A GENERALISATION OF WITTEN'S CONJECTURE FOR THE PIXTON CLASS AND THE NONCOMMUTATIVE KDV HIERARCHY

ALEXANDR BURYAK [1] AND PAOLO ROSSI [2]

[1]*Faculty of Mathematics, National Research University Higher School of Economics, 6 Usacheva Street, Moscow, Russian Federation, 119048; Center for Advanced Studies, Skolkovo Institute of Science and Technology, 1 Nobel Street, Moscow, Russian Federation, 143026; Faculty of Mechanics and Mathematics, Lomonosov Moscow State University, GSP-1, Moscow, Russian Federation, 119991*
(aburyak@hse.ru)

[2]*Dipartimento di Matematica 'Tullio Levi-Civita', Università degli Studi di Padova, Via Trieste 63, 35121 Padova, Italy*
(paolo.rossi@math.unipd.it)

Abstract In this paper, we formulate and present ample evidence towards the conjecture that the partition function (i.e. the exponential of the generating series of intersection numbers with monomials in psi classes) of the Pixton class on the moduli space of stable curves is the topological tau function of the noncommutative Korteweg-de Vries hierarchy, which we introduced in a previous work. The specialisation of this conjecture to the top degree part of Pixton's class states that the partition function of the double ramification cycle is the tau function of the dispersionless limit of this hierarchy. In fact, we prove that this conjecture follows from the double ramification/Dubrovin–Zhang equivalence conjecture. We also provide several independent computational checks in support of it.

Contents

1. Introduction

The Witten–Kontsevich theorem [17, 22] states that the partition function $\exp\left(\varepsilon^{-2}\mathcal{F}^{\mathrm{W}}\right)$ of the trivial cohomological field theory on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves of genus g with n marked points,

$$\mathcal{F}^{\mathrm{W}}(t_0,t_1,\ldots,\varepsilon) := \sum_{\substack{g,n\geq 0 \\ 2g-2+n>0}} \frac{\varepsilon^{2g}}{n!} \sum_{d_1,\ldots,d_n\geq 0} \left(\int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n}\psi_i^{d_i}\right) \prod_{i=1}^{n} t_{d_i},$$

is the topological tau function of the Korteweg–de Vries hierarchy. In particular, this means that $u = u^{\mathrm{W}} = \frac{\partial^2 \mathcal{F}^{\mathrm{W}}}{(\partial t_0)^2}$ satisfies the infinite system of compatible partial differential equations (PDEs):

$$\frac{\partial u}{\partial t_1} = \partial_x\left(\frac{u^2}{2} + \frac{\varepsilon^2}{12}u_{xx}\right), \quad \frac{\partial u}{\partial t_2} = \partial_x\left(\frac{u^3}{6} + \frac{\varepsilon^2}{24}(2uu_{xx}+u_x^2) + \frac{\varepsilon^4}{240}u_{xxxx}\right), \quad \ldots,$$

where $x = t_0$ and whose generic member is given in terms of a simple and well-known Lax representation. In this paper, we propose a generalisation of this result involving Pixton's class $\sum_{j=0}^{g} P_g^j(A)$, a family of nonhomogeneous tautological classes on $\overline{\mathcal{M}}_{g,n}$ depending on an n-tuple of integers $A = (a_1,\ldots,a_n) \in \mathbb{Z}^n$ with $\sum a_i = 0$, given explicitly in terms of a subtle combinatorial formula introduced by A. Pixton, whose top degree term $P_g^g(A)$, in cohomological degree $2g$, equals (up to a constant) the double ramification (DR) cycle $\mathrm{DR}_g(A)$ (i.e. the cohomological representative of (a compactification of) the locus of genus g curves whose marked points support a principal divisor [16]). This family of tautological classes forms a partial cohomological field theory $c_{g,n}$ with an infinite dimensional phase space $V = \mathrm{span}(\{e_a\}_{a\in\mathbb{Z}})$. Consider the generating series:

$$\mathcal{F}^{\mathrm{P}}(t_*^*,\varepsilon,\mu) := \sum_{\substack{g,n\geq 0 \\ 2g-2+n>0}} \sum_{j=0}^{g} \frac{\varepsilon^{2g}\mu^{2j}}{n!} \sum_{\substack{A=(a_1,\ldots,a_n)\in\mathbb{Z}^n \\ \sum a_i=0}}$$

$$\sum_{d_1,\ldots,d_n\geq 0} \left(\int_{\overline{\mathcal{M}}_{g,n}} 2^{-j} P_g^j(A) \prod_{i=1}^{n}\psi_i^{d_i}\right) \prod_{i=1}^{n} t_{d_i}^{a_i},$$

and let:

$$(w^{\mathrm{P}})^a := \frac{\partial^2 \mathcal{F}^{\mathrm{P}}}{\partial t_0^0 \partial t_0^{-a}}, \quad a \in \mathbb{Z}, \qquad w^{\mathrm{P}} := \sum_{a \in \mathbb{Z}} (w^{\mathrm{P}})^a e^{iay}, \qquad u^{\mathrm{P}} := \frac{S(\varepsilon \mu \partial_x)}{S(i \varepsilon \mu \partial_x \partial_y)} w^{\mathrm{P}},$$

where $S(z) := \frac{e^{z/2} - e^{-z/2}}{z}$. Then our main Conjecture 2 states that $u = u^{\mathrm{P}}$ satisfies the infinite system of compatible PDEs:

$$\frac{\partial u}{\partial t_1} = \partial_x \left(\frac{u * u}{2} + \frac{\varepsilon^2}{12} u_{xx} \right),$$

$$\frac{\partial u}{\partial t_2} = \partial_x \left(\frac{u * u * u}{6} + \frac{\varepsilon^2}{24} (u * u_{xx} + u_x * u_x + u_{xx} * u) + \frac{\varepsilon^4}{240} u_{xxxx} \right),$$

$$\vdots$$

where $x = t_0^0$, which is a noncommutative analogue of the Korteweg-de Vries (KdV) hierarchy above, with respect to the noncommutative Moyal product:

$$f * g := f \ \exp\left(\frac{i\varepsilon\mu}{2} (\overleftarrow{\partial_x}\overrightarrow{\partial_y} - \overleftarrow{\partial_y}\overrightarrow{\partial_x}) \right) g$$

for functions f,g on a 2-dimensional torus with coordinates x,y [6]. Notice that, together with the string and dilaton equation, the above noncommutative KdV (ncKdV) equations determine uniquely the generating series $\mathcal{F}^{\mathrm{P}}$.

This conjecture specialises to:

$$\mathcal{F}^{\mathrm{DR}}(t_*^*, \varepsilon) := \sum_{\substack{g,n \geq 0 \\ 2g-2+n>0}} \frac{\varepsilon^{2g}}{n!} \sum_{\substack{A=(a_1,\ldots,a_n)\in\mathbb{Z}^n \\ \sum a_i = 0}} \sum_{d_1,\ldots,d_n \geq 0} \left(\int_{\overline{\mathcal{M}}_{g,n}} \mathrm{DR}_g(A) \prod_{i=1}^n \psi_i^{d_i} \right) \prod_{i=1}^n t_{d_i}^{a_i},$$

and:

$$(w^{\mathrm{DR}})^a := \frac{\partial^2 \mathcal{F}^{\mathrm{DR}}}{\partial t_0^0 \partial t_0^{-a}}, \quad a \in \mathbb{Z}, \qquad w^{\mathrm{DR}} := \sum_{a \in \mathbb{Z}} (w^{\mathrm{DR}})^a e^{iay}, \qquad u^{\mathrm{DR}} := \frac{S(\varepsilon \partial_x)}{S(i \varepsilon \partial_x \partial_y)} w^{\mathrm{DR}},$$

with $u = u^{\mathrm{DR}}|_{\varepsilon \mapsto \varepsilon\mu}$ satisfying the dispersionless ncKdV hierarchy:

$$\frac{\partial u}{\partial t_n} = \partial_x \left(\frac{u^{*(n+1)}}{(n+1)!} \right), \quad n \geq 1, \tag{1.1}$$

which is Conjecture 1. In this paper, we provide a proof that the two above conjectures follow from the much more general DR/Dubrovin-Zhang (DZ) equivalence conjecture, which states that the double ramification hierarchy (introduced in [1] and further studied in [5]) and the DZ hierarchy (introduced in [11]) are equivalent up to a very specific change of coordinates in the corresponding phase space [1–4], together with the results of [6], where we proved that the double ramification hierarchy for the Pixton class is indeed the ncKdV hierarchy. In particular, this proves our conjectures at the approximation up to ε^2. Moreover, we provide several independent computational checks for Conjectures 1 and 2 themselves.

2. Double ramification cycles and the dispersionless ncKdV hierarchy

In this section, we recall the definition of the double ramification cycles on the moduli spaces of stable curves and present Conjecture 1 describing an integrable system controlling the intersections of monomials in psi classes with the double ramification cycles.

All the cohomology and homology groups of topological spaces will be taken with complex coefficients.

2.1. Double ramification cycles

For a pair of nonnegative integers (g,n) in the stable range (i.e. satisfying $2g+2-n > 0$), let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable algebraic curves of genus g with n marked points labeled by the set $[n] := \{1,\ldots,n\}$. Denote by $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n})$ the first Chern class of the line bundle $\mathcal{L}_i$ over $\overline{\mathcal{M}}_{g,n}$ formed by the cotangent lines at the i-th marked point on stable curves. The classes ψ_i are called the *psi classes*. Denote by $\mathbb{E}$ the rank g *Hodge vector bundle* over $\overline{\mathcal{M}}_{g,n}$ whose fibers are the spaces of holomorphic one-forms on stable curves. Let $\lambda_j := c_j(\mathbb{E}) \in H^{2j}(\overline{\mathcal{M}}_{g,n})$. Let $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ be the moduli space of smooth pointed curves and denote by $\mathcal{M}_{g,n}^{\mathrm{ct}} \subset \overline{\mathcal{M}}_{g,n}$ the locus of stable curves with no nonseparating nodes.

Consider an n-tuple of integers $A = (a_1,\ldots,a_n)$, such that $\sum a_i = 0$, it will be called a *vector of double ramification data*. Suppose first that not all the numbers a_i are equal to zero. Let:

$$\mathcal{Z}_g(A) \subset \mathcal{M}_{g,n}$$

be the locus parameterising the isomorphism classes of pointed smooth curves $(C;p_1,\ldots,p_n)$ satisfying the condition $\mathcal{O}_C(\sum_{i=1}^n a_i p_i) \cong \mathcal{O}_C$, which is algebraic and defines $\mathcal{Z}_g(A)$ canonically as a substack of $\mathcal{M}_{g,n}$ of dimension $2g-3+n$. Naively, the double ramification cycle $\mathrm{DR}_g(A)$ is defined as the cohomology class on $\overline{\mathcal{M}}_{g,n}$ that is Poincaré dual to a compactification of $\mathcal{Z}(A)$ in $\overline{\mathcal{M}}_{g,n}$. A rigorous definition is the following (see, e.g. [16]).

The positive parts of A define a partition $\mu = (\mu_1,\ldots,\mu_{l(\mu)})$. The negative parts of A define a second partition $\nu = (\nu_1,\ldots,\nu_{l(\nu)})$. Since the parts of A sum to 0, the partitions μ and ν must be of the same size. We now allow the case $|\mu| = |\nu| = 0$. Let $n_0 := n - l(\mu) - l(\nu)$. The moduli space:

$$\overline{\mathcal{M}}_{g,n_0}(\mathbb{P}^1,\mu,\nu)^{\sim}$$

parameterises stable relative maps of connected algebraic curves of genus g to rubber $\mathbb{P}^1$ with ramification profiles μ,ν over the points $0,\infty \in \mathbb{P}^1$, respectively. There is a natural map:

$$\mathrm{st}\colon \overline{\mathcal{M}}_{g,n_0}(\mathbb{P}^1,\mu,\nu)^{\sim} \to \overline{\mathcal{M}}_{g,n}$$

forgetting everything except the marked domain curve. The moduli space $\overline{\mathcal{M}}_{g,n_0}(\mathbb{P}^1,\mu,\nu)^{\sim}$ possesses a virtual fundamental class $\left[\overline{\mathcal{M}}_{g,n_0}(\mathbb{P}^1,\mu,\nu)^{\sim}\right]^{\mathrm{vir}}$, which is a homology class of

degree $2(2g-3+n)$. The *double ramification cycle*:

$$\mathrm{DR}_g(A) \in H^{2g}(\overline{\mathcal{M}}_{g,n})$$

is defined as the Poincaré dual to the push-forward $\mathrm{st}_* \left[\overline{\mathcal{M}}_{g,n_0}(\mathbb{P}^1, \mu, \nu)^\sim \right]^{\mathrm{vir}} \in H_{2(2g-3+n)}(\overline{\mathcal{M}}_{g,n})$.

Let us list some properties of the double ramification cycles (see, e.g. [16]). In genus 0, we have:

$$\mathrm{DR}_0(A) = 1 \in H^0(\overline{\mathcal{M}}_{g,n}).$$

If all the numbers a_i are equal to zero, then we have:

$$\mathrm{DR}_g(0, \ldots, 0) = (-1)^g \lambda_g \in H^{2g}(\overline{\mathcal{M}}_{g,n}).$$

There is a very simple explicit formula for the restriction of the double ramification cycle to the moduli space $\mathcal{M}_{g,n}^{\mathrm{ct}}$. For $J \subset [n]$ and $0 \le h \le g$ in the stable range $2h-1+|J| > 0$ and $2(g-h)-1+(n-|J|) > 0$, denote by $\delta_h^J \in H^2(\overline{\mathcal{M}}_{g,n})$ the Poincaré dual to the substack of $\overline{\mathcal{M}}_{g,n}$ formed by stable curves with a separating node at which two stable components meet, one of genus h and with marked points labeled by $|J|$, and the other of genus $g-h$ and with marked points labeled by the complement $[n]\backslash J$. We adopt the convention $\delta_h^J := 0$ if at least one of the stability conditions $2h-1+|J| > 0$ and $2(g-h)-1+(n-|J|) > 0$ is not satisfied. Let $a_J := \sum_{j \in J} a_j$. Introduce a degree 2 cohomology class $\theta_g(A)$ on $\overline{\mathcal{M}}_{g,n}$ by:

$$\theta_g(A) := \sum_{j=1}^n \frac{a_j^2 \psi_j}{2} - \frac{1}{4} \sum_{h=0}^g \sum_{J \subset [n]} a_J^2 \delta_h^J \in H^2(\overline{\mathcal{M}}_{g,n}).$$

Then we have the formula:

$$\mathrm{DR}_g(A)|_{\mathcal{M}_{g,n}^{\mathrm{ct}}} = \frac{1}{g!} \theta_g(A)^g \Big|_{\mathcal{M}_{g,n}^{\mathrm{ct}}}, \tag{2.1}$$

which is called Hain's formula. More properties of the double ramification cycles will be presented in Section 3.

2.2. The noncommutative KdV hierarchy

The classical construction of the KdV hierarchy as the system of Lax equations (see, e.g. [9]):

$$\frac{\partial L}{\partial t_n} = \frac{\varepsilon^{2n}}{(2n+1)!!} \left[\left(L^{n+1/2} \right)_+, L \right], \quad n \ge 1,$$

where $L := \partial_x^2 + 2\varepsilon^{-2} u$, u is a function of $x, t_1, t_2, \ldots, \varepsilon$ is a formal parameter and $(2n+1)!! := (2n+1) \cdot (2n-1) \cdots 3 \cdot 1$ admits generalisations, called noncommutative KdV hierarchies, where one doesn't have the pairwise commutativity of the x-derivatives of the dependent variable u. In what follows, we will work with a specific example from the class of noncommutative KdV hierarchies.

Let u_{k_1,k_2}, $k_1,k_2 \in \mathbb{Z}_{\geq 0}$, ε and μ be formal variables and consider the space $\widehat{\mathcal{A}} :=$ $\mathbb{C}[[u_{*,*},\varepsilon,\mu]]$, whose elements will be called *differential polynomials in two space variables*. Consider a gradation on $\widehat{\mathcal{A}}$ given by:

$$\deg u_{k_1,k_2} := (k_1,k_2), \qquad \deg \varepsilon := (-1,0), \qquad \deg \mu := (0,-1).$$

We will denote by $\widehat{\mathcal{A}}^{[(d_1,d_2)]} \subset \widehat{\mathcal{A}}$ the space of differential polynomials of degree (d_1,d_2). The space $\widehat{\mathcal{A}}$ is endowed with operators ∂_x and ∂_y of degrees $(1,0)$ and $(0,1)$, respectively, defined by:

$$\partial_x := \sum_{k_1,k_2 \geq 0} u_{k_1+1,k_2} \frac{\partial}{\partial u_{k_1,k_2}}, \qquad \partial_y := \sum_{k_1,k_2 \geq 0} u_{k_1,k_2+1} \frac{\partial}{\partial u_{k_1,k_2}}.$$

We see that $u_{k_1,k_2} = \partial_x^{k_1} \partial_y^{k_2} u$. We will denote $u_{0,0}$ simply by u.

The algebra $\widehat{\mathcal{A}}$ is also endowed with the *Moyal star product* defined by:

$$f * g := f \, \exp\left(\frac{i\varepsilon\mu}{2}(\overleftarrow{\partial_x}\overrightarrow{\partial_y} - \overleftarrow{\partial_y}\overrightarrow{\partial_x})\right) g = \sum_{n \geq 0} \sum_{k_1+k_2=n} \frac{(-1)^{k_2}(i\varepsilon\mu)^n}{2^n k_1! k_2!}(\partial_x^{k_1}\partial_y^{k_2}f)(\partial_x^{k_2}\partial_y^{k_1}g), \tag{2.2}$$

where $f,g \in \mathbb{C}[[u_{*,*},\varepsilon,\mu]]$. The Moyal star product is associative, and it is graded: if $\deg f = (i_1,i_2)$ and $\deg g = (j_1,j_2)$, then $\deg(f*g) = (i_1+j_1,i_2+j_2)$. Note also that when $\mu = 0$, the Moyal star product becomes the usual multiplication:

$$(f*g)|_{\mu=0} = f|_{\mu=0} \cdot g|_{\mu=0}. \tag{2.3}$$

Let us now consider the algebra of pseudo-differential operators of the form:

$$A = \sum_{i \leq n} a_i * \partial_x^i, \quad n \in \mathbb{Z}, \quad a_i \in \mathbb{C}[[u_{*,*},\mu]][[\varepsilon,\varepsilon^{-1}], \tag{2.4}$$

with the multiplication $\circ$ given by:

$$(a * \partial_x^i) \circ (b * \partial_x^j) := \sum_{k \geq 0} \binom{i}{k} (a * \partial_x^k b) * \partial_x^{i+j-k}, \quad a,b \in \mathbb{C}[[u_{*,*},\mu]][[\varepsilon,\varepsilon^{-1}], \quad i,j \in \mathbb{Z}.$$

The positive part of a pseudo-differential operator (2.4) is defined by $A_+ := \sum_{0 \leq i \leq n} a_i * \partial_x^i$, and, as in the classical theory of pseudo-differential operators, a pseudo-differential operator A of the form $\partial_x^2 + \sum_{i<2} a_i * \partial_x^i$ has a unique square root of the form $\partial_x + \sum_{i<1} b_i * \partial_x^i$, which we denote by $A^{\frac{1}{2}}$.

Consider the operator $L := \partial_x^2 + 2\varepsilon^{-2}u$. The *noncommutative KdV hierarchy* with respect to the Moyal star product (2.2) is defined by (see, e.g. [10, 15]):

$$\frac{\partial L}{\partial t_n} = \frac{\varepsilon^{2n}}{(2n+1)!!}\left[\left(L^{n+1/2}\right)_+,L\right], \quad n \geq 1. \tag{2.5}$$

The ncKdV hierarchy is integrable in the sense that its flows pairwise commute. Explicitly, the first two equations of the hierarchy are:

$$\frac{\partial u}{\partial t_1} = \partial_x \left(\frac{u * u}{2} + \frac{\varepsilon^2}{12} u_{xx} \right),$$

$$\frac{\partial u}{\partial t_2} = \partial_x \left(\frac{u * u * u}{6} + \frac{\varepsilon^2}{24} (u * u_{xx} + u_x * u_x + u_{xx} * u) + \frac{\varepsilon^4}{240} u_{xxxx} \right).$$

For any $n \geq 1$, the right-hand side of (2.5) has the form $\partial_x P_n$, where $P_n \in \widehat{\mathcal{A}}^{[(0,0)]}$. Moreover, $P_n = \sum_{g=0}^{n} P_{n,g}$, where $P_{n,g}$ is a linear combination of the monomials $\varepsilon^{2g} u_{d_1} * \cdots * u_{d_{n+1-g}}$ with $d_1 + \cdots + d_{n+1-g} = 2g$. The leading term $P_{n,0}$ is equal to $\frac{u^{*(n+1)}}{(n+1)!}$. The hierarchy:

$$\frac{\partial u}{\partial t_n} = \partial_x \left(\frac{u^{*(n+1)}}{(n+1)!} \right), \quad n \geq 1, \tag{2.6}$$

will be called the *dispersionless noncommutative KdV (dncKdV) hierarchy*.

Note that because of (2.3), the noncommutative KdV hierarchy becomes the classical KdV hierarchy when $\mu = 0$.

We are now ready to present our first conjecture. Let us introduce formal variables t_d^a, $a \in \mathbb{Z}$, $d \geq 0$ and consider the generating function:

$$\mathcal{F}^{\mathrm{DR}}(t_*^*, \varepsilon) := \sum_{\substack{g, n \geq 0 \\ 2g-2+n > 0}} \frac{\varepsilon^{2g}}{n!} \sum_{\substack{A = (a_1, \ldots, a_n) \in \mathbb{Z}^n \\ \sum a_i = 0}}$$

$$\sum_{d_1, \ldots, d_n \geq 0} \left(\int_{\overline{\mathcal{M}}_{g,n}} \mathrm{DR}_g(A) \prod_{i=1}^{n} \psi_i^{d_i} \right) \prod_{i=1}^{n} t_{d_i}^{a_i} \in \mathbb{C}[[t_*^*, \varepsilon]].$$

Introduce a formal power series:

$$S(z) := \frac{e^{z/2} - e^{-z/2}}{z} = 1 + \frac{z^2}{24} + \frac{z^4}{1920} + O(z^6),$$

and let:

$$(w^{\mathrm{DR}})^a := \frac{\partial^2 \mathcal{F}^{\mathrm{DR}}}{\partial t_0^0 \partial t_0^{-a}} \in \mathbb{C}[[t_*^*, \varepsilon]], \quad a \in \mathbb{Z},$$

$$w^{\mathrm{DR}} := \sum_{a \in \mathbb{Z}} (w^{\mathrm{DR}})^a e^{iay} \in \mathbb{C}[[t_*^*, \varepsilon]][[e^{iy}, e^{-iy}]],$$

$$u^{\mathrm{DR}} := \frac{S(\varepsilon \partial_x)}{S(i\varepsilon \partial_x \partial_y)} w^{\mathrm{DR}} \in \mathbb{C}[[t_*^*, \varepsilon]][[e^{iy}, e^{-iy}]]. \tag{2.7}$$

Conjecture 1. *The function $u^{\mathrm{DR}}\big|_{\varepsilon \mapsto \varepsilon\mu}$ satisfies the dispersionless noncommutative KdV hierarchy (2.6), where we identify $t_d = t_d^0$ and $x = t_0^0$.*

Let us analyse the system of equations that this conjecture gives for the generating series $\mathcal{F}^{\mathrm{DR}}$ in a bit more detail. For any $a \in \mathbb{Z}$, introduce a formal power series:

$$T(a,z) := \frac{S(z)}{S(az)} = 1 + \frac{1-a^2}{24}z^2 + \frac{3-10a^2+7a^4}{5760}z^4 + O(z^6) = \sum_{g \geq 0} Q_g(a)z^{2g}. \qquad (2.8)$$

Here, $Q_g(a)$ are polynomials in a. If we decompose:

$$u^{\mathrm{DR}} = \sum_{a \in \mathbb{Z}} (u^{\mathrm{DR}})^a e^{iay}, \qquad (2.9)$$

then the transformation (2.7) simply means that:

$$(u^{\mathrm{DR}})^a = (w^{\mathrm{DR}})^a + \sum_{g \geq 1} \varepsilon^{2g} Q_g(a) \partial_x^{2g} (w^{\mathrm{DR}})^a, \quad a \in \mathbb{Z}.$$

Also, using the decomposition (2.9), we can rewrite the equations of the dncKdV hierarchy as a system of evolutionary PDEs with one spatial variable x and infinitely many times t_d, $d \geq 1$, for the functions $(u^{\mathrm{DR}})^a$, $a \in \mathbb{Z}$. For example, the first equation of the dncKdV hierarchy, $\frac{\partial u}{\partial t_1} = \partial_x \left(\frac{u*u}{2} \right)$, via Conjecture 1, gives the following PDEs for the functions $(u^{\mathrm{DR}})^a$:

$$\frac{\partial (u^{\mathrm{DR}})^a}{\partial t_1} = \sum_{g \geq 0} \frac{\varepsilon^{2g}}{2^{2g}} \sum_{\substack{a_1, a_2 \in \mathbb{Z} \\ a_1+a_2=a}} \sum_{\substack{k_1, k_2 \geq 0 \\ k_1+k_2=2g}} \frac{(-1)^{k_1}}{k_1! k_2!} a_1^{k_2} a_2^{k_1} \partial_x \left(\partial_x^{k_1} (u^{\mathrm{DR}})^{a_1} \partial_x^{k_2} (u^{\mathrm{DR}})^{a_2} \right), \quad a \in \mathbb{Z}.$$

3. The Pixton class and the full ncKdV hierarchy

Here, we recall Pixton's very explicit construction of a nonhomogeneous cohomology class on $\overline{\mathcal{M}}_{g,n}$, with nontrivial terms in degree $0, 2, 4, \ldots, 2g$. By a result of [16], the degree $2g$ part of this class coincides with the double ramification cycle. We then present Conjecture 2, which generalises Conjecture 1 and says that the intersection numbers of Pixton's class with monomials in psi classes are controlled by the full noncommutative KdV hierarchy.

Let us first recall a standard way to construct cohomology classes on $\overline{\mathcal{M}}_{g,n}$ in terms of stable graphs. A *stable graph* is the following data:

$$\Gamma = (V, H, L, g \colon V \to \mathbb{Z}_{\geq 0}, v \colon H \to V, \iota \colon H \to H),$$

where:

(1) V is a set of *vertices* with a genus function $g \colon V \to \mathbb{Z}_{\geq 0}$,

(2) H is a set of *half-edges* equipped with a vertex assignment $v \colon H \to V$ and an involution ι,

(3) the set of *edges* E is defined as the set of orbits of ι of length 2,

(4) the set of *legs* L is defined as the set of fixed points of ι and is placed in bijective correspondence with the set $[n]$, the leg corresponding to the marking $i \in [n]$ will be denoted by l_i,

(5) the pair (V, E) defines a connected graph,

(6) the stability condition $2g(v) - 2 + n(v) > 0$ is satisfied at each vertex $v \in V$, where $n(v)$ is the valence of Γ at v including both half-edges and legs.

An *automorphism* of Γ consists of automorphisms of the sets V and H that leave invariant the structures L, g, v and ι. Denote by $\mathrm{Aut}(\Gamma)$ the authomorphism group of Γ. The *genus* of a stable graph Γ is defined by $g(\Gamma) := \sum_{v \in V} g(v) + h^1(\Gamma)$. Denote by $G_{g,n}$ the set of isomorphism classes of stable graphs of genus g with n legs.

For each stable graph $\Gamma \in G_{g,n}$, there is an associated moduli space:

$$\overline{\mathcal{M}}_\Gamma := \prod_{v \in V} \overline{\mathcal{M}}_{g(v), n(v)},$$

and a canonical map:

$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n},$$

that is given by the gluing of the marked points corresponding to the two halves of each edge in $E(\Gamma)$. Each half-edge $h \in H(\Gamma)$ determines a cotangent line bundle $\mathcal{L}_h \to \overline{\mathcal{M}}_\Gamma$. If $h \in L(\Gamma)$, then $\mathcal{L}_h$ is the pull back via ξ_Γ of the corresponding cotangent line bundle over $\overline{\mathcal{M}}_{g,n}$. Let $\psi_h := c_1(\mathcal{L}_h) \in H^2(\overline{\mathcal{M}}_\Gamma)$. The Pixton class will be described as a linear combination of cohomology classes of the form:

$$\xi_{\Gamma *}\left(\prod_{h \in H} \psi_h^{d(h)} \right),$$

where $\Gamma \in G_{g,n}$ and $d \colon H(\Gamma) \to \mathbb{Z}_{\geq 0}$.

Let $A = (a_1, \ldots, a_n)$ be a vector of double ramification data. Let $\Gamma \in G_{g,n}$ and $r \geq 1$. A *weighting mod r* of Γ is a function:

$$w \colon H(\Gamma) \to \{0, \ldots, r-1\},$$

that satisfies the following three properties:

(1) for any leg $l_i \in L(\Gamma)$, we have $w(l_i) = a_i \bmod r$;

(2) for any edge $e = \{h, h'\} \in E(\Gamma)$, we have $w(h) + w(h') = 0 \bmod r$;

(3) for any vertex $v \in V(\Gamma)$, we have $\sum_{h \in H(\Gamma),\, v(h) = v} w(h) = 0 \bmod r$.

Denote by $W_{\Gamma, r}$ the set of weightings mod r of Γ. We have $|W_{\Gamma, r}| = r^{h^1(\Gamma)}$.

We denote by $P_g^{d,r}(A) \in H^{2d}(\overline{\mathcal{M}}_{g,n})$ the degree $2d$ component of the cohomology class:

$$\sum_{\Gamma \in G_{g,n}} \sum_{w \in W_{\Gamma, r}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \frac{1}{r^{h^1(\Gamma)}} \xi_{\Gamma *}$$

$$\left[\prod_{i=1}^n \exp(a_i^2 \psi_{l_i}) \prod_{e = \{h, h'\} \in E(\Gamma)} \frac{1 - \exp\left(-w(h) w(h') (\psi_h + \psi_{h'}) \right)}{\psi_h + \psi_{h'}} \right] \tag{3.1}$$

in $H^*(\overline{\mathcal{M}}_{g,n})$. Note that the factor $\frac{1 - \exp\left(-w(h) w(h') (\psi_h + \psi_{h'}) \right)}{\psi_h + \psi_{h'}}$ is well defined since the denominator formally divides the numerator. In [16], the authors proved that for fixed

g, A and d the class $P_g^{d,r}$ is polynomial in r for all sufficiently large r. Denote by $P_g^d(A)$ the constant term of the associated polynomial in r.

The restriction of the class $P_g^j(A)$ to $\mathcal{M}_{g,n}^{\mathrm{ct}}$ is given by:

$$P_g^j(A)\big|_{\mathcal{M}_{g,n}^{\mathrm{ct}}} = \frac{2^j}{j!}\theta_g(A)^j\big|_{\mathcal{M}_{g,n}^{\mathrm{ct}}}.$$

In [16], the authors proved that:

$$\mathrm{DR}_g(A) = 2^{-g}P_g^g(A).$$

In [8], the authors proved that the class $P_g^d(A)$ vanishes for $d > g$. In [16, page 10], the authors remark, *'For $d < g$, the classes $P_g^d(A)$ do not yet have a geometric interpretation'*. Our next conjecture shows that the intersection numbers of these classes with monomials in psi classes have an elegant structure from the point of view of integrable systems.

Let us introduce the following generating series:

$$\mathcal{F}^{\mathrm{P}}(t_*^*, \varepsilon, \mu) := \sum_{\substack{g,n \geq 0 \\ 2g-2+n > 0}} \sum_{j=0}^{g} \frac{\varepsilon^{2g}\mu^{2j}}{n!} \sum_{\substack{A=(a_1,\ldots,a_n) \in \mathbb{Z}^n \\ \sum a_i = 0}}$$

$$\sum_{d_1,\ldots,d_n \geq 0} \left(\int_{\overline{\mathcal{M}}_{g,n}} 2^{-j} P_g^j(A) \prod_{i=1}^{n} \psi_i^{d_i} \right) \prod_{i=1}^{n} t_{d_i}^{a_i},$$

and let:

$$(w^{\mathrm{P}})^a := \frac{\partial^2 \mathcal{F}^{\mathrm{P}}}{\partial t_0^0 \partial t_0^{-a}}, \quad a \in \mathbb{Z},$$

$$w^{\mathrm{P}} := \sum_{a \in \mathbb{Z}} (w^{\mathrm{P}})^a e^{iay},$$

$$u^{\mathrm{P}} := \frac{S(\varepsilon\mu\partial_x)}{S(i\varepsilon\mu\partial_x\partial_y)} w^{\mathrm{P}}.$$

Conjecture 2. *The function u^{P} satisfies the full noncommutative KdV hierarchy (2.5), where we recall that we identify $t_d^0 = t_d$ and $t_0^0 = x$.*

Note that since:

$$\left(\mathcal{F}^{\mathrm{P}}\big|_{\substack{\varepsilon \mapsto \varepsilon\tau \\ \mu \mapsto \tau^{-1}}} \right)\bigg|_{\tau=0} = \mathcal{F}^{\mathrm{DR}}.$$

Conjecture 1 immediately follows from Conjecture 2.

Note also that since $P_g^0(A) = 1$, we have:

$$\mathcal{F}^{\mathrm{P}}\big|_{\mu = t_*^{\neq 0} = 0} = \mathcal{F}^{\mathrm{W}},$$

where $\mathcal{F}^{\mathrm{W}}$ is the classical generating series of intersection numbers on $\overline{\mathcal{M}}_{g,n}$ considered by Witten in [22]:

$$\mathcal{F}^{\mathrm{W}}(t_0, t_1, \dots, \varepsilon) := \sum_{\substack{g,n \geq 0 \\ 2g-2+n > 0}} \frac{\varepsilon^{2g}}{n!} \sum_{d_1, \dots, d_n \geq 0} \left(\int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i} \right) \prod_{i=1}^{n} t_{d_i}.$$

Clearly, we have:

$$(w^{\mathrm{P}})^a \big|_{\mu = t_*^{\neq 0} = 0} = (u^{\mathrm{P}})^a \big|_{\mu = t_*^{\neq 0} = 0} = \begin{cases} \frac{\partial^2 \mathcal{F}^{\mathrm{W}}}{(\partial t_0)^2}, & \text{if } a = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, after the specialisation $\mu = t_*^{\neq 0} = 0$, Conjecture 2 says that the function $\frac{\partial^2 \mathcal{F}^{\mathrm{W}}}{\partial t_0^2}$ is a solution of the classical KdV hierarchy, which is the celebrated conjecture of Witten [22], first proved by Kontsevich [17].

4. A relation with the DR/DZ equivalence conjecture

The goal of this section is to show that Conjecture 2 follows from the so-called DR/DZ equivalence conjecture proposed in [2] and a result of [6], where the authors proved that the DR hierarchy corresponding to the partial cohomological field theory formed by the classes $\exp(\mu^2 \theta_g(A))$ coincides with the noncommutative KdV hierarchy. In particular, since the DR/DZ equivalence conjecture is proved at the approximation up to genus 1 [2, 4], this proves Conjecture 2 at the approximation up to genus 1.

4.1. Partial cohomological field theories

Recall the following definition, which is a generalisation first considered in [19] of the notion of a cohomological field theory from [18].

Definition 4.1. A *partial cohomological field theory (CohFT)* is a system of linear maps:

$$c_{g,n} \colon V^{\otimes n} \to H^{\mathrm{even}}(\overline{\mathcal{M}}_{g,n}),$$

for (g,n) in the stable range, where V is an arbitrary finite dimensional $\mathbb{C}$-vector space called the *phase space*, together with a special element $e \in V$ called the *unit* and a symmetric nondegenerate bilinear form $\eta \in (V^*)^{\otimes 2}$ called the *metric*, such that, fixing a basis $e_1, \dots, e_{\dim V}$ in V, the following axioms are satisfied:

(i) The maps $c_{g,n}$ are equivariant with respect to the S_n-action permuting the n copies of V in $V^{\otimes n}$ and the n marked points in $\overline{\mathcal{M}}_{g,n}$, respectively.

(ii) Let $\pi \colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the map that forgets the last marked point. Then

$$\pi^* c_{g,n}(\otimes_{i=1}^{n} e_{\alpha_i}) = c_{g,n+1}(\otimes_{i=1}^{n} e_{\alpha_i} \otimes e), \quad 1 \leq \alpha_1, \dots, \alpha_n \leq \dim V.$$

Moreover, $c_{0,3}(e_\alpha \otimes e_\beta \otimes e) = \eta(e_\alpha \otimes e_\beta) =: \eta_{\alpha\beta}$ for $1 \leq \alpha, \beta \leq \dim V$.

(iii) For decompositions $I \sqcup J = [n]$, $|I| = n_1$, $|J| = n_2$ and $g_1 + g_2 = g$ with $2g_1 - 1 + n_1 > 0$, $2g_2 - 1 + n_2 > 0$, let:

$$\text{gl}: \overline{\mathcal{M}}_{g_1, n_1+1} \times \overline{\mathcal{M}}_{g_2, n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2, n_1+n_2} \tag{4.1}$$

be the corresponding gluing map. Then:

$$\text{gl}^* c_{g,n}(\otimes_{i=1}^n e_{\alpha_i}) = c_{g_1, n_1+1}(\otimes_{i \in I} e_{\alpha_i} \otimes e_\mu) \eta^{\mu\nu} c_{g_2, n_2+1}(\otimes_{j \in J} e_{\alpha_j} \otimes e_\nu),$$
$$1 \le \alpha_1, \ldots, \alpha_n \le \dim V, \tag{4.2}$$

where $\eta^{\alpha\beta}$ are the entries of the matrix $(\eta_{\alpha\beta})^{-1}$.

Definition 4.2. A *CohFT* is a partial CohFT $c_{g,n}: V^{\otimes n} \to H^{\text{even}}(\overline{\mathcal{M}}_{g,n})$, such that the following extra axiom is satisfied:

(iv) $\text{gl}^* c_{g+1, n}(\otimes_{i=1}^n e_{\alpha_i}) = c_{g, n+2}(\otimes_{i=1}^n e_{\alpha_i} \otimes e_\mu \otimes e_\nu)\eta^{\mu\nu}$ for $1 \le \alpha_1, \ldots, \alpha_n \le \dim V$, where $\text{gl}: \overline{\mathcal{M}}_{g, n+2} \to \overline{\mathcal{M}}_{g+1, n}$ is the gluing map that increases the genus by identifying the last two marked points.

Note that a notion of infinite rank partial CohFT (i.e. a partial CohFT with an infinite dimensional phase space V) requires some care. One needs to clarify what is meant by the matrix $(\eta^{\alpha\beta})$ and to make sense of the, a priori infinite, sum over μ and ν, both appearing in Axiom (iii). One possibility is demanding that the image of the linear map $V^{\otimes(n-1)} \to H^*(\overline{\mathcal{M}}_{g,n}) \otimes V^*$ induced by $c_{g,n}: V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n})$ is contained in $H^*(\overline{\mathcal{M}}_{g,n}) \otimes \eta^\sharp(V)$, where $\eta^\sharp: V \to V^*$ is the injective map induced by the bilinear form η. Then in Axiom (iii), instead of using an undefined bilinear form $(\eta^{\alpha\beta})$ on V^*, one can use the bilinear form on $\eta^\sharp(V)$ induced by η. This solves the problem with convergence.

A useful special case is the following. Consider a vector space V with a countable basis $\{e_\alpha\}_{\alpha \in \mathbb{Z}}$, and suppose that for any (g, n) in the stable range and each $e_{\alpha_1}, \ldots, e_{\alpha_{n-1}} \in V$, the set $\{\beta \in \mathbb{Z} \,|\, c_{g,n}(\otimes_{i=1}^{n-1} e_{\alpha_i} \otimes e_\beta) \ne 0\}$ is finite. In particular, this implies that the matrix $\eta_{\alpha\beta}$ is row- and column-finite (each row and each column have a finite number of nonzero entries), which is equivalent to $\eta^\sharp(V) \subseteq \text{span}(\{e^\alpha\}_{\alpha \in \mathbb{Z}})$, where $\{e^\alpha\}_{\alpha \in \mathbb{Z}}$ is the dual 'basis'. Let us further demand that the injective map $\eta^\sharp: V \to \text{span}(\{e^\alpha\}_{\alpha \in \mathbb{Z}})$ is surjective too (i.e. that a unique two-sided row- and column-finite matrix $(\eta^{\alpha\beta})$, inverse to $(\eta_{\alpha\beta})$, exists; it represents the inverse map $(\eta^\sharp)^{-1}: \text{span}(\{e^\alpha\}_{\alpha \in \mathbb{Z}}) \to V$). Then the equation appearing in Axiom (iii) is well defined with the double sum only having a finite number of nonzero terms. Such a partial CohFT will be called a *tame partial CohFT of infinite rank*.

4.2. The DR/DZ equivalence conjecture

Let us fix a positive integer N.

4.2.1. Differential polynomials. Let us introduce formal variables u_i^α, $\alpha = 1, \ldots, N$, $i = 0, 1, \ldots$. Following [11] (see also [21]), we define the ring of *differential polynomials* $\mathcal{A}_N$ in the variables $u^1, \ldots, u^N$ as the ring of polynomials $f(u^*, u_1^*, u_2^*, \ldots)$ in the variables u_i^α, $i > 0$, with coefficients in the ring of formal power series in the variables $u^\alpha = u_0^\alpha$:

$$\mathcal{A}_N := \mathbb{C}[[u^*]][u_{\ge 1}^*].$$

Remark 4.3. This way, we define a model of the loop space of a vector space V of dimension N by describing its ring of functions. In particular, it is useful to think of the variables $u^\alpha := u_0^\alpha$ as the components $u^\alpha(x)$ of a formal loop $u\colon S^1 \to V$ in a fixed basis $e_1, \ldots, e_N$ of V. Then the variables $u_1^\alpha := u_x^\alpha, u_2^\alpha := u_{xx}^\alpha, \ldots$ are the components of the iterated x-derivatives of a formal loop.

A gradation on $\mathcal{A}_N$, which we denote by deg, is introduced by $\deg u_i^\alpha := i$. The homogeneous component of $\mathcal{A}_N$ of degree d is denoted by $\mathcal{A}_N^{[d]}$. The operator:

$$\partial_x := \sum_{i \geq 0} u_{i+1}^\alpha \frac{\partial}{\partial u_i^\alpha}$$

increases the degree by 1.

Differential polynomials can also be described using another set of formal variables, corresponding heuristically to the Fourier components p_k^α, $k \in \mathbb{Z}$, of the functions $u^\alpha = u^\alpha(x)$. We define a change of variables:

$$u_j^\alpha = \sum_{k \in \mathbb{Z}} (ik)^j p_k^\alpha e^{ikx}, \tag{4.3}$$

which allows us to express a differential polynomial $f(u, u_x, u_{xx}, \ldots) \in \mathcal{A}_N$ as a formal Fourier series in x. In the latter expression, the coefficient of e^{ikx} is a power series in the variables p_j^α with the sum of the subscripts in each monomial in p_j^α equal to k.

Consider the extension $\widehat{\mathcal{A}}_N := \mathcal{A}_N[[\varepsilon]]$ of the space $\mathcal{A}_N$ with a new variable ε of degree $\deg \varepsilon := -1$. Abusing the terminology, we still call its elements *differential polynomials*. Let $\widehat{\mathcal{A}}_N^{[k]} \subset \widehat{\mathcal{A}}_N$ denote the subspace of differential polynomials of degree k.

4.2.2. The DR hierarchy of a partial CohFT. Consider an arbitrary partial CohFT:

$$c_{g,n}\colon V^{\otimes n} \to H^{\mathrm{even}}(\overline{\mathcal{M}}_{g,n}).$$

Following [1, 2], we will present the construction of the DR hierarchy and the DR/DZ equivalence conjecture. Formally, the results presented here were obtained in [1, 2] for a CohFT, but, as it was already remarked in [2, Section 9.1], the construction of the DR hierarchy works without any change for an arbitrary partial CohFT, and all the results that we discuss here are true for an arbitrary partial CohFT with the same proofs.

Let $N := \dim V$, and let us fix a basis $e_1, \ldots, e_N \in V$. Introduce the following generating series:

$$P_{\beta,d}^\alpha := \sum_{\substack{g \geq 0 \\ n \geq 1}} \frac{(-\varepsilon^2)^g}{n!} \sum_{a_1, \ldots, a_n \in \mathbb{Z}} \left(\int_{\mathrm{DR}_g(-\sum_{i=1}^n a_i, 0, a_1, \ldots, a_n)} \lambda_g \psi_2^d \eta^{\alpha\gamma} c_{g,n+2}(e_\gamma \otimes e_\beta \otimes_{i=1}^n e_{\alpha_i}) \right)$$

$$\prod_{i=1}^n p_{a_i}^{\alpha_i} e^{i(\sum_{j=1}^n a_j)x}, \tag{4.4}$$

for $\alpha, \beta = 1, \ldots, N$ and $d = 0, 1, 2, \ldots$. The expression on the right-hand side of (4.4) can be uniquely written as a differential polynomial from $\widehat{\mathcal{A}}_N^{[0]}$ using the change of variables (4.3). Concretely, it can be done in the following way. From Hain's formula (2.1), it follows that the restriction $\mathrm{DR}_g\left(-\sum_{i=1}^n a_i, a_1, \ldots, a_n\right)\big|_{\mathcal{M}_{g,n+1}^{\mathrm{ct}}}$ is a homogeneous polynomial in $a_1, \ldots, a_n$ of degree $2g$ with the coefficients in $H^{2g}(\mathcal{M}_{g,n+1}^{\mathrm{ct}})$. This property, together with the fact that λ_g vanishes on $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}^{\mathrm{ct}}$ (see, e.g. [12, Section 0.4]), implies that the integral:

$$\int_{\mathrm{DR}_g\left(-\sum_{i=1}^n a_i, 0, a_1, \ldots, a_n\right)} \lambda_g \psi_2^d \eta^{\alpha\gamma} c_{g,n+2}(e_\gamma \otimes e_\beta \otimes_{i=1}^n e_{\alpha_i}) \tag{4.5}$$

is a homogeneous polynomial in $a_1, \ldots, a_n$ of degree $2g$, which we denote by:

$$Q_{\beta,d,g;\alpha_1,\ldots,\alpha_n}^{\alpha}(a_1, \ldots, a_n) = \sum_{\substack{b_1,\ldots,b_n \geq 0 \\ b_1+\ldots+b_n = 2g}} Q_{\beta,d,g;\alpha_1,\ldots,\alpha_n}^{\alpha;b_1,\ldots,b_n} a_1^{b_1} \ldots a_n^{b_n}.$$

Then we have:

$$P_{\beta,d}^{\alpha} = \sum_{\substack{g \geq 0 \\ n \geq 1}} \frac{\varepsilon^{2g}}{n!} \sum_{\substack{b_1,\ldots,b_n \geq 0 \\ b_1+\ldots+b_n = 2g}} Q_{\beta,d,g;\alpha_1,\ldots,\alpha_n}^{\alpha;b_1,\ldots,b_n} u_{b_1}^{\alpha_1} \ldots u_{b_n}^{\alpha_n}.$$

The system of PDEs:

$$\frac{\partial u^\alpha}{\partial t_d^\beta} = \partial_x P_{\beta,d}^{\alpha}, \quad 1 \leq \alpha, \beta \leq N, \quad d \geq 0 \tag{4.6}$$

is called the *DR hierarchy*. The flows of the hierarchy pairwise commute. Let $A^\alpha e_\alpha := e \in V$. The flow $\frac{\partial}{\partial t^{\mathrm{II}}} := A^\alpha \frac{\partial}{\partial t_0^\alpha}$ is given by:

$$\frac{\partial u^\alpha}{\partial t_0^{\mathrm{II}}} = u_x^\alpha. \tag{4.7}$$

Remark 4.4. The DR hierarchy is actually Hamiltonian, and in [1, 2], it is introduced via a sequence of local functionals. However, since we don't need the Hamiltonian structure in this paper, we introduce directly the equations of the DR hierarchy.

Because of (4.7), as a solution of the DR hierarchy, we can consider an N-tuple of formal power series $u^\alpha(t_*^*, \varepsilon) \in \mathbb{C}[[t_*^*, \varepsilon]]$, $1 \leq \alpha \leq N$, satisfying the system (4.6) after the identification of the flows ∂_x and $\frac{\partial}{\partial t_0^{\mathrm{II}}}$. The *string solution* $(u^{\mathrm{str}})^\alpha(t_*^*, \varepsilon) \in \mathbb{C}[[t_*^*, \varepsilon]]$ of the DR heirarchy is defined as the unique solution satisfying the initial condition:

$$(u^{\mathrm{str}})^\alpha\big|_{t_n^\gamma = \delta_{n,0} A^\gamma x} = A^\alpha x.$$

The *potential* of our partial CohFT is defined by:

$$\mathcal{F}(t_*^*, \varepsilon) := \sum_{\substack{g,n \geq 0 \\ 2g-2+n > 0}} \frac{\varepsilon^{2g}}{n!} \sum_{\substack{1 \leq \alpha_1,\ldots,\alpha_n \leq N \\ d_1,\ldots,d_n \geq 0}} \left(\int_{\overline{\mathcal{M}}_{g,n}} c_{g,n}(\otimes_{i=1}^n e_{\alpha_i}) \prod_{i=1}^n \psi_i^{d_i} \right) \prod_{i=1}^n t_{d_i}^{a_i} \in \mathbb{C}[[t_*^*, \varepsilon]].$$

The exponent $\exp\left(\varepsilon^{-2}\mathcal{F}\right)$ is traditionally called the *partition function*.

Define:

$$(w^{\text{top}})_n^\alpha := \eta^{\alpha\mu} \frac{\partial^n}{(\partial t_0^{\mathbb{1}})^n} \frac{\partial^2 \mathcal{F}}{\partial t_0^\mu \partial t_0^{\mathbb{1}}}, \quad 1 \leq \alpha \leq N.$$

In [2, Proposition 7.2], the authors proved that there exists a unique differential polynomial $\mathcal{P} \in \widehat{\mathcal{A}}_N^{[-2]}$, such that the power series $\mathcal{F}^{\text{red}} \in \mathbb{C}[[t_*^*, \varepsilon]]$ defined by:

$$\mathcal{F}^{\text{red}} := \mathcal{F} + \mathcal{P}|_{u_n^\gamma = (w^{\text{top}})_n^\gamma},$$

satisfies the following vanishing property:

$$\text{Coef}_{\varepsilon^{2g}} \left. \frac{\partial^n \mathcal{F}^{\text{red}}}{\partial t_{d_1}^{\alpha_1} \dots \partial t_{d_n}^{\alpha_n}} \right|_{t_*^* = 0} = 0, \quad \text{if} \quad \sum_{i=1}^n d_i \leq 2g - 2. \tag{4.8}$$

The power series $\mathcal{F}^{\text{red}}$ is called the *reduced potential* of our partial CohFT.

The differential polynomials $\widetilde{u}^\alpha \in \widehat{\mathcal{A}}_N^{[0]}$ defined by:

$$\widetilde{u}_\alpha := \eta_{\mathbb{1}\mu} P_{\alpha,0}^\mu$$

are called the *normal coordinates* of the DR hierarchy. The differential polynomials $\widetilde{u}^\alpha := \eta^{\alpha\nu} \widetilde{u}_\nu$ are also called the normal coordinates.

The following conjecture was presented in [2, Conjecture 7.5].

Conjecture 3. *We have:*

$$\frac{\partial^2 \mathcal{F}^{\text{red}}}{\partial t_0^{\mathbb{1}} \partial t_0^\alpha} = \widetilde{u}_\alpha|_{u_n^\gamma = (u^{\text{str}})_n^\gamma}, \quad 1 \leq \alpha \leq N. \tag{4.9}$$

Remark 4.5. To be precise, Conjecture 7.5 from [2] claims that:

$$\mathcal{F}^{\text{DRH}} = \mathcal{F}^{\text{red}}, \tag{4.10}$$

where $\mathcal{F}^{\text{DRH}}$ is the potential of the DR hierarchy, see Section 4.2 in [2] for the construction. Let us explain why it is equivalent to Conjecture 3. In one direction, equation (4.9) immediately follows from (4.10) and the definition of $\mathcal{F}^{\text{DRH}}$. Conversely, equation (4.9) implies that $\frac{\partial^2 \mathcal{F}^{\text{red}}}{\partial(t_0^{\mathbb{1}})^2} = \frac{\partial^2 \mathcal{F}^{\text{DRH}}}{\partial(t_0^{\mathbb{1}})^2}$, which, using the string equations for $\mathcal{F}^{\text{red}}$ [2, Proposition 7.2] and $\mathcal{F}^{\text{DRH}}$ [2, Proposition 6.3], gives that $\mathcal{F}^{\text{red}} = \mathcal{F}^{\text{DRH}}$ (see [14, Lemma 3.1]).

4.3. The Pixton class as a partial cohomological field theory

Proposition 4.6. *The classes:*

$$c_{g,n}(\otimes_{i=1}^n e_{a_i}) := \sum_{d=0}^g 2^{-d} \mu^{2d} P_g^d(a_1, \dots, a_n) \tag{4.11}$$

form an infinite rank tame partial cohomological field theory with the phase space $V = \text{span}(\{e_a\}_{a \in \mathbb{Z}})$, *the unit* e_0 *and the metric given in the basis* $\{e_a\}_{a \in \mathbb{Z}}$ *by* $\eta_{ab} = \delta_{a+b,0}$.

Proof. Since $c_{g,n}(\otimes_{i=1}^n e_{a_i}) = 0$ unless $\sum_{i=1}^n a_i = 0$, the tameness property is clear.

To prove the axioms from Definition 4.1, the crucial observation is that formula (3.1) is very close to the formula for the action of a Givental R-matrix on a topological

field theory (see, e.g. [20, Section 2] for an introduction to these techniques). Let $V_r := \operatorname{span}(\{e_0, \ldots, e_{r-1}\})$ and fix a bilinear form $\eta_r(e_a, e_b) := \frac{1}{r}\delta_{a+b=0 \bmod r}$ on V_r. Starting with the topological field theory $\omega_{g,n} \colon V_r^{\otimes n} \to H^0(\overline{\mathcal{M}}_{g,n})$, where:

$$\omega_{g,n}(e_{a_1} \otimes \ldots \otimes e_{a_n}) := r^{2g-1}\delta_{a_1 + \ldots + a_n = 0 \bmod r},$$

and applying the Givental R-matrix:

$$R(z) := \exp\left(-\operatorname{diag}_{a=0}^{r-1}(a^2)2^{-1}\mu^2 z\right),$$

we obtain the CohFT:

$$\Omega_{g,n}^r(e_{a_1} \otimes \ldots \otimes e_{a_n}) = \sum_{\Gamma \in G_{g,n}} \sum_{w \in W_{\Gamma,r}} \frac{r^{2g-1-h^1(\Gamma)}}{|\operatorname{Aut}(\Gamma)|} \xi_{\Gamma*}\left[\prod_{i=1}^{n}\exp(a_i^2 2^{-1}\mu^2\psi_{l_i})\right.$$

$$\left. \times \prod_{e=\{h,h'\}\in E(\Gamma)} 2^{-1}\mu^2 \frac{1 - \exp\left(2^{-1}\mu^2\left(w(h)^2\psi_h + w(h')^2\psi_{h'}\right)\right)}{2^{-1}\mu^2(\psi_h + \psi_{h'})}\right],$$

whose unit is e_0 and where $W_{\Gamma,r}$ are the same weightings appearing in formula (3.1). In particular, the factor $r^{2g-1-h^1(\Gamma)}$ comes from the product of the factors $r^{2g(v)-1}$ appended to each vertex $v \in V(\Gamma)$ times the factors r appended to each edge (from the η_r^{-1} in the edge contributions), since:

$$\sum_{v \in V(\Gamma)}(2g(v)-1) + |E| = |E| - |V| + 2$$

$$\sum_{v \in V(\Gamma)}g(v) = (h^1(\Gamma)-1) + 2(g - h^1(\Gamma)) = 2g - 1 - h^1(\Gamma).$$

Dividing the classes $\Omega_{g,n}^r(\otimes_{i=1}^{n}e_{a_i})$ by r^{2g-1} preserves the property of being a partial CohFT. Therefore, the classes:

$$\widetilde{\Omega}_{g,n}^r(e_{a_1} \otimes \ldots \otimes e_{a_n}) := \sum_{\Gamma \in G_{g,n}} \sum_{w \in W_{\Gamma,r}} \frac{1}{|\operatorname{Aut}(\Gamma)|}\frac{1}{r^{h^1(\Gamma)}} \xi_{\Gamma*}\left[\prod_{i=1}^{n}\exp(a_i^2 2^{-1}\mu^2\psi_{l_i})\right.$$

$$\left. \times \prod_{e=\{h,h'\}\in E(\Gamma)} 2^{-1}\mu^2 \frac{1 - \exp\left(2^{-1}\mu^2\left(w(h)^2\psi_h + w(h')^2\psi_{h'}\right)\right)}{2^{-1}\mu^2(\psi_h + \psi_{h'})}\right] \tag{4.12}$$

form a partial CohFT with the same phase space V_r, the metric $\widetilde{\eta}_r(e_a, e_b) = \delta_{a+b=0 \bmod r}$ and the unit e_0. Note that in this formula we have:

$$w(h)^2 = -w(h)w(h') + rw(h), \qquad w(h')^2 = -w(h)w(h') + rw(h').$$

Note also that the class $\widetilde{\Omega}_{g,n}^r(e_{a_1} \otimes \ldots \otimes e_{a_n})$ is zero unless $a_1 + \ldots + a_n = 0 \bmod r$.

For an integer a, let us denote by $\widetilde{a} \in \{0, \ldots, r-1\}$ the unique number, such that $a = \widetilde{a}$ mod r. If $r > |a|$, then, clearly,

$$\widetilde{a} = \begin{cases} a, & \text{if } a \geq 0, \\ r + a, & \text{if } a < 0. \end{cases}$$

Consider an n-tuple $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ satisfying $a_1 + \ldots + a_n = 0$, and let $\widetilde{A} := (\widetilde{a}_1, \ldots, \widetilde{a}_n)$. Comparing formulas (3.1) and (4.12), and using Proposition 3" in [16], we conclude that both classes $\widetilde{\Omega}^r_{g,n}(\otimes^n_{i=1} e_{\widetilde{a}_i})$ and $\sum_{d \geq 0} 2^{-d} \mu^{2d} P^{d,r}_g(A)$ are polynomials in r (for r sufficiently large) having the same constant term, which is equal to the class $c_{g,n}(\otimes^n_{i=1} e_{a_i})$ (one should notice that the factors $2^{-1} \mu^2$ appended to each psi class and each edge of a stable graph in (4.12) globally produce a factor $2^{-d} \mu^{2d}$). The proposition can now be easily derived from that.

To prove Axiom (iii) from Definition 4.1 for the classes $c_{g,n}(\otimes^n_{i=1} e_{a_i})$, consider the gluing map (4.1) with respect to the bilinear form $\widetilde{\eta}_r$. We have:

$$\mathrm{gl}^* \widetilde{\Omega}^r_{g,n}(\otimes^n_{i=1} e_{\widetilde{a}_i}) = \widetilde{\Omega}^r_{g_1, n_1+1}(\otimes_{i \in I} e_{\widetilde{a}_i} \otimes e_{\widetilde{-a_I}}) \widetilde{\Omega}^r_{g_2, n_2+1}(\otimes_{j \in J} e_{\widetilde{a}_j} \otimes e_{\widetilde{-a_J}}).$$

Considering both sides as polynomials in r (for r sufficiently large) and taking the constant terms, we obtain:

$$\mathrm{gl}^* c_{g,n}(\otimes^n_{i=1} e_{a_i}) = c_{g_1, n_1+1}(\otimes_{i \in I} e_{a_i} \otimes e_{-a_I}) c_{g_2, n_2+1}(\otimes_{j \in J} e_{a_j} \otimes e_{-a_J}),$$

as required. Proofs of Axioms (i) and (ii) are the same, and we omit them. $\qquad\square$

4.4. The DR/DZ equivalence conjecture implies Conjectures 1 and 2

Consider the partial CohFT given by the Pixton class,

$$c_{g,n}(\otimes^n_{i=1} e_{a_i}) := \sum_{d=0}^g 2^{-d} \mu^{2d} P^d_g(a_1, \ldots, a_n)$$

and the corresponding DR hierarchy.

Remark 4.7. Strictly speaking, we discussed the construction of the DR hierarchy only for partial CohFTs with a finite dimensional phase space. However, it is not hard to understand that, for a partial CohFT of infinite rank, tameness is a sufficient condition for all the constructions and results to remain true. More precisely, while the definition of the Hamiltonians of the DR hierarchy works even without the tameness hypothesis for any infinite rank CohFT (at the cost of replacing the spaces of differential polynomials and local functionals with a space of formal power series in all formal variables u^*_* and ε), the construction of the equations of the DR hierarchy (4.6) already requires dealing with the existence of the matrix $(\eta^{\alpha\beta})$ and the convergence of the infinite sum appearing in formula (4.4). From there on, through the proof of compatibility of the equations of the DR hierarchy (commutativity of Hamiltonians) to the existence of the potential of the DR hierarchy $\mathcal{F}^{\mathrm{DRH}}$ featured in the DR/DZ equivalence conjecture, several constructions and results present the very same problem. It is immediate to see that the tameness hypothesis

is always sufficient to ensure that $(\eta^{\alpha\beta})$ exists and that all infinite sums always have only a finite number of nonzero terms.

Proposition 4.8. *The normal coordinates of the DR hierarchy are given by:*

$$\widetilde{u}^\alpha = u^\alpha + \sum_{g \geq 1} \varepsilon^{2g} \frac{(\mu\alpha)^{2g}}{2^{2g}(2g+1)!} u^\alpha_{2g}, \quad \alpha \in \mathbb{Z}. \tag{4.13}$$

Proof. To compute the normal coordinates $\widetilde{u}^\alpha$, one has to compute the integrals:

$$\int_{\overline{\mathcal{M}}_{g,n+2}} \mathrm{DR}_g\left(-\sum_{i=1}^n a_i, 0, a_1, \ldots, a_n\right) \lambda_g 2^{-d} P_g^d(0, -\alpha, \alpha_1, \ldots, \alpha_n),$$

where $n \geq 1$, $a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_n \in \mathbb{Z}$, $0 \leq d \leq g$, which by degree reasons can be nonzero only if $g - 1 + n = d$. Therefore, only the integrals with $n = 1$ and $d = g$ give a nontrivial contribution, that is, the integrals:

$$\int_{\overline{\mathcal{M}}_{g,3}} \mathrm{DR}_g(-a, 0, a)\, \lambda_g \mathrm{DR}_g(0, -\alpha, \alpha) \overset{\text{[6, Theorem 2.1]}}{=\!=\!=\!=\!=\!=\!=} \frac{(\alpha a)^{2g}}{2^{2g}(2g+1)!},$$

which gives formula (4.13). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 4.9. *The reduced potential $\mathcal{F}^{\mathrm{P,red}}$ of our partial CohFT is equal to:*

$$\mathcal{F}^{\mathrm{P,red}} = \mathcal{F}^{\mathrm{P}} + \sum_{g \geq 1} \frac{(\varepsilon\mu)^{2g}}{2^{2g}(2g+1)!}(w^{\mathrm{top}})^0_{2g-2}.$$

Proof. Equivalently, we have to check that:

$$\mathcal{F}^{\mathrm{P,red}} = S(\varepsilon\mu\partial_x)\mathcal{F}^{\mathrm{P}},$$

where we identify $x = t_0^0$.

For $d \in \mathbb{Z}$, denote by $S_d \subset \mathbb{C}[[t_*^*, \varepsilon]]$ the space of formal power series F satisfying the condition:

$$\mathrm{Coef}_{\varepsilon^{2g}} \left.\frac{\partial^n F}{\partial t_{d_1}^{\alpha_1} \ldots \partial t_{d_n}^{\alpha_n}}\right|_{t_*^* = 0} = 0, \quad \text{if} \quad \sum_{i=1}^n d_i \leq 2g + d.$$

By degree reasons, we have:

$$\mathcal{F}^{\mathrm{P}} - \sum_{g \geq 1}(\varepsilon\mu)^{2g}\underbrace{\left((-1)^g \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \psi_1^{2g-2}\right)}_{=: b_g} t_{2g-2}^0 \in S_{-2}.$$

Let $b_0 := 1$. Using the string equation:

$$\frac{\partial \mathcal{F}^{\mathrm{P}}}{\partial t_0^0} = \sum_{a \geq 0} t_{a+1}^\alpha \frac{\partial \mathcal{F}^{\mathrm{P}}}{\partial t_a^\alpha} + \frac{1}{2}\sum_{\alpha \in \mathbb{Z}} t_0^\alpha t_0^{-\alpha},$$

we obtain:

$$\partial_x^{2h}\mathcal{F}^{\mathrm{P}} - \sum_{g \geq 0}(\varepsilon\mu)^{2g}(-1)^g b_g t_{2g+2h-2}^0 \in S_{2h-2} \quad \text{for } h \geq 1$$

$$\Rightarrow \quad S(\varepsilon\mu\partial_x)\mathcal{F}^{\mathrm{P}} - \sum_{g \geq 1}(\varepsilon\mu)^{2g} t_{2g-2}^0 \left(\sum_{g_1+g_2=g} \frac{(-1)^{g_1}b_{g_1}}{2^{2g_2}(2g_2+1)!} \right) \in S_{-2}.$$

By [13, Theorem 2], the numbers b_g are given by:

$$1 + \sum_{g \geq 1}b_g z^{2g} = \frac{iz}{e^{iz/2} - e^{-iz/2}} = \frac{1}{S(iz)},$$

which implies that $S(\varepsilon\mu\partial_x)\widehat{\mathcal{F}}^{\mathrm{DR}} \in S_{-2}$, as required. $\qquad\square$

Theorem 4.10. *Suppose that Conjecture 3 is true for the partial CohFT given by the Pixton class. Then Conjecture 2 is true.*

Proof. Conjecture 3, together with Proposition 4.8, implies that:

$$\frac{\partial^2 \mathcal{F}^{\mathrm{P,red}}}{\partial t_0^0 \partial t_0^{-\alpha}} = S(\mu\varepsilon\alpha\partial_x)(u^{\mathrm{str}})^{\alpha}.$$

On the other hand, from Proposition 4.9, it follows that:

$$\frac{\partial^2 \mathcal{F}^{\mathrm{P,red}}}{\partial t_0^0 \partial t_0^{-\alpha}} = S(\varepsilon\mu\partial_x)(w^{\mathrm{P}})^{\alpha}.$$

Therefore,

$$(u^{\mathrm{str}})^{\alpha} = \frac{S(\varepsilon\mu\partial_x)}{S(i\varepsilon\mu\alpha\partial_x)}(w^{\mathrm{P}})^{\alpha} \quad \Rightarrow \quad u^{\mathrm{P}} = \sum_{\alpha \in \mathbb{Z}}(u^{\mathrm{str}})^{\alpha}e^{i\alpha y}.$$

In [6, Theorem 4.1], the author proved that $\sum_{\alpha \in \mathbb{Z}}(u^{\mathrm{str}})^{\alpha}e^{i\alpha y}$ satisfies the noncommutative KdV hierarchy. This implies that u^{P} satisfies the noncommutative KdV hierarchy, as required. $\qquad\square$

Corollary 4.11. *Conjectures 1 and 2 are true at the approximation up to ε^2.*

Proof. In [4], the authors proved that Conjecture 3 is true at the approximation up to ε^2. Together with Theorem 4.10, this gives the corollary. $\qquad\square$

5. A prediction for the integrals $\int_{\overline{\mathcal{M}}_{g,2}} 2^{-j}P_g^j(a,-a)\psi_1^{3g-1-j}$

In this section, using Conjecture 2, we will present an explicit formula for the generating series of integrals $\int_{\overline{\mathcal{M}}_{g,2}} 2^{-j}P_g^j(a,-a)\psi_1^{3g-1-j}$. We will then check several special cases of this formula.

Proposition 5.1. *Suppose that Conjecture 2 is true. Then we have:*

$$\sum_{1 \le j \le g} \left(\int_{\overline{\mathcal{M}}_{g,2}} 2^{-j} P_g^j(a, -a) \psi_1^{3g-1-j} \right) \mu^{2j} z^{3g-1-j} = \frac{1}{z} \left(\frac{S(a\mu z)}{S(\mu z)} e^{\frac{z^3}{24}} - 1 \right). \qquad (5.1)$$

Proof. By the string equation:

$$\frac{\partial \mathcal{F}^{\mathrm{P}}}{\partial t_0^0} = \sum_{a \ge 0} t_{a+1}^\alpha \frac{\partial \mathcal{F}^{\mathrm{P}}}{\partial t_a^\alpha} + \frac{1}{2} \sum_{\alpha \in \mathbb{Z}} t_0^\alpha t_0^{-\alpha}, \qquad (5.2)$$

equation (5.1) is equivalent to:

$$\sum_{0 \le j \le g} \left(\underbrace{\int_{\overline{\mathcal{M}}_{g,3}} 2^{-j} P_g^j(a, -a, 0) \psi_1^{3g-j}}_{=:T_g^j(a)} \right) \mu^{2j} z^{3g-j} = \frac{S(a\mu z)}{S(\mu z)} e^{\frac{z^3}{24}}$$

$$\Leftrightarrow \quad \sum_{0 \le j \le g} T_g^j(a) \mu^{2j} z^{g-j} = \frac{S(a\mu)}{S(\mu)} e^{\frac{z}{24}}. \qquad (5.3)$$

From (5.2), it also follows that:

$$\partial_x^n (w^{\mathrm{P}})^\alpha = \delta^{\alpha,0} \delta_{n,1} + \sum_{0 \le j \le g} \varepsilon^{2g} \mu^{2j} T_g^j(\alpha) t_{3g-j+n}^\alpha + O\left((t_*^*)^2 \right).$$

Therefore, $\partial_x^n (u^{\mathrm{P}})^\alpha$ has the form:

$$\partial_x^n (u^{\mathrm{P}})^\alpha = \delta^{\alpha,0} \delta_{n,1} + \sum_{0 \le j \le g} \varepsilon^{2g} \mu^{2j} R_g^j(\alpha) t_{3g-j+n}^\alpha + O\left((t_*^*)^2 \right), \qquad (5.4)$$

where $R_g^j(\alpha) = \sum_{h=0}^j T_{g-h}^{j-h}(\alpha) Q_h(\alpha)$ (recall that $Q_h(\alpha)$ was defined in (2.8)) or, equivalently,

$$\sum_{0 \le j \le g} R_g^j(\alpha) \mu^{2j} z^{g-j} = \left(\sum_{0 \le j \le g} T_g^j(\alpha) \mu^{2j} z^{g-j} \right) \left(\sum_{h \ge 0} Q_h(\alpha) \mu^{2h} \right).$$

Since $\sum_{h \ge 0} Q_h(\alpha) \mu^{2h} = \frac{S(\mu)}{S(\alpha\mu)}$, we see that (5.3) is equivalent to the equation:

$$R_g^j(\alpha) = \frac{\delta^{j,0}}{24^g g!}. \qquad (5.5)$$

Equation (5.5) is obvious for $g = 0$. The property (5.4) implies that:

$$\frac{1}{2} \partial_x (u^{\mathrm{P}} * u^{\mathrm{P}}) + \frac{\varepsilon^2}{12} u_{xxx}^{\mathrm{P}} = \sum_{\alpha \in \mathbb{Z}} \sum_{0 \le j \le g} \varepsilon^{2g} \mu^{2j} \left(R_g^j(\alpha) + \frac{1}{12} R_{g-1}^j(\alpha) \right) t_{3g-j}^\alpha e^{i\alpha y} + O\left((t_*^*)^2 \right),$$

where we adopt the convention $R_g^j(\alpha) := 0$ for $j > g$ or $g < 0$. On the other hand, the dilaton equation:

$$\frac{\partial \mathcal{F}^{\mathrm{P}}}{\partial t_1^0} = \sum_{a \geq 0} t_a^\alpha \frac{\partial \mathcal{F}^{\mathrm{P}}}{\partial t_a^\alpha} + \varepsilon \frac{\partial \mathcal{F}^{\mathrm{P}}}{\partial \varepsilon} - 2\mathcal{F}^{\mathrm{P}} + \frac{1}{24}$$

implies that:

$$\frac{\partial(u^{\mathrm{P}})}{\partial t_1^0} = \sum_{\alpha \in \mathbb{Z}} \sum_{0 \leq j \leq g} \varepsilon^{2g} \mu^{2j} (2g+1) R_g^j(\alpha) t_{3g-j}^\alpha e^{i\alpha y} + O\left((t_*^*)^2\right).$$

Using now the first equation of the noncommutative KdV hierarchy:

$$\frac{\partial u^{\mathrm{P}}}{\partial t_1^0} = \frac{1}{2}(u^{\mathrm{P}} * u^{\mathrm{P}}) + \frac{\varepsilon^2}{12} u_{xxx}^{\mathrm{P}},$$

we obtain:

$$R_g^j(\alpha) = \frac{1}{24g} R_{g-1}^j \quad \text{for } g \geq 1,$$

which gives (5.5) and proves the proposition. $\qquad \square$

Let us now check several special cases of formula (5.1). First of all, note that the following two specialisations of (5.1) appeared in the literature before:

(1) If we put $\mu = 0$ in (5.1), we obtain:

$$\sum_{g \geq 1} \left(\fint_{\overline{\mathcal{M}}_{g,2}} \psi_1^{3g-1} \right) z^{3g-1} = \frac{1}{z} \left(e^{\frac{z^3}{24}} - 1 \right)$$

$$\Leftrightarrow \quad \sum_{g \geq 1} \left(\fint_{\overline{\mathcal{M}}_{g,1}} \psi_1^{3g-2} \right) z^{3g-2} = \frac{1}{z^2} \left(e^{\frac{z^3}{24}} - 1 \right),$$

which is a classical formula (see, e.g. [12, equation (6)]).

(2) Multiplying both sides of (5.1) by z, substituting $\mu \mapsto \mu z^{-1}$ and putting $z = 0$, we obtain:

$$\sum_{1 \leq j \leq g} \left(\fint_{\overline{\mathcal{M}}_{g,2}} \mathrm{DR}_g(a, -a) \psi_1^{2g-1} \right) \mu^{2j} = \frac{S(a\mu)}{S(\mu)} - 1,$$

which was proved in [7, Theorem 1].

The two special cases discussed above involved the integrals with the classes $P_g^j(A)$, where $j = 0$ or $j = g$. Consider now an example with $0 < j < g$. For a fixed integer a and r big enough, we have:

$$P_2^{1,r}(a, -a) = a^2(\psi_1 + \psi_2) + |a|(r - |a|) \; 1 \!-\!\underbrace{1}\!-\!\underbrace{1}\!-\! 2 \; + \frac{r^2 - 1}{12} \; 1 \!-\!\overset{\frown}{\underbrace{1}}\!-\! 2 \; ,$$

which gives:

$$P_2^1(a, -a) = a^2 \left(\psi_1 + \psi_2 - \;{}_1 \!-\!\!\!\boxed{1}\!\!\!-\!\!\!\boxed{1}\!\!\!-\; {}_2 \; \right) - \frac{1}{12} \; {}_1 \!-\!\!\!\overset{\frown}{\underset{\smile}{\boxed{1}}}\!\!\!-\; {}_2 \, ,$$

where we refer a reader to [4, Section 2.1] for our pictorial notation for the cohomology classes on $\overline{\mathcal{M}}_{g,n}$. Therefore,

$$\int_{\overline{\mathcal{M}}_{2,2}} P_2^1(a, -a)\psi_1^4 = \frac{a^2 - 1}{288},$$

which agrees with formula (5.1).

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