha$ ,  $i, j = 1, \dots, n$ , and consider system (4.19) also on  $[-\tau^*, +\infty)$ .

Let us set  $v_i(t) = -e^{-\alpha t}$  for  $i = 1, \dots, n - 1$ , and  $v_n(t) = e^{-\alpha t}$  in condition (1) of Theorem 4.1. We obtain that this condition is satisfied if  $\alpha$  satisfies the following system

of the inequalities:

$$\alpha \leq p_{ii}(t) e^{\alpha \tau_{ii}(t)} + \sum_{j=1, i \neq j}^n p_{ij}(t) e^{\alpha \tau_{ij}(t)}, \quad i = 1, \dots, n-1, \quad t \in [0, +\infty), \quad (4.21)$$

$$p_{nn}^+(t) e^{\alpha \tau_{nn}(t)} - \sum_{j=1}^{n-1} p_{nj}(t) e^{\alpha \tau_{nj}(t)} \leq \alpha, \quad t \in [0, +\infty). \quad (4.22)$$

Now by virtue of Theorem 4.1 all elements of the  $n$ th row of the Cauchy matrix satisfy the inequalities  $C_{nj}(t, s) \geq 0$  for  $j = 1, \dots, n-1$ , and  $C_{nn}(t, s) > 0$  for  $0 \leq s \leq t < +\infty$ .

The Cauchy matrices of system (4.19) on the interval  $[0, +\infty)$  and on the interval  $[-\tau^*, +\infty)$  clearly coincide in the triangle  $0 \leq s \leq t < +\infty$ . This completes the proof.  $\square$

For the ordinary differential system

$$x'_i(t) + \sum_{j=1}^n p_{ij}(t) x_j(t) = f_i(t), \quad i = 1, \dots, n, \quad t \in [0, +\infty), \quad (4.23)$$

Theorem 4.2 implies the following assertion.

**THEOREM 4.3.** *Let the conditions be fulfilled:*

- (1)  $p_{ij} \leq 0$  for  $i \neq j$ ,  $i, j = 1, \dots, n-1$ ;
- (2)  $p_{jn} \geq 0$ ,  $p_{nj} \leq 0$  for  $j = 1, \dots, n-1$ ;
- (3) *There exists a positive  $\alpha$  such that*

$$p_{nn}^+(t) - \sum_{j=1}^{n-1} p_{nj}(t) \leq \alpha \leq \min_{1 \leq i \leq n-1} \left\{ p_{ii}(t) + \sum_{j=1, i \neq j}^n p_{ij}(t) \right\}, \quad t \in [0, +\infty). \quad (4.24)$$

*Then the elements of the  $n$ th row of the Cauchy matrix of system (4.23) satisfy the inequalities  $C_{nn}(t, s) > 0$ ,  $C_{nj}(t, s) \geq 0$  for  $j = 1, \dots, n-1$ ,  $0 \leq s \leq t < +\infty$ .*

Consider now the following ordinary differential system of the second order

$$\begin{aligned} x'_1(t) + p_{11}(t)x_1(t) + p_{12}(t)x_2(t) &= f_1(t), \\ x'_2(t) + p_{21}(t)x_1(t) + p_{22}(t)x_2(t) &= f_2(t), \end{aligned} \quad t \in [0, +\infty). \quad (4.25)$$

**THEOREM 4.4.** *Let the following conditions be fulfilled:*

- (1)  $p_{11} \geq 0$ ,  $p_{12} \geq 0$ ,  $p_{21} \leq 0$ ,  $p_{22} \geq 0$ ;
- (2) *There exists a positive  $\alpha$  such that*

$$p_{22}(t) - p_{21}(t) \leq \alpha \leq p_{11}(t) - p_{12}(t), \quad t \in [0, +\infty). \quad (4.26)$$

*Then the elements of the second row of the Cauchy matrix of system (4.25) satisfy the inequalities  $C_{21}(t, s) \geq 0$ ,  $C_{22}(t, s) > 0$  for  $0 \leq s \leq t < +\infty$ .*

REMARK 4.2. If coefficients  $p_{ij}$  are constants the second condition in Theorem 4.4 is as follows:

$$p_{22} - p_{21} \leq p_{11} - p_{12}. \tag{4.27}$$

REMARK 4.3. Let us demonstrate that inequality (4.27) (and consequently inequality (4.26)) is best possible in a corresponding case. It is known that for each fixed  $s$  the  $2 \times 2$  matrix  $C(t, s)$  is a fundamental matrix  $X(t)$  of system (4.25) satisfying the condition  $C(s, s) = E$ , where  $E$  is the unit  $2 \times 2$  matrix. Theorem 4.4 claims that elements in the second row of the fundamental matrices are positive. The characteristic equation of the system

$$\begin{aligned} x'_1(t) + p_{11}x_1(t) + p_{12}x_2(t) &= 0, \\ x'_2(t) + p_{21}x_1(t) + p_{22}x_2(t) &= 0, \end{aligned} \quad t \in [0, +\infty), \tag{4.28}$$

with constant coefficients is as follows:

$$\lambda^2 + (p_{11} + p_{22})\lambda + p_{11}p_{22} - p_{12}p_{21} = 0, \tag{4.29}$$

and its roots are real if and only if

$$(p_{11} - p_{22})^2 \geq -4p_{12}p_{21}. \tag{4.30}$$

Let us instead of inequality (4.27) consider

$$p_{22} - p_{21} \leq p_{11} - p_{12} + \varepsilon, \tag{4.31}$$

where  $\varepsilon$  is any positive constant. We can set  $p_{11} = p_{22}$ , then the inequality becomes of the form  $p_{12} - p_{21} \leq \varepsilon$ . If  $p_{12}p_{21} < 0$ , then inequality (4.30) is not satisfied and consequently each element of the fundamental and the Cauchy matrices oscillates.

Let us prove the following assertions, giving an efficient test of non-negativity of elements in the  $n$ th row of the Cauchy matrix in case when the coefficients  $|p_{nj}|$  are small enough for  $j = 1, \dots, n - 1$ .

THEOREM 4.5. *Let the following conditions be fulfilled:*

- (1)  $p_{ij} \leq 0$  for  $i \neq j, i, j = 1, \dots, n - 1$ ;
- (2)  $p_{jn} \geq 0, p_{nj} \leq 0$  for  $j = 1, \dots, n - 1$ ;
- (3)  $\tau_{ij} = 0$  for  $i = 1, \dots, n, j = 1, \dots, n - 1, \tau_{nn} = const$ ;
- (4) *The inequalities*

$$p_{nn}^+(t)\tau_{nn} \exp \left\{ \tau_{nn} \sum_{j=1}^{n-1} |p_{nj}|^* \right\} \leq \frac{1}{e}, \quad t \in [0, +\infty) \tag{4.32}$$

and

$$\frac{1}{\tau_{nn}} + \sum_{j=1}^{n-1} |p_{nj}|^* \leq \min_{1 \leq i \leq n-1} \left\{ p_{ii}(t) + \sum_{j=1, i \neq j}^n p_{ij}(t) \right\}, \quad t \in [0, +\infty) \tag{4.33}$$

are fulfilled.

Then the elements of the  $n$ th row of the Cauchy matrix of system (4.19) satisfy the inequalities  $C_{nm}(t, s) > 0$ ,  $C_{nj}(t, s) \geq 0$  for  $j = 1, \dots, n - 1$ ,  $0 \leq s \leq t < +\infty$ .

*Proof.* Consider the left part

$$p_{nm}^+(t) e^{\alpha \tau_{nm}(t)} - \sum_{j=1}^{n-1} p_{nj}(t) e^{\alpha \tau_{nj}(t)} \leq \alpha, \quad t \in [0, +\infty) \tag{4.34}$$

of inequality (4.20). Using condition (3), we obtain that the inequality

$$p_{nm}^+(t) \leq \left\{ \alpha - \sum_{j=1}^{n-1} |p_{nj}|^* \right\} e^{-\alpha \tau_{nm}}, \quad [0, +\infty) \tag{4.35}$$

is sufficient for the truth of inequality (4.34). The right-hand side of the inequality gets its maximum for  $\alpha = \frac{1}{\tau_{nm}} + \sum_{j=1}^{n-1} |p_{nj}|^*$ . Substituting this  $\alpha$  into (4.35) and the right part of (4.20), we obtain inequalities (4.32) and (4.33).  $\square$

REMARK 4.4. It should be noted that inequality (4.32) is best possible in the following sense. If  $p_{nj} = 0$  for  $j = 1, \dots, n - 1$ ,  $p_{nm} = \text{const} > 0$ , then inequality (4.32) becomes

$$p_{nm} \tau_{nm} \leq \frac{1}{e}, \quad t \in [0, +\infty), \tag{4.36}$$

and  $C_{nm}(t, s) = c_n(t, s)$ , where  $c_n(t, s)$  is the Cauchy function of the diagonal equation

$$x_n'(t) + p_{nn}x(t - \tau_{nn}) = 0, \quad t \in [0, +\infty). \tag{4.37}$$

The opposite inequality  $p_{nm} \tau_{nm} > 1/e$  implies oscillation of all solutions [13], and by virtue of Theorem 2.1  $c_n(t, s)$  changes its sign. Now it is clear that we cannot substitute

$$p_{nm}^+(t) \tau_{nm} \exp \left\{ \tau_{nm} \sum_{j=1}^{n-1} |p_{nj}|^* \right\} \leq \frac{1 + \varepsilon}{e}, \quad t \in [0, +\infty), \tag{4.38}$$

where  $\varepsilon$  is any positive number instead of inequality (4.32).

Let us consider the second-order scalar differential equation

$$(Ny)(t) \equiv y''(t) + p_{11}(t)y'(t - \tau_{11}(t)) + p_{12}(t)y(t - \tau_{12}(t)) = f_1(t), \quad t \in [0, +\infty), \tag{4.39}$$

where  $y(\theta) = y'(\theta) = 0$  for  $\theta < 0$ , and the corresponding differential system of the second order

$$\begin{aligned} x_1'(t) + p_{11}(t)x_1(t - \tau_{11}(t)) + p_{12}(t)x_2(t - \tau_{12}(t)) &= f_1(t), \\ x_2'(t) - x_1(t) &= 0, \end{aligned} \quad t \in [0, +\infty), \tag{4.40}$$

where  $x_1(\theta) = x_2(\theta) = 0$  for  $\theta < 0$ .

It should be noted that the element  $C_{21}(t, s)$  of the Cauchy matrix of system (4.40) coincides with the Cauchy function  $W(t, s)$  of the second-order scalar equation (4.39) and  $C_{11}(t, s) = W_t'(t, s)$ . If a function  $y(t)$  is the solution of the Cauchy problem  $(Ny)(t) = 0$ ,  $t \in [0, +\infty)$ ,  $y(0) = 1$ ,  $y'(0) = 0$ , then  $C_{22}(t, 0) = y(t)$  and  $C_{12}(t, 0) = y'(t)$ .

**THEOREM 4.6.** Assume that  $p_{12} \geq 0$ ,  $p_{11}^* \tau_{11}^* \leq 1/e$  and there exists a positive number  $\alpha$  such that  $\alpha \tau_{11}^* \leq 1/e$  and

$$\alpha^2 + p_{12}(t) e^{\alpha \tau_{12}(t)} \leq \alpha p_{11}(t) e^{\alpha \tau_{11}(t)}, \quad t \in [0, +\infty). \quad (4.41)$$

Then the elements of the second row of the Cauchy matrix of system (4.40) satisfy the inequalities  $C_{21}(t, s) \geq 0$ ,  $C_{22}(t, s) > 0$  for  $0 \leq s \leq t < +\infty$ .

*Proof.* In order to prove Theorem 4.6 we set  $v_1(t) = -\alpha e^{-\alpha t}$ ,  $v_2(t) = e^{-\alpha t}$  in assertion (1) of Theorem 4.1.  $\square$

**THEOREM 4.7.** Assume that  $p_{12} \geq 0$ ,  $p_{11}^* \tau_{11}^* \leq 1/e$ ,  $\tau_{11} \geq \tau_{12}$  and

$$4p_{12}(t) \leq p_{11*}^2, \quad t \in [0, +\infty). \quad (4.42)$$

Then the elements of the second row of the Cauchy matrix of system (4.40) satisfy the inequalities  $C_{21}(t, s) \geq 0$ ,  $C_{22}(t, s) > 0$  for  $0 \leq s \leq t < +\infty$ .

*Proof.* In order to prove Theorem 4.7 we set  $\alpha = p_{11*}/2$  in Theorem 4.6.  $\square$

**REMARK 4.5.** Inequality (4.42) is best possible in the following sense. Let us consider the system with constant coefficients

$$\begin{aligned} x_1'(t) + p_{11}x_1(t) + p_{12}x_2(t) &= f_1(t), \\ x_2'(t) - x_1(t) &= 0, \end{aligned} \quad t \in [0, +\infty). \quad (4.43)$$

The characteristic equation for this system has real roots if and only if inequality (4.42) is fulfilled.

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