POLYTOPES OVER GF(2) AND THEIR RELEVANCE FOR THE CUBIC SURFACE GROUP

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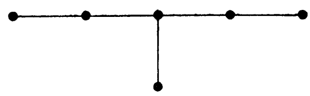
1. Introduction. In the preceding paper, Edge represented the celebrated "cubic surface group" of order 72.6! = 51840 as the group of automorphisms of a senary quadratic form over the field of residue-classes mod 2. The object of this sequel is to compare Edge's finite space with a real space, thus identifying his non-ruled quadric in PG(5, 2) with a modular counterpart of the semi-regular polytope 2_{21} which was discovered by Gosset in 1897.

2. The automorphisms of a real quadratic form. The positive definite senary quadratic form

$$E_6 = (x^1)^2 - x^1x^2 + (x^2)^2 - x^2x^3 + (x^3)^2 - x^3x^4 + (x^4)^2 - x^4x^5 + (x^5)^2 - x^3x^6 + (x^6)^2$$

is conveniently symbolized by the "graph"



in which the nodes represent the "square" terms while the branches represent the "product" terms (**4**, pp. 192, 297). The form evidently possesses six involutory automorphisms (or "automorphs") $R_1, \ldots, R_6$, defined as follows. Suppose the kth node of the graph is joined to the ith, jth, etc. Then R_k leaves all the x's unchanged except x^k, which it transforms into

$$-x^k + x^i + x^j + \ldots ;$$

for instance, R_3 is

$$(x^1, x^2, x^3, x^4, x^5, x^6) \to (x^1, x^2, x^2 - x^3 + x^4 + x^6, x^4, x^5, x^6).$$

The six R's generate a group of order 51840 which is the whole group of automorphisms (omitting the trivial automorphism that replaces every x^i by $-x^i$) (**5**, p. 422). To identify this with the group of the 27 lines on the general cubic surface, we observe that its generators satisfy the relations $R_k^2 = 1$ and either

$$R_iR_jR_i = R_jR_iR_j \quad \text{or} \quad R_iR_j = R_jR_i$$

according as the ith and jth nodes of the graph are, or are not, joined. These

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same relations are satisfied by certain permutations of the 27 lines on the cubic surface **(3**, p. 458**)**, namely:

$$R_1 = (1\ 2) = \begin{pmatrix} a_1 & b_1 & c_{23} & c_{24} & c_{24} & c_{26} \\ a_2 & b_2 & c_{13} & c_{14} & c_{15} & c_{16} \end{pmatrix},$$

$$R_2 = (2\ 3),\ R_3 = (3\ 4),\ R_4 = (4\ 5),\ R_5 = (5\ 6),$$

$$R_6 = \begin{pmatrix} c_{23} & c_{13} & c_{12} & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & c_{56} & c_{46} & c_{45} \end{pmatrix}.$$

The Diophantine equation

$$E_6 = 1$$

has 36 pairs of solutions, which are easily derived from any one of them, say (1, 0, 0, 0, 0, 0) or

$$\begin{matrix} 10000 \\ 0 \end{matrix},$$

by applying the antomorphisms; for example, by applying R_2, R_3, R_4, R_5, R_6 successively we obtain

$$\begin{matrix} 11000 \\ 0 \end{matrix} \quad \begin{matrix} 11100 \\ 0 \end{matrix} \quad \begin{matrix} 11110 \\ 0 \end{matrix} \quad \begin{matrix} 11111 \\ 0 \end{matrix} \quad \begin{matrix} 11111 \\ 1 \end{matrix}.$$

When regarded as points in an affine 6-space with a Euclidean metric given by the form E_6, these 72 solutions are the vertices of Elte's six-dimensional uniform polytope 1_{22} **(1**, p. 372; **2**, p. 472; **3**, p. 470; **4**, p. 210; **5**, p. 414**)**.

3. The transition from the real field to the field with two elements. Most of the above remarks remain valid if we regard the x's as marks of the field GF(2), that is, if we work in the finite affine 6-space EG(6, 2). Since $-1 = 1$, the minus signs in E_6 can now be replaced by plus signs, and the description of the generating automorphisms becomes still simpler: R_k "increases" the co-ordinate x^k by the sum of its (one, two, or three) neighbours. Another simplification is that the 36 pairs of opposite solutions of $E_6 = 1$ become 36 single solutions which we can identify with the 36 vertices of an "elliptic" 1_{22}, or with the 36 double-sixes on the cubic surface.

Since GF(2) has only the two elements 0, 1, every point of EG(6, 2) that does not satisfy $E_6 = 1$ must satisfy $E_6 = 0$. One such point is the origin. The remaining 27 are the vertices of Gosset's six-dimensional polytope 2_{21} **(3**, p. 466; **4**, p. 202**)** for which it is natural to use the same symbols that Schläfli invented for the 27 lines. In fact, the point

$$\begin{matrix} \mathbf{01000} \\ \mathbf{1} \end{matrix}$$

must be a_1, since it is invariant under all the generating automorphisms

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except R_1; and by applying these automorphisms in an appropriate order, we obtain the whole set of 27 as follows:

a_1	a_2	a_3	a_4	a_5
01000	11000	10000	10100	10110
1	1	1	1	1

a_6	c_{56}	c_{46}	c_{36}
10111	10100	10110	10010
1	0	0	0

c_{26}	c_{16}	c_{15}	c_{14}	c_{13}
11010	01010	01011	01001	01101
0	0	0	0	0

c_{12}	c_{23}	c_{24}	c_{25}
00101	11101	11001	11011
0	0	0	0

c_{35}	c_{45}	c_{34}	b_1	b_2
10011	10111	10001	11101	01101
0	0	0	1	1

b_3	b_4	b_5	b_6
00101	00001	00011	00010
1	1	1	1

By regarding the six x's as homogeneous co-ordinates, we thus obtain a partition of the $2^6 - 1 = 63$ points of the finite projective 5-space PG(5, 2) into 27 which lie on the quadric $E_6 = 0$ and 36 which do not. In other words:

In PG(5, 2), *a non-ruled quadric is a polytope* 2_{21}, *and the rest of the space is the "semi-reciprocal" polytope* 1_{22}.

4. The polytope 2_{21}. It is known (**1**, p. 414; **3**, p. 465; **4**, pp. 202, 203) that the uniform polytope 2_{21} has 27 vertices

$$a_1, \ldots, a_6;\ b_1, \ldots, b_6;\ c_{12}, \ldots, c_{56},$$

216 edges such as a_1a_2; and 720 triangular faces such as $a_1a_2a_3$. Referring to Table II of Edge (**6**, p. 643), we readily identify these with his 27 points m, 216 lines c, and 720 planes j.

The polytope has five-dimensional cells of two kinds: 36 pairs of opposite simplexes $2_{20} = \alpha_5$ such as

$$a_1\ a_2\ a_3\ a_4\ a_5\ a_6, \qquad b_1\ b_2\ b_3\ b_4\ b_5\ b_6$$

and 27 cross-polytopes $2_{11} = \beta_5$ such as

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56}. \end{matrix}$$

Edge (**6**, p. 626) has made the important observation that, when any one of the 72 simplexes α_5 is used as simplex of reference, the quadric has the symmetrical equation

$$\sum_{i<j} x_i x_j = 0.$$

When the 6 + 6 vertices of a pair of opposite simplexes have been removed, the remaining 15 vertices belong to a polytope $0_{31} = t_1\alpha_5$ (**1**, pp. 360, 372) such as

$$c_{12}\ c_{13}\ \ldots\ c_{56},$$

lying in one of Edge's 36 hyperplanes P.

Other central sections of 2_{21} include 45 triangles such as

$$a_1\ b_2\ c_{12}$$

(Edge's 45 lines g), which correspond to the 45 triangles on the cubic surface, and 120 triangular double-prisms $\{3\} \times \{3\}$ (**3**, pp. 463, 466) such as

$$\begin{matrix} c_{23} & c_{15} & c_{46} \\ a_2 & b_1 & c_{12} \\ b_3 & a_5 & c_{35} \end{matrix}$$

(Edge's 120 spaces k), which correspond to the 120 pairs of Steiner trihedra. The latter fall interestingly into 40 sets of three (**3**, p. 466).

5. The polytope 1_{22}. The "elliptic" 1_{22}, which differs from the ordinary 1_{22} (**1**, p. 414) in that pairs of opposite elements are identified, has 36 vertices. In PG(5, 2), these are Edge's 36 points p. Each is the centre from which two opposite simplexes α_5 of 2_{21} are in perspective; for example, the simplexes $a_1\ a_2\ a_3\ a_4\ a_5\ a_6$ and $b_1\ b_2\ b_3\ b_4\ b_5\ b_6$ of § 4 are evidently in perspective from the vertex

$$\begin{matrix} 10101 \\ 0 \end{matrix}$$

of 1_{22}.

The principal central sections of the ordinary 1_{22} are given in Table IV of (**2**, p. 476). These, with their numbers inserted, are as follows:

36	$\{\ \}$	36	$e\alpha_5$
120	$\{6\}$	120	$\{6\} + \{6\}$
270	$\left\{\begin{matrix}3\\4\end{matrix}\right\}$	270	$\{4\}$
45	$\{3, 4, 3\}$		
216	$e\alpha_4$		
27	$t_1\beta_5$		

The "36 { }" are merely the diagonals joining pairs of opposite vertices, and so correspond to Edge's 36 points p. The 120 hexagons (which reduce to triangles when opposites are identified) are his 120 lines s. The 270 cuboctahedra (which reduce to complete quadrilaterals) are his 270 planes h. The 45 $\{3, 4, 3\}$ (which reduce to desmic triads of tetrahedra) are his 45 spaces χ. The 216 $e\alpha_4$ (which reduce to complete pentahedra) are his 216 spaces λ. The 27 $t_1\beta_5$ (**1**, pp. 354-360) are his 27 spaces M. The 36 $e\alpha_5$ (**1**, p. 366) are his 36 spaces P. The 120 pairs of hexagons in completely orthogonal planes are his 120 spaces κ. Finally, the 270 squares (which reduce to point pairs) are his 270 lines t.

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