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TWO-TRANSITIVE ACTIONS ON CONJUGACY CLASSES

MICHAEL J.J. BARRY AND MICHAEL B. WARD

Every group acts transitively by conjugation on each of its conjugacy classes of elements. It is natural to wonder when this action becomes multiply transitive. In this paper, we determine all finite groups which act faithfully and 2–transitively on a conjugacy class of elements. We also give some consequences including a solvability criterion based on what fraction of elements belong to conjugacy classes upon which the group acts faithfully and 2–transitively.

1. INTRODUCTION

Certain natural group actions are always transitive. It is interesting to investigate when those actions become 2-transitive. For example, in **[1]**, finite groups G acting 2-transitively by conjugation on $Syl_p(G)$, the set of Sylow p-subgroups of G for some prime p, were determined, modulo the kernel of the action. Herein, we determine finite groups G acting faithfully and 2-transitively by conjugation on $cl_G(x)$, the conjugacy class of x in G, for some $x \in G$.

The following construction illustrates the sort of groups that can occur. Let W be a finite dimensional vector space over a finite field and let T be the group of translations of W. Suppose C is a subgroup of $GL(W)$ with $Z(C) \neq 1$ which acts transitively on the nonzero vectors of W. Of course, T acts transitively on W and so the natural semidirect product $G = TC$ acts 2–transitively on W. Consequently, C is a maximal subgroup of G. Therefore, C, the stabiliser of the zero vector, is also the centraliser in G of any nonidentity element $z \in Z(C)$. It is then a routine exercise to show the action of G on W is equivalent to the action of G on $cl_G(z)$ by conjugation. Therefore, G acts 2-transitively on $cl_G(z)$.

Consider two concrete examples of that general construction. In the first example, G is nonsolvable (except when $q = 3$) while, in the second, G is solvable.

EXAMPLE 1. Let W be a finite dimensional vector space over the finite field $\mathbb{F}_q$ for some prime power $q > 2$ and let $C = GL(W)$. Here $Z(C)$ consists of the $q - 1$ scalar matrices.

EXAMPLE 2. Suppose $q = p^n$ for some prime p. Consider $W = \mathbb{F}_q$ as an n-dimensional vector space over $\mathbb{F}_p$. Let C be the group of linear transformations of $\mathbb{F}_q$ induced by multiplication by elements of $\mathbb{F}_q$. In this example, $Z(C) = C$ is cyclic of order $q - 1$.

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In Section 2 we state our main result classifying finite groups acting faithfully and 2–transitively on $cl_G(x)$ for some $x \in G$ and begin its proof, which is concluded in Sections 3 and 4. For the solvable groups, we use Huppert's classification of solvable 2–transitive groups [**7**]. In the nonsolvable case, we use the classification of nonsolvable 2-transitive groups which is a consequence of the Classification of Finite Simple Groups and the work of Hering [**5, 6**] and Curtis, Kantor and Seitz [**2**]. Finally, in Section 5, we give some consequences, including an amusing solvability criterion.

2. The Main Result

NOTATION: $\mathbb{F}_q$ denotes a finite field with q elements where $q = p^r$ for some prime p, $(\mathbb{F}_q)^k$ the vector space of k-tuples of $\mathbb{F}_q$-elements, $A\Gamma L(k,q)$ the group of affine semilinear transformations of $(\mathbb{F}_q)^k$, $\Gamma L(k,q) \leqslant A\Gamma L(k,q)$ the subgroup of semilinear transformations of $(\mathbb{F}_q)^k$, and $T(k,q) \leqslant A\Gamma L(k,q)$ the full group of translations of $(\mathbb{F}_q)^k$.

We recall that $\Gamma L(k,q)$ is the semidirect product $GL(k,q)\langle\sigma\rangle$, where σ is the semilinear transformation induced by the Galois automorphism $a \mapsto a^p$ of $\mathbb{F}_q$, and that $A\Gamma L(k,q)$ is the semidirect product $T(k,q)\Gamma L(k,q)$. In view of Huppert's classification, the case $k = 1$ plays a special role. There, $GL(1,q) = \langle\omega\rangle$ where ω is the linear transformation induced by multiplication by $\overline{\omega}$, a generator of the multiplicative group of $\mathbb{F}_q$. We fix those meanings of σ and ω.

THEOREM 1. *Suppose G is a finite group, g is an element of G, and G acts (by conjugation) faithfully and 2-transitively on the conjugacy class $cl_G(g)$. Then G has a unique minimal normal subgroup M which is elementary Abelian, and G is the semidirect product $MC_G(x)$ for any $x \in cl_G(g)$. In addition, if $|M| = p^n$ where p is a prime, there is a prime-power q and natural number k with $q^k = p^n = |M|$ such that G acting on $cl_G(g)$ is permutation isomorphic to a subgroup H of $A\Gamma L(k,q)$ acting on $(\mathbb{F}_q)^k$, with M mapped to $T(k,q)$, and $C_G(g)$ mapped to H_0, the point stabiliser in H of the zero vector of $(\mathbb{F}_q)^k$, which is a subgroup of $\Gamma L(k,q)$. The possibilities for H_0 are the groups in the following list having the property $Z(H_0) \neq 1$.*

1. *$H_0 \leqslant \Gamma L(1,q)$ such that H_0 is transitive on the set of nonzero elements of $(\mathbb{F}_q)^k$.*
2. *$H_0 \leqslant GL(k,q) \leqslant \Gamma L(k,q)$ where $(k,q) = (2,3)$, $(2,5)$, $(2,7)$, $(2,11)$, $(2,23)$ or $(4,3)$, and H_0 is one of the 13 exceptional groups listed in Huppert's classification of solvable 2-transitive groups,* [**7**, p. 126-127].
3. *$SL(k,q) \trianglelefteq H_0 \leqslant \Gamma L(k,q)$, excluding the solvable cases $(k,q) = (1,q)$, $(2,2), (2,3)$.*
4. *$Sp(k,q) \trianglelefteq H_0 \leqslant \Gamma L(k,q)$ where $q^k = p^n$ and n is even.*
5. *$G_2(2^m) \trianglelefteq H_0 \leqslant \Gamma L(6,2^m)$ where $2^{6m} = p^n$.*

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6. *$E \trianglelefteq H_0 \leqslant \Gamma L(4,3)$, E is extra special of order 32, $C_{H_0}(E) = Z(H_0)$, and $H_0/\big(E \cdot Z(H_0)\big)$ is faithfully represented on $E/Z(E)$, and isomorphic to a nonsolvable subgroup of S_5.*
7. *$SL(2,5) \cong H_0^{(\infty)} \trianglelefteq H_0 \leqslant \Gamma L(2,q)$ where $q^2 = p^n$ and $q = 9$, 11, 19, 29, or 59.*
8. *$SL(2,13) \cong H_0 \leqslant \Gamma L(6,3)$.*

Conversely, if H_0 is on the list above and there exists a nonidentity $h \in Z(H_0)$, then the semidirect product $H = H_0T(k,q)$ acts faithfully and 2-transitively on the conjugacy class $cl_H(h) = \{tht^{-1} \mid t \in T(k,q)\}$.

PROOF: We start by proving the converse. Each $H = H_0T(k,q)$ acts faithfully and 2-transitively on $(\mathbb{F}_q)^k$. If $h \in Z(H_0) - \{1\}$, then $C_H(h) = H_0$, and the action of H on $cl_H(h)$ by conjugation is readily shown to be equivalent to the action of H on $(\mathbb{F}_q)^k$.

Now suppose G acts (by conjugation) faithfully and 2-transitively on some conjugacy class $cl_G(g)$. By a result of Burnside [**4**, Theorem 4.1B], the socle M of G is either (i) a regular elementary Abelian p-group for some prime p, or (ii) a nonregular nonabelian simple group. In both cases, M is the unique minimal normal subgroup of G, and in case (ii) $C_G(M) = 1$ [**4**, Theorems 4.1B and 4.2B], allowing one to assume that $M \trianglelefteq G \leqslant \mathrm{Aut}(M)$. Solvable groups of the first type have been classified by Huppert in [**7**], while Hering in [**5**], with a little cleanup ruling out sporadic groups as composition factors in [**1**], classified the nonsolvable groups of the first type. Those of the second type have been classified in [**11**] and [**2**]. Since the point stabiliser in our situation must be $C_G(g)$, we are looking for point stabilisers with a non-trivial centre. We shall show in the next section that all examples of the second type have point stabilisers with trivial centre. An examination of the lists of Huppert [**7**] and Hering [**5**, pp. 443-444] for point stabilisers with possible nontrivial centres yields our theorem. ∎

3. Ruling out Almost Simple Groups

PROPOSITION 1. *Suppose G is a 2-transitive nonsolvable group acting on a set Ω and M is a nonabelian simple group with $M \trianglelefteq G \leqslant \mathrm{Aut}(M)$. Then $Z(G_\alpha) = 1$ for all $\alpha \in \Omega$.*

PROOF: By the classification of finite simple groups M is an alternating group, a simple group of Lie type, or a sporadic group. When M is alternating, the classical reference is Maillet [**11**]. We start by disposing of each of these six possibilities.

ALT1 In this case, $n \geqslant 5$ and $G = M = A_n$ or $G = S_n$ in the natural representation of degree n. Hence $G_\alpha = A_{n-1}$ or $G_\alpha = S_{n-1}$, respectively, both of which are centreless.

ALT2 G is A_6 or S_6 in the extra representation of degree 6. Again, G_α is A_5 or S_5.

ALT3 $G = M = A_5$ or $G = S_5$ in their degree 6 representation. This is really a special case of CKS(ii)(a) below since $A_5 \cong PSL(2,5)$.

ALT4 $G = M = A_6$ or $G = S_6$ in their degree 10 representation. This is a special case of CKS(ii)(a) below since $A_6 \cong PSL(2,9)$.

ALT5 $G = A_8$ in either of its degree 15 representations. This is a special case of CKS(i) below since $A_8 \cong PSL(4,2)$.

ALT6 $G = A_7$ in the restriction of either of the degree 15 representations from A_8. Here G_α is $PSL(3,2)$ which is centreless.

The possibilities for G when M is simple of Lie type have been described in [**2**]. We now dispose of each of these possibilities, using the numbering of [**2**]. For consistency with the references used below, in this proof we write functions on the left.

In CKS(i), $M = PSL(n,q)$ with $n \geqslant 3$, and we have $PSL(n,q) \leqslant G \leqslant P\Gamma L(n,q)$. Now G has two possible actions, one on lines, and a dual action on hyperplanes. Because of duality it will suffice to deal with the action on lines. We shall show that $C_{P\Gamma L(n,q)}(PSL(n,q)_\alpha) = 1$. Using a bar to denote the images of transformations of $\Gamma L(n,q)$ in $P\Gamma L(n,q)$, we must show that if $g \in \Gamma L(n,q)$ is such that

$$\overline{g}\,\overline{\sigma}\,\overline{g}^{-1} = \overline{\sigma}$$

for all σ in the preimage H in $SL(n,q)$ of $PSL(n,q)_\alpha$, then $\overline{g} = 1$. Let v be a nonzero vector in $V = (\mathbb{F}_q)^n$ such that α is the line $\mathbb{F}_q v$. Our proof will be in two steps. First we shall show for every line L in V there is a transvection in H fixing v with residual line L. Then we shall apply an argument of O'Meara in [**12**] to conclude that $\overline{g} = 1$.

In what follows, for $v \in V$ and ρ in the dual of V, $\tau_{v,\rho}\colon V \to V$ is defined by

$$\tau_{v,\rho}(x) = x + \rho(x)v.$$

Then $\tau_{v,\rho}$ is a transvection if and only if $\rho(v) = 0$.

Let $\{v_1 = v, \ldots, v_n\}$ be a basis for V. Let L be a typical line in $V = (\mathbb{F}_q)^n$. Write $L = \mathbb{F}_q w$ and $w = \sum \alpha_i v_i$. Let $\{\rho_1, \ldots, \rho_n\}$ be the dual basis to $\{v_1, \ldots, v_n\}$, that is, $\rho_i(v_j)$ is 1 when $i = j$ and is 0 otherwise. If $w \in \mathbb{F}_q v_1$, then τ_{v_1,ρ_2} is a transvection (because $\rho_2(v_1) = 0$) fixing $v = v_1$ with residual line L. If $w \notin \mathbb{F}_q v_1$, then $u = w - \alpha_1 v_1 \neq 0$. Let $\{w_1 = v, w_2 = u, \ldots, w_n\}$ be a basis for V with $\operatorname{span}\{w_2, \ldots, w_n\} = \operatorname{span}\{v_2, \ldots, v_n\}$. Let $\{\rho'_1, \ldots, \rho'_n\}$ be the dual basis to $\{w_1, \ldots, w_n\}$. Then τ_{w,ρ'_3} is a transvection (because $\rho'_3(w) = 0$) fixing $v = w_1$ with residual line L.

Let L be a typical line in V, let τ be a transvection in H with residual line L. Then $g\tau g^{-1}$ is a transvection with residual line gL. Therefore $\overline{\tau}$ and $\overline{g}\,\overline{\tau}\,\overline{g}^{-1}$ are projective transvections with residual lines L and gL respectively. But $\overline{\tau} = \overline{g}\,\overline{\tau}\,\overline{g}^{-1}$. Hence $gL = L$ for all lines L. This forces g to be a radiation, giving $\overline{g} = 1$.

CKS (ii) In each of the four cases covered here, $M \leqslant G \leqslant \operatorname{Aut}(M)$ where M is a simple rank one group of Lie type and M_α is a Borel subgroup B. We shall show that if τ an automorphism of M satisfying $\tau(b) = b$ for all $b \in B$, then τ is the identity automorphism on M. The group B is a semidirect product UH, where U is a p-Sylow subgroup for the natural characteristic p of M, $B = N_M(U)$, and $\big(|U|, |H|\big) = 1$. Now $\tau = i \circ d \circ f$ or $\tau = i \circ f$, where i, d, and f are inner, diagonal, and field automorphisms, respectively. Since f, and d when present, map B to B, it follows that i does too. Hence, if i is conjugation by an element g, we see that $g \in N_G(B) = B$. Our strategy then is to show, by computing the action of τ on a typical element of U, that f is the identity. Furthermore, we show that $d = 1$ by showing d, if present, can realised as conjugation by an element of H, a contradiction since d is supposed to be an outer automorphism. This means that τ is conjugation by an element of B. But, in these cases, $C_B(B) = 1$. Thus τ is the identity.

In what follows of this case, we use the notation for Chevalley and twisted groups as in [**15**], particularly that of Lemma 63 for the twisted groups. This lemma and the results on conjugation by a diagonal element [**15**, p. 196] are used extensively in the calculations.

In CKS(ii)(a), $M = PSL(2,q) = A_1(q)$. Here $\tau = i \circ d \circ f$, where i, d, and f are inner, diagonal, and field automorphisms, respectively, and i is conjugation by an element g of B. (Note that q must be odd for d to be present.) We can write $g = x_{\alpha_1}(t)h_{\alpha_1}(s)$ (using the notation of [**15**]) for some $t \in \mathbb{F}_q$ and some $s \in \mathbb{F}_q^*$, while a typical element of U has the form $x_{\alpha_1}(t_1)$ for $t_1 \in \mathbb{F}_q$. So

$$\begin{aligned}
x_{\alpha_1}(t_1) &= \tau \cdot x_{\alpha_1}(t_1) \\
&= (i \circ d \circ f) \cdot x_{\alpha_1}(t_1) \\
&= (i \circ d) \cdot x_{\alpha_1}\big(f(t_1)\big) \\
&= i \cdot x_{\alpha_1}\big(d_1 f(t_1)\big) \qquad \text{for some } d_1 \text{ in } \mathbb{F}_q^* \\
&= x_{\alpha_1}(t)h_{\alpha_1}(s)x_{\alpha_1}\big(d_1 f(t_1)\big)h_{\alpha_1}(s^{-1})x_{\alpha_1}(-t) \\
&= x_{\alpha_1}(t)x_{\alpha_1}\big(s^2 d_1 f(t_1)\big)x_{\alpha_1}(-t) \\
&= x_{\alpha_1}\big(s^2 d_1 f(t_1)\big)
\end{aligned}$$

for all $t_1 \in \mathbb{F}_q$.

Setting $t_1 = 1$ we get $d_1 s^2 = 1$. This says that the diagonal automorphism d can actually be realised as conjugation by the element $h_{\alpha_1}(s^{-1})$ of H. Since the equation $t_1 = f(t_1)$ now holds for all $t_1 \in \mathbb{F}_q$, f must be the identity field automorphism. Thus τ is conjugation by an element of B, and, as argued above, must be the identity automorphism.

In CKS(ii)(b), $M = PSU(3,q) = {}^2A_2(q)$. Now $\tau = i \circ d \circ f$, where i, d, and f are inner, diagonal, and field automorphisms, respectively, and i is conjugation by an element g of B. (Note that 3 must divide $q+1$ for d to be present.) We can write $g = (t,u)d(s)$ where $t, u \in \mathbb{F}_{q^2}$, $s \in F_{q^2}^*$, and $u + u^q + t^{q+1} = 0$, while a typical element of U has the form (t_1, u_1). Then

$$\begin{aligned}(t_1,u_1) &= (i\circ d\circ f)\cdot(t_1,u_1)\\ &= (i\circ d)\cdot\big(f(t_1),f(u_1)\big)\\ &= i\cdot\big(d_1f(t_1),d_1^{q+1}f(u_1)\big) \qquad \text{for some } d_1 \text{ in } \mathbb{F}_q^*\\ &= g\big(d_1f(t_1),d_1^{q+1}f(u_1)\big)g^{-1}\\ &= (t,u)\big(s^{2-q}d_1f(t_1),s^{1+q}d_1^{q+1}f(u_1)\big)(t,u)^{-1}\\ &= \Big(s^{2-q}d_1f(t_1),s^{1+q}d_1^{q+1}f(u_1)-t^qs^{2-q}d_1f(t_1)+\big(s^{2-q}d_1f(t_1)\big)^qt\Big)\end{aligned}$$

for all $(t_1, u_1) \in \mathbb{F}_{q^2}^2$ satisfying $u_1 + u_1^q + t_1^{q+1} = 0$. Setting $t_1 = 1$ gives $s^{2-q}d_1 = 1$. Note that this implies that $s^{1+q}d_1^{q+1} = 1$, using the fact that $s^{q^2-1} = 1$. This says that the diagonal automorphism d can be realised as conjugation by the element $d(s^{-1})$ of H. Since the equation $t_1 = f(t_1)$ now holds for all $t_1 \in \mathbb{F}_{q^2}$, f must be the identity field automorphism. Thus τ is conjugation by an element of B, and, as argued above, must be the identity automorphism.

In CKS(ii)(c), $M = Sz(q) = {}^2B_2(q)$, and the natural characteristic p is 2. Here $\tau = i \circ f$, where i and f are inner and field automorphisms, respectively, and i is conjugation by an element g of B. We can write $g = (t, u)d(s)$ where t and u are elements of $\mathbb{F}_q$ and $s \in \mathbb{F}_q^*$, while a typical element of U has the form (t_1, u_1). Then

$$\begin{aligned}(0,u_1) &= (i\circ f)\cdot(0,u_1)\\ &= i\cdot(0,f(u_1))\\ &= \big(0,s^{2\theta}f(u_1)\big)\end{aligned}$$

for all $u_1 \in \mathbb{F}_q$. Setting $u_1 = 1$ gives $s^{2\theta} = 1$, from which it follows that $s = 1$ because $s^{2\theta}$ is s raised to a power of 2, whereas s has odd order. This implies that f is the identity field automorphism. Thus τ is conjugation by an element of B, and, as argued above, must be the identity automorphism.

In CKS(ii)(d), $M = {}^2G_2(q)$, and the natural characteristic p is 3. Here $\tau = i \circ f$, where i and f are inner and field automorphisms, respectively, and i is conjugation by an element g of B. We can write $g = (t, u, v)d(s)$ where t, u and v are elements of $\mathbb{F}_q$ and $s \in \mathbb{F}_q^*$, while a typical element of U has the form (t_1, u_1, v_1). Then

$$\begin{aligned}(0,0,v_1) &= (i\circ f)\cdot(0,0,v_1)\\ &= i\cdot\big(0,0,f(v_1)\big)\\ &= \big(0,0,sf(v_1)\big)\end{aligned}$$

for all $v_1 \in \mathbb{F}_q$. Setting $v_1 = 1$ gives $s = 1$. This implies that f is the identity field automorphism. Thus τ is conjugation by an element of B, and, as argued above, must be the identity automorphism.

CKS (iii) G is $PSL(2,4) \cong PSL(2,5) \cong A_5$ or $P\Gamma L(2,4) \cong PGL(2,5) \cong S_5$. The degree is 5, and this case has been dealt with in CKS(ii)(a).

CKS (iv) G is $PSL(2,9) \cong A_6$ or $PSL(2,9) \cdot \mathrm{Aut}\, \mathbb{F}_9 \cong S_6$. The degree is 6. This case has been dealt with in ALT1 and ALT2.

CKS (v) G is $PSL(2,11)$, and $G_\alpha = A_5$ which is centreless. The degree is 11.

CKS (vi) G is $P\Gamma L(2,8) \cong {}^2G_2(3) \cong \mathrm{Aut}(L_2(8)) \cong L_2(8) \cdot 3$ [**10**, p. 172]. The degree is 28, G_α is the normaliser of a Sylow 3-subgroup, and this case has been dealt with in CKS(ii)(d).

CKS (vii) G is $PSL(3,2) \cong PSL(2,7)$ or Aut $PSL(3,2) \cong PGL(2,7)$. The degree is 8, and this case has been dealt with in CKS(ii)(a).

CKS (viii) G is $PSL(4,2) \cong A_8$ or Aut $PSL(4,2) \cong S_8$. The degree is 8, and this case has been dealt with in ALT1.

CKS (ix) G is $Sp(n,2)$ in one of its 2-transitive representations of degree $2^{n-1}(2^n \pm 1)$, with the stabiliser of a point being $O^{\pm}(2n,2)$, both of which are centreless. This last assertion follows from [**3**, p. 64]. For $n = 2$, $Sp(4,2) \cong S_6$ is not simple, and we are talking about representations of S_6 of degree 6 and 10, which have been dealt with in ALT1, ALT2, and ALT4.

CKS (x) G is $G_2(2) \cong PSU(3,3) \cdot \mathrm{Aut}\, \mathbb{F}_9$ or Aut $G_2(2) \cong P\Gamma L(3,3)$. The degree is 28, and this case has been dealt with in CKS(ii)(b).

We now deal with the possibilities when M is sporadic. The possibilities for G are

1. $M_{11}, M_{12}, M_{22}, \mathrm{Aut}(M_{22}), M_{23}$, or M_{24} acting on their associated Steiner systems.
2. M_{11} in an exceptional 3-transitive action of degree 12 with point stabiliser $PSL(2,11)$.
3. HS in an action of degree 176, with point stabiliser $U_3(5): 2$.
4. Co_3 in an action of degree 276 with point stabiliser $McL : 2 \cong \mathrm{Aut}(McL)$, by Dixon and Mortimer [**4**].

In all these cases the point stabiliser has trivial centre. The conclusion is obvious when the point stabiliser is a simple group. In the other cases, including $G = \mathrm{Aut}(M_{22})$, one can see by consulting the Atlas that a point stabiliser in M has much greater order than that of any centraliser in M of an element of G, and thus the point stabilisers in G must have trivial centre. □

4. Point Stabilisers with Nontrivial Centers

In this section we discuss when the groups H_0 in Theorem 1 have nontrivial centres, in other words, when H_0 is the centraliser of some element of H.

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If $H_0 \leqslant \Gamma L(1,q)$ such that H_0 is transitive on the set of nonzero elements of $(\mathbb{F}_q)^k$, then $H_0 \cong \langle \omega^m, \sigma^v\omega^j\rangle$ with m, v and j satisfying the number theoretic conditions given in [**14**, Theorem 2.3 with k replaced by m] and $q = p^n$ where p is a prime. In particular, m is a divisor of n. We shall show $H_0 = C_H(t)$ if and only if t is the linear transformation induced by multiplication by some element of the multiplicative group of the fixed field of σ^v. Sufficiency is clear. We prove necessity.

As $t \in H_0 \leqslant \Gamma L(1,p^n) = \langle \sigma, \omega\rangle$, $t = \sigma^i\omega^l$ with $0 \leqslant i < n$. Since $\omega^m \in C_H(t)$, $\sigma^i\omega^l = (\sigma^i\omega^l)^{\omega^m} = \sigma^i(\omega^{-m})^{\sigma^i}\omega^m\omega^l = \sigma^i\omega^{-mp^i}\omega^m\omega^l = \sigma^i\omega^{-m(p^i-1)}\omega^l$. Thus, $1 = \omega^{-m(p^i-1)}$ and so $(p^n-1) \mid m(p^i-1)$.

First we shall show $i = 0$. Assume $i > 0$. Suppose r is a Zsigmondy prime for p and n, then $r \mid (p^n-1)$, but r does not divide n nor p^i-1. However, r not dividing n implies r does not divide m. Hence, since $r \mid (p^n-1) \mid m(p^i-1)$, $r \mid (p^i-1)$, a contradiction. Thus, there is no Zsigmondy prime for p and n, so by Zsigmondy's Theorem [**9**, Theorem IX.8.3, p. 508, for instance], either p is a Mersenne prime and $n = 2$ or else $p = 2$ and $n = 6$. In the former case, $m \mid n$ implies $m \leqslant 2$ and $i < n$ implies $i = 1$. Therefore, $(p^2-1) \mid 2(p-1)$, that is, $(p+1) \mid 2$, another contradiction. Finally, if $p = 2$ and $n = 6$, then $3^2\dot{7} = (2^6-1) \mid m(2^i-1) \mid 6(2^i-1)$ which, by inspection, cannot occur since $i < 6$. Thus, $i = 0$, which means $t = \omega^l$.

Recall σ normalises $\langle\omega\rangle$ and $\sigma^v\omega^j$ centralises t. Hence, $t = t^{\sigma^v\omega^j} = (\omega^l)^{\sigma^v\omega^j} = (\omega^l)^{\sigma^v} = t^{\sigma^v}$. That shows $\overline{\omega}^l$ is in the fixed field of σ^v as claimed where $\overline{\omega}$ is a generator of the multiplicative group of $\mathbb{F}_{p^n}$ and ω is the linear transformation induced by multiplication by $\overline{\omega}$.

In each of Huppert's 13 exceptional groups, $Z(H_0)$ contains the scalar matrices which lie in H_0. In particular, $-I \in Z(H_0)$ where I is the identity matrix. To see that, Huppert gives matrix generators for H_0. In all the 2–dimensional cases, a simple matrix calculation shows t commutes with the generators denoted A and B by Huppert if and only if $t \in Z\big(GL(k,p)\big)$ and $-I = B^2 \in Z(H_0)$. In the 4–dimensional cases, we get the same result from the fact that t commutes with generators A, B, C and D. In addition, $-I = A^2 \in Z(H_0)$.

In the nonsolvable cases 3, 4 and 5, we shall next show $Z(H_0)$ has nontrivial centre if and only if H_0 contains a nonidentity scalar matrix αI with α in the fixed field of σ^j where $GL(k,q)H_0/GL(k,q) = \langle GL(k,q)\sigma^j\rangle$.

First, in the case $SL(k,q) \trianglelefteq H_0 \leqslant \Gamma L(k,q)$, excluding the solvable cases $(k,q) = (1,q), (2,2), (2,3)$, $C_{\Gamma L(k,q)}\big(SL(k,q)\big) = RL(k,q)$ where $RL(k,q) := \{\alpha I \mid \alpha \in \mathbb{F}_q^*\}$ is the set of $k \times k$ scalar matrices [**12**]. Thus,

$$Z(H_0) \leqslant C_{\Gamma L(k,q)}\big(SL(k,q)\big) \cap H_0 = RL(k,q) \cap H_0$$

Therefore, elements of $Z(H_0)$ are scalar matrices. Furthermore, it is easy to see αI is in $Z(H_0)$ if and only if α is in the fixed field of σ^j. The claim follows.

In the case $Sp(k,q) \trianglelefteq H_0 \leqslant \Gamma L(k,q)$, $C_{\Gamma L(k,q)}\big(Sp(k,q)\big) = RL(k,q)$ [**13**]. Thus, we can deduce the claim as above.

In the case $G_2(2^m) \trianglelefteq H_0 \leqslant \Gamma L(6,2^m)$ where $2^{6m} = p^n$, one can show (see below) that $C_{\Gamma L(6,2^m)}\big(G_2(2^m)\big) = RL(6,2^m)$ which again is sufficient to establish the claim.

Since we do not know a reference for the precise result $C_{\Gamma L(6,2^m)}\big(G_2(2^m)\big) = RL(6,2^m)$ of the previous paragraph, we shall give a sketch of its proof. Let λ_1 denote the fundamental dominant weight corresponding to the short root in a root system of type G_2. If F is a field of characteristic 2, it has been pointed out to us by Gary Seitz that the action of $G_2(F)$ on $V = V(\lambda_1)$, the rational irreducible 6-dimensional module of highest weight λ_1, preserves a non-singular symplectic form on V, thus giving rise to an embedding of $G_2(F)$ in $Sp(6,V)$. Taking a basis of weight vectors in V allows us to write down matrices for the generators of $G_2(F)$. From this we can compute that the only semilinear transformations on V which commute with these generators are the scalars.

In the case $E \trianglelefteq H_0 \leqslant \Gamma L(4,3)$, $Z(E) \leqslant C_{H_0}(E) = Z(H_0)$ gives $Z(H_0)$ nontrivial.

In the case $SL(2,5) \cong H_0^{(\infty)} \trianglelefteq H_0 \leqslant \Gamma L(2,q)$, where $q^2 = p^n$ and $q = 9,11,19,29,59$, we see $Z(H_0^{(\infty)})$ has order 2 and is characteristic in $H_0^{(\infty)}$ which is characteristic in H_0. Thus $Z(H_0^{(\infty)})$ is characteristic, and hence normal, in H_0. If x is the nonidentity element in $Z(H_0^{(\infty)})$, then any conjugate of x by any element of H_0 must equal x, putting x in $Z(H_0)$.

Finally, in the case $SL(2,13) \cong H_0 \leqslant \Gamma L(6,3)$, $Z(H_0)$ has order 2.

5. Some Consequences

Suppose a group acts faithfully and 2–transitively on one of its conjugacy classes of elements. Upon which other classes might it also act 2–transitively? We begin this section by answering that question.

Lemma 1. *Suppose M is a minimal normal self-centralising subgroup of G and $C_G(x)$ is a complement to M in G for some $x \in G$. Then $C_G(x)$ is a maximal subgroup of G and all complements to M in G are conjugate in G.*

Proof: Let K be any complement to M in G, then K is a maximal subgroup of G. For assume $K \leqslant L \leqslant G$, then $L = K(L \cap M)$. Since $L \cap M$ is normalised by L and centralised by M, which is elementary Abelian by hypothesis, $L \cap M$ is normal in G. Hence, $L \cap M = 1$ or M. Thus, K is maximal.

Note, furthermore, that $Z(G) \leqslant C_G(x) \cap C_G(M) = C_G(x) \cap M = 1$. It follows that $K = C_G\big(Z(K)\big)$.

Next we shall show $\big(|Z(K)|, |M|\big) = 1$. Let $|M| = p^n$ where p is a prime. Suppose p divides $|Z(K)|$. Let y be a nonidentity p–element in $Z(K)$, then $K \leqslant C_G(y)$. Therefore, $C_G(y) = K$ by the maximality of K and the fact that G has trivial centre. Hence, $Z\big(\langle y\rangle M\big) \leqslant C_G(y) \cap C_G(M) = K \cap M = 1$, a contradiction since $\langle y\rangle M$ is a p–group.

Suppose K and J are complements to M in G. Set $Z/M = Z(G/M)$, then $Z = M(K\cap Z)$. Moreover, $[K\cap Z, K] \leqslant M\cap K = 1$, which implies $K\cap Z \leqslant Z(K)$. Conversely, $Z(K) \leqslant K\cap Z$ since $[Z(K), G] = [Z(K), KM] \leqslant [Z(K), M] \leqslant M$. Therefore, $K\cap Z = Z(K)$ and $Z = MZ(K)$. Similarly, $Z = MZ(J)$.

From above, $Z(K)$ and $Z(J)$ are Hall p'–subgroups of Z. As Z is a solvable group, $Z(K)^w = Z(J)$ for some $w \in Z$. Consequently, $K^w = C_G(Z(K))^w = C_G(Z(J)) = J$. Thus, all complements to M in G are conjugate as was to be shown. $\square$

Proposition 2. *Suppose G acts faithfully and 2–transitively on $cl_G(x)$ for some $x \in G$, then G acts faithfully and 2–transitively on $cl_G(y)$ if and only if y is conjugate to a nonidentity element of $Z(C_G(x))$. Furthermore, no two distinct elements of $Z(C_G(x))$ are conjugate in G.*

Proof: Let M be the minimal normal subgroup of G. By Theorem 1, M is Abelian. (That is the only place our classification is used in this proof. The rest is elementary.) By [**8**, II.1.4 and II.3.2], M is self–centralising; $C_G(x)$ is a complement to M in G and $C_G(x)$ is a maximal subgroup of G.

If G acts faithfully and 2–transitively on $cl_G(y)$, then [**8**, II.3.2] also implies $C_G(y)$ is a complement to M. Hence, $C_G(x) = C_G(y^t)$ for some $t \in G$ by Lemma 1. Thus, $y^t \in Z(C_G(x))$.

For the converse, suppose $y^t \in Z(C_G(x))$ for some $t \in G$, then, by the maximality of $C_G(x)$, either $C_G(y^t) = C_G(x)$ or $y^t \in Z(G)$. The latter is impossible since M is self-centralising. By the faithful action of G on $cl_G(x)$ we have

$$1 = \bigcap_{g\in G} C_G(x)^g = \bigcap_{g\in G} C_G(y^t)^g = \bigcap_{g\in G} C_G(y)^g.$$

Therefore, G also acts faithfully on $cl_G(y)$.

To show G acts 2–transitively on $cl_G(y)$, we shall show $C_G(y)$ acts transitively on $cl_G(y) - \{y\}$. Let y^g and y^h be distinct elements of $cl_G(y) - \{y\}$. It is easy to calculate that $x^{t^{-1}g}$ and $x^{t^{-1}h}$ are distinct elements of $cl_G(x) - \{x^{t^{-1}}\}$. By the 2–transitivity of G on $cl_G(x)$, $x^{t^{-1}gw} = x^{t^{-1}h}$ for some $w \in C_G(x^{t^{-1}}) = C_G(y)$. Thus, $gwh^{-1} \in C_G(x^{t^{-1}}) = C_G(y)$ and $y^{gw} = y^h$.

Finally, suppose z, z^g are distinct nonidentity elements of $Z(C_G(x))$ for some $g \in G$, then $\langle C_G(x), C_G(x^{g^{-1}})\rangle \leqslant C_G(z)$. By the maximality of $C_G(x)$, it must be that $C_G(x) = C_G(x^{g^{-1}})$, for otherwise $z \in Z(G)$. Thus, $g^{-1} \in N_G(C_G(x)) = C_G(x)$, from which it follows that $z^g = z$. Therefore, no two distinct nonidentity elements of $Z(C_G(x))$ are conjugate in G. $\square$

We conclude this section with a solvability criterion based on the fraction of elements of a group belonging to classes upon which the group acts faithfully and 2–transitively.

Theorem 2. *Suppose G is a finite group and at least $1/12$ of the elements of G belong to conjugacy classes upon which G acts faithfully and 2–transitively, then G is* *solvable. Moreover, by choosing a solvable G appropriately, the fraction of such elements may be made arbitrarily close to* 1.

PROOF: Let $F = \{x \in G : G \text{ acts faithfully and 2-transitively on } cl_G(x)\}$. For any $x \in F$ we have, by Proposition 2, $|F| = \left(\left|Z\left(C_G(x)\right)\right| - 1\right)|M|$ where M is the minimal normal subgroup of G. Therefore,

$$\frac{|F|}{|G|} = \frac{\left(\left|Z\left(C_G(x)\right)\right| - 1\right)|M|}{\left|C_G(x)\right||M|} < \frac{\left|Z\left(C_G(x)\right)\right|}{\left|C_G(x)\right|}.$$

Assume G is nonsolvable, then we have by Theorem 1, $C_G(x)$ is nonsolvable. As in Hering [5], we set $S = C_G(x)^{(\infty)}$ and by [5, Theorem 5.8(g)], we have $Z\left(C_G(x)\right) \leqslant C_G(S)$ is a subgroup of the set of scalar matrices $RL(k,q)$ with k and q as in Theorem 1. Thus,

$$\frac{|F|}{|G|} < \frac{\left|RL(k,q)\right|}{\left|C_G(x)\right|} = \frac{q-1}{\left|C_G(x)\right|}.$$

In case 3 of Theorem 1, $C_G(x) \geqslant SL(n,q)$. Suppose $n = 2$, then $q \geqslant 4$, so

$$\frac{|F|}{|G|} < \frac{q-1}{\left|SL(n,q)\right|} \leqslant \frac{1}{q(q+1)} \leqslant \frac{1}{20}.$$

If $n > 2$, then

$$\frac{|F|}{|G|} < \frac{q-1}{\left|SL(n,q)\right|} \leqslant \frac{1}{(q^3-1)(q+1)} \leqslant \frac{1}{21}.$$

Cases 4, 5, 6 and 8 are handled similarly. For case 7, we note that by the 2–transitivity of G on $cl_G(x)$, which has order q^2, we have $q^2 - 1$ divides $\left|C_G(x)\right|$. In this case, we have $\left|C_G(x)^{(\infty)}\right| = 120$. Combining these gives $\left|C_G(x)\right| \geqslant 120r$ where $r = 2, 1, 3, 7, 29$ for $q = 9, 11, 19, 29, 59$ respectively. We calculate $\dfrac{|F|}{|G|} < \dfrac{1}{12}$ in all cases. Therefore, if $\dfrac{|F|}{|G|} \geqslant \dfrac{1}{12}$, then G is solvable. By considering the solvable group $G = \Gamma L(1,q)$, we have $\dfrac{|F|}{|G|} = \dfrac{q-2}{q-1}$, which can be made arbitrarily close to 1 by choosing q sufficiently large. $\square$

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Department of Mathematics
Allegheny College
Meadville PA 16335
United States of America
e-mail: mbarry@alleg.edu

Department of Mathematics
Western Oregon University
Monmouth OR 97361
United States of America
e-mail: wardm@wou.edu