THE GENERALIZED INVERSE $A_{T,S}^{(2)}$ OF A MATRIX OVER AN ASSOCIATIVE RING

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Abstract

In this paper we establish the definition of the generalized inverse $A_{T,S}^{(2)}$, which is a $[2]$ inverse of a matrix $A$ with prescribed image $T$ and kernel $S$ over an associative ring, and give necessary and sufficient conditions for the existence of the generalized inverse $A_{T,S}^{[1,2]}$ and some explicit expressions for $A_{T,S}^{[1,2]}$ of a matrix $A$ over an associative ring, which reduce to the group inverse or $[1]$ inverses. In addition, we show that for an arbitrary matrix $A$ over an associative ring, the Drazin inverse $A_d$, the group inverse $A_s$ and the Moore-Penrose inverse $A_P$, if they exist, are all the generalized inverse $A_{T,S}^{(2)}$.


1. Introduction

It is well known that, over the field of complex numbers, the Moore-Penrose inverse, the Drazin inverse, the group inverse and so on, are all the generalized inverse $A_{T,S}^{(2)}$, which is a $[2]$ inverse of a matrix $A$ with prescribed range $T$ and null space $S$ (see [2, 10]). Y. Wei in [11] gave an explicit expression for the generalized inverse $A_{T,S}^{(2)}$ which reduces to the group inverse.

There are some results on generalized inverses of matrices, such as the Drazin inverse, the group inverse and the Moore-Penrose inverse, over an associative ring (see, for example, [3–8]). These results include necessary and sufficient conditions for the existence of these generalized inverses. In [8], Corollary 1 implies that over an associative ring, a von Neumann regular matrix $A$ has a group inverse if and only if $A^2 A^{(1)} + I - A A^{(1)}$ is invertible, if and only if $A^{(1)} A^2 + I - A^{(1)} A$ is invertible. Recently,
similar results about the Moore-Penrose inverse and the Drazin inverse appeared in [6, 7]. This is a motivation for our research.

Throughout this paper, $R$ denotes an associative ring with identity $1$ and $R^{m \times n}$ denotes the set of $m \times n$ matrices over $R$. In particular, we write $R^m$ for $R^{m \times 1}$ and $M_n(R)$ for $R^{n \times n}$, the ring of square $n \times n$ matrices over $R$. By a module we mean a right $R$-module. If $S$ is an $R$-submodule of an $R$-module $M$ then we write $S \subseteq M$.

Let $A \in R^{m \times n}$. We denote the image of $A$ (that is $\{Ax | x \in R^n\}$) by $R(A)$ and the kernel of $A$ (that is $\{x \in R^n | Ax = 0\}$) by $N(A)$.

An $m \times n$ matrix $A$ over $R$ is said to be von Neumann regular if there exists an $n \times m$ matrix $X$ over $R$ such that

1. $AXA = A$.

In this case $X$ is called a $\{1\}$ inverse of $A$ and is denoted by $A^{(1)}$.

An $n \times n$ matrix $A$ over $R$ is said to be Drazin invertible if for some positive integer $k$ there exists a matrix $X$ over $R$ such that

2. $A^kXA = A^k$,
3. $XAX = X$,
4. $AX =XA$.

If $X$ exists then it is unique and is called the Drazin inverse of $A$ and denoted by $A_d$. If $k$ is the smallest positive integer such that $X$ and $A$ satisfy (2), (3) and (4), then it is called the Drazin index and denoted by $k = \text{Ind}(A)$. If $k = 1$ then $A_d$ is denoted by $A_g$ and is called the group inverse of $A$.

Let $*$ be an involution on the matrices over $R$. Recall that an $m \times n$ matrix $A$ over $R$ is said to be Moore-Penrose invertible (with respect to $*$) if there exists an $n \times m$ matrix $X$ such that (1) and (3) hold and

5. $(AX)^* = AX$,
6. $(XA)^* =XA$.

If $X$ exists then it is unique and is called the Moore-Penrose inverse of $A$ and denoted by $A^*$.

If a matrix $X$ satisfies condition (3) then $X$ is called a $\{2\}$ inverse of $A$. If a matrix $X$ over an associative ring with prescribed image $T$ and kernel $S$, and show that for an arbitrary matrix $A$ over an associative ring the Drazin inverse $A_d$, the group inverse $A_g$ and the Moore-Penrose inverse $A^*$, if they exist, are all the generalized inverse $A^{(2)}_{T,S}$. In Section 3, we give necessary and sufficient conditions for the existence of the generalized inverse $A^{(1,2)}_{T,S}$. In Section 4 we study some explicit expressions for $A^{(1,2)}_{T,S}$ of a matrix $A$ over an associative ring, which reduce to the group inverse or $\{1\}$ inverses, and some equivalent conditions for the existence of $A^{(1,2)}_{T,S}$.
2. The generalized inverse $A_{T,S}^{(2)}$

Suppose that $L, M \subset R^n$ and $L \oplus M = R^n$. Then every $x \in R^n$ can be uniquely written as $x = x_1 + x_2$, where $x_1 \in L, x_2 \in M$. Thus 

$$P_{L,M}x = x_1$$

defines a homomorphism $P_{L,M} : R^n \rightarrow R^n$ called the projection of $R^n$ on $L$ along $M$. This homomorphism can be represented by a matrix with respect to the standard basis of $R^n$, since the module $R^n$ is free. The symbol $P_{L,M}$ is used to denote the matrix as well.

About $P_{L,M}$, we have the following results, whose proof is analogous to that over the field of complex numbers.

**Lemma 2.1.** If $L, M \subset R^n$ and $L \oplus M = R^n$ then

(i) $P_{L,M}A = A$ if and only if $R(A) \subset L$.

(ii) $AP_{L,M} = A$ if and only if $N(A) \supset M$.

We now characterize the $(2)$ inverse of a matrix $A$ over $R$ with prescribed image $T$ and kernel $S$. The proof of the following theorem is analogous to that of [13, Theorem 1].

**Theorem 2.2.** Let $A$ be an $m \times n$ matrix over an associative ring $R$ with identity and $T \subset R^n$ and $S \subset R^m$. Then the following conditions are equivalent.

(i) There exists some $X \in R^{n \times m}$ such that

\[(2.1) \quad XAX = X, \quad R(X) = T, \quad N(X) = S.\]

(ii) $AT \oplus S = R^m$ and $N(A) \cap T = \{0\}$.

If these conditions are satisfied then $X$ is unique.

**Proof.** (i)$\Rightarrow$(ii) Since $XAX = X$, $AX$ is an idempotent homomorphism from $R^n$ to $R^m$. So, by [1, Lemma 5.6],

$$R(AX) \oplus N(AX) = R^m.$$ 

It is easy to see that $R(AX) = AR(X) = AT$ and $N(AX) = N(X) = S$. Hence

$$AT \oplus S = R^m.$$ 

Next we will show that $N(A) \cap T = \{0\}$. Let $x \in N(A) \cap T$. Then $Ax = 0$ and there exists a $y \in R^m$ such that $x = Xy$. So $x = Xy = XAXy = XAx = 0$. Therefore we have $N(A) \cap T = \{0\}.$
(ii)⇒(i) Obviously \( A|_T \) is an epimorphism from \( T \) to \( AT \). Since \( N(A|_T) = N(A) \cap T = 0 \), \( A|_T \) is a monomorphism and so \( A|_T \) has an inverse \( (A|_T)^{-1} : AT \to T \).

From \( AT \oplus S = R^m \), we know that any \( y \in R^m \), can be uniquely written as \( y = y_1 + y_2 \), where \( y_1 \in AT \), \( y_2 \in S \). So we define \( X : R^m \to R^n \) by \( Xy = (A|_T)^{-1}y_1 \).

Obviously \( X \) is a homomorphism and satisfies

\[
\begin{aligned}
Xy &= (A|_T)^{-1}y, & \text{if } y \in AT; \\
Xy &= 0, & \text{if } y \in S.
\end{aligned}
\]

Because \( R^m \) and \( R^n \) are both free modules, there exists a matrix of the homomorphism \( X \) with respect to the standard bases of \( R^m \) and \( R^n \), and we write \( X \) for the matrix as well. It is easy to see that \( R(X) = T \) and \( N(X) = S \) by \( AT \oplus S = R^m \).

For every \( y \in R^m = AT \oplus S \) we have \( y = y_1 + y_2 \) where \( y_1 \in AT \), \( y_2 \in S \). Then

\[
XAXy = XAXy_1 = XA(A|_T)^{-1}y_1 = Xy_1 = Xy.
\]

This implies that \( XAX = X \).

Now we prove the uniqueness. Suppose that \( X_1 \) and \( X_2 \) both satisfy (2.1). Then \( X_1A \) and \( AX_2 \) are idempotent matrices of order \( m \) and \( n \) respectively, and

\[
\begin{aligned}
X_1A &= P_{R(X_1A),N(X_1A)} = P_{R(X_1),N(X_1A)} = P_{T,N(X_1A)}, \\
AX_2 &= P_{R(AX_2),N(AX_2)} = P_{R(AX_2),N(X_2)} = P_{R(AX_2),S}.
\end{aligned}
\]

By Lemma 2.1, we deduce that

\[
X_2 = P_{T,N(X_1A)}X_2 = (X_1A)X_2 = X_1(AX_2) = X_1P_{R(AX_2),S} = X_1
\]

A matrix \( X \in R^{n \times m} \) is called the generalized inverse which is a \( \{2\} \) inverse of a matrix \( A \) over \( R \) with prescribed image \( T \) and kernel \( S \) if it satisfies the equivalent conditions in Theorem 2.2, and is denoted by \( A^{(2)}_{T,S} \).

By (2.2), we have that

\[
A^{(2)}_{T,S} = (A|_T)^{-1}P_{AT,S}.
\]

From the proof of uniqueness in the theorem above and Lemma 2.1, we have the following corollary.

**COROLLARY 2.3.** Let \( A \) and \( G \) be matrices over an associative ring \( R \). If the generalized inverse \( A^{(2)}_{T,S} \) exists, then

(i) \( A^{(2)}_{T,S}AG = G \) if and only if \( R(G) \subseteq T \);

(ii) \( GA^{(2)}_{T,S} = G \) if and only if \( N(G) \supseteq S \).

About the generalized inverse, we also have the following property.
THEOREM 2.4. Let $A$ be a matrix over $R$. If $A^{(2)}_{T,S}$ exists and there exists a matrix $G$ over $R$ satisfying $R(G) = T$ and $N(G) = S$ then there exists a matrix $W$ over $R$ such that

\begin{align}
GAGW &= G, \\
A^{(2)}_{T,S}AGW &= A^{(2)}_{T,S}.
\end{align}

PROOF. Suppose $A^{(2)}_{T,S}$ exists with $R(G) = T$ and $N(G) = S$ for a matrix $G$. Then $AR(G) \oplus N(G) = R^m$ and so there exists an epimorphism $R^m \to N(G) \to 0$. By [1, Theorem 8.1], $N(G)$ has a finite spanning set whose elements constitute a matrix, denoted by $L$. Thus $GL = 0$, and the columns of $(AG, L)$ generate $R^m$, that is, there exists a matrix $(W^T, W_1^T)^T$ such that

$$AGW + LW_1 = I_m.$$ 

If we multiply the left hand side by $G$ and $A^{(2)}_{T,S}$ respectively, then we obtain (2.4) and (2.5).

The following theorem shows that for an arbitrary matrix $A$ over an associative ring, $A^\dagger$, $A_\gamma$ and $A_\alpha$, if they exist, are all the generalized inverse $A^{(2)}_{T,S}$.

THEOREM 2.5. (i) Let $A$ be an $m \times n$ matrix over $R$ and let $\ast$ be an involution on the matrices over $R$. If $A^\dagger$ exists, then $A^\dagger = A^{(2)}_{R(A^\dagger), N(A^\dagger)}$.

(ii) Let $A$ be an $n \times n$ matrix over $R$, and $k = \text{Ind}(A)$. If $A_\gamma$ exists, then $A_\gamma = A^{(2)}_{R(A_\gamma), N(A_\gamma)}$.

(iii) Let $A$ be an $n \times n$ matrix over $R$. If $A_\alpha$ exists, then $A_\alpha = A^{(2)}_{R(A), N(A)}$.

PROOF. (i) Since $A^\dagger \in A[1, 2]$ and $A^\dagger \ast \in A^\ast[1, 2]$, we easily see that

\begin{align}
R(A^\dagger) &= R(A^\dagger A) = R((A^\dagger A)^*) = R(A^\ast A^\dagger) = R(A^\ast), \\
N(A^\dagger) &= N(AA^\dagger) = N((AA^\dagger)^*) = N(A^\ast A^\dagger) = N(A^\ast),
\end{align}

and $N(A) = N(A^\dagger A)$.

Since $AA^\dagger$ and $A^\dagger A$ are idempotent, we have

$$R^m = R(AA^\dagger) \oplus N(AA^\dagger) = AR(A^\dagger) \oplus N(AA^\dagger) = AR(A^\ast) \oplus N(A^\ast)$$

and

$$N(A) \cap R(A^\ast) = N(A^\dagger A) \cap R(A^\dagger A) = \{0\}$$

by [1, Lemma 5.6]. So, by Theorem 2.2, $A^{(2)}_{R(A^\ast), N(A^\ast)}$ exists and $A^\dagger = A^{(2)}_{R(A^\ast), N(A^\ast)}$. 


(ii) Firstly, we shall show that

\[ R(A_d) = R(A A_d) = R(A^l) \quad \text{and} \quad N(A_d) = N(A A_d) = N(A^l) \]

for any positive integer \( l \geq k \). Since

\[ R(A_d) = R(A A_d^2) \subset R(A A_d) = R(A_d A) \subset R(A_d), \]

we have \( R(A_d) = R(A A_d) \) and so

\[ R(A A_d) = AR(A_d) = AR(A A_d) = A^2 R(A_d). \]

It is easy to obtain inductively that

\[ R(A A_d) = A^h R(A_d) \]

for any positive integer \( h \). This gives us that \( R(A_d) = R(A A_d) = R(A^l) \) for any positive integer \( l \geq k \). Also, since for any positive integer \( l \geq k \),

\[ N(A_d) \subset N(A^{l+1} A_d) = N(A^l) \subset N(A_d A^l) = N(A_d A) \subset N(A_d^2) A = N(A_d), \]

we get that \( N(A_d) = N(A A_d) = N(A^l) \).

Since \( A A_d \) is idempotent, by [1, Lemma 5.6], we have

\[ R^n = R(A A_d) \oplus N(A A_d) = AR(A^k) \oplus N(A^k) = R^n. \]

Since

\[ N(A) \cap R(A^k) \subset N(A^k) \cap R(A^{k+1}) = \{0\}, \]

\( A_{R(A^k), N(A^k)}^{(2)} \) exists and \( A_{R(A^k), N(A^k)}^{(2)} = A_d \) by Theorem 2.2.

(iii) Take \( k = 1 \) in (ii).

\[ \square \]

3. The generalized inverse \( A_{T,S}^{(1,2)} \)

If the generalized inverse \( A_{T,S}^{(2)} \) satisfies \( A A_{T,S}^{(2)} A = A \) then it is called the generalized inverse which is a \( \{1,2\} \) inverse of a matrix \( A \) over \( R \) with prescribed image \( T \) and kernel \( S \), and is denoted by \( A_{T,S}^{(1,2)} \). (Its uniqueness is guaranteed by the following theorem.)

**THEOREM 3.1.** Let \( A \) be an \( m \times n \) matrix over an associative ring \( R \) with identity and \( T \subset R^n \) and \( S \subset R^m \). Then the following conditions are equivalent.

(i) \( AT \oplus S = R^m, \ R(A) \cap S = \{0\} \) and \( N(A) \cap T = \{0\} \).

(ii) \( R(A) \oplus S = R^m, \ N(A) \oplus T = R^n \).

(iii) There exists some \( X \in R^{n \times m} \) such that

\[ AXA = A, \quad XAX = X, \quad R(X) = T, \quad N(X) = S. \]

If these conditions are satisfied then \( X \) is unique.
The generalized inverse $A_{T,S}^{(2)}$

**PROOF.** (ii) $\implies$ (i) It is obvious that $R(A) \cap S = \{0\}$ and $N(A) \cap T = \{0\}$. To obtain $AT \oplus S = R^m$, it suffices to prove $AT = R(A)$.

Obviously, $AT \subset R(A)$. For any $x \in R(A)$, we have $x = Ay$, where $y \in R^n$. Since $N(A) \oplus T = R^n$, we can write $y = y_1 + y_2$, where $y_1 \in N(A), y_2 \in T$. Thus,

$$x = Ay = Ay_1 + Ay_2 = Ay_2 \in AT,$$

and therefore $R(A) \subset AT$. Consequently, $AT = R(A)$.

(i) $\implies$ (iii) By Theorem 2.2, from $AT \oplus S = R^m$ and $N(A) \cap T = \{0\}$, we know that $X = A_{T,S}^{(2)}$ exists and that $R(X) = T$ and $N(X) = S$. We shall show $AXA = A$.

Since $XAX = X$, we have $XAXA = XA$ and then $X(AXA - A) = 0$. So

$$R(AXA - A) \subset R(A) \cap N(X) = R(A) \cap S = \{0\}.$$ 

Hence $AXA = A$.

(iii) $\implies$ (ii) From (iii), we have $(AX)^2 = AX$, and

$$N(X) \subset N(AX) \subset N(XAX) = N(X),$$

$$N(AX) \subset N(XAX) = N(A) \subset N(AX),$$

$$R(XA) \subset R(X) = R(XAX) \subset R(XA),$$

$$R(A) \subset R(AXA) \subset R(AX).$$

So

$$N(AX) = N(X) = S, \quad N(AX) = N(A),$$

$$R(XA) = R(X) = T, \quad R(AX) = R(A).$$

By [1, Lemma 5.6] and the four equations above, we reach (ii).

By Theorem 2.2, $X$ is unique.

The next result is concerning the equivalent conditions in Theorem 3.1.

**THEOREM 3.2.** Let $A$ be an $m \times n$ matrix over an associative ring $R$ with identity and $T \subset R^n$ and $S \subset R^m$.

(i) If $N(A) + T = R^n$ then $AT = R(A)$.

(ii) If $AT \oplus S = R^m$ then

$$AT = R(A) \quad \text{if and only if} \quad R(A) \cap S = \{0\}.$$ 

**PROOF.** (i) From the proof of the theorem above (ii) implies (i).

(ii) Suppose that $R(A) \cap S = \{0\}$. Obviously, $AT \subset R(A)$. Now we will show the inclusion in reverse. For any $x \in R(A)$,

$$x = x_1 + x_2 \in R^m = AT \oplus S,$$

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where $x_1 \in AT$, $x_2 \in S$. By $AT \subset R(A)$, $x_1 \in R(A)$. So
\[ x_2 = x - x_1 \in R(A) \cap S = \{0\}. \]
Therefore, $x_2 = 0$ and then $x = x_1 \in AT$. Hence $R(A) \subset AT$.

Conversely, suppose that $AT = R(A)$. Since $AT \oplus S = R^m$ and $AT = R(A)$, we have $R(A) \cap S = AT \cap S = \{0\}$. \qed

We denote the maximal order of a nonvanishing minor of $A$ over a commutative ring $R$ by $\rho(A)$. This is called the determinantal rank of $A$. Obviously $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$ (see [9, Theorem 2.3]). When $R$ is the complex number field, $\rho(A) = \operatorname{rank}(A)$.

**Theorem 3.3.** Let $A$ be an $m \times n$ matrix over an integral domain $R$ and $T \subset R^n$ and $S \subset R^m$ be free submodules. If $AT \oplus S = R^m$ then the following conditions are equivalent.

(i) $N(A) \cap T = \{0\}$ and $R(A) \cap S = \{0\}$.

(ii) $\dim(T) = \rho(A)$ and $\dim(S) = m - \dim(T)$.

**Proof.** Suppose that (i) holds and let the columns of $U$ be a basis of $T$. From the proof of [13, Theorem 2], we have $\dim(T) = \dim(AT) = \rho(AU) \leq \rho(A)$ and $\dim(S) = m - \dim(T)$. By Theorem 3.2, $AT = R(A)$. Thus there exists a matrix $X$ over $R$ such that $A = AUX$. Thus $\rho(A) \leq \rho(AU) = \dim(AT)$. Therefore $\rho(A) = \dim(AT) = \dim(T)$.

Conversely, suppose that (ii) holds. We have that $\dim(T) = \dim(AT)$ from the proof of [13, Theorem 2]. Thus $\rho(A) = \dim(T) = \dim(AT)$. By [12, Lemma 1], the maximal number of linearly independent columns of $A$ is $\dim(AT)$. Since $AT \subset R(A)$, $R(A) + S = R^m$. Over the quotient field $F$ of $R$, $AT = R(A)$ because $\rho(A) = \dim(AT)$, and $R(A) \oplus S = R^m$. Therefore $x$ and $y$ are linear independent over $F$ for any $x \in R(A)$, $y \in S$.

On the other hand, over an integral domain $R$, suppose that $0 \neq z \in R(A) \cap S$. Then there exist $r_i \in R$, $i = 1, \ldots, s$, such that
\[ z = \sum_{i=1}^{s} \beta_i r_i, \]
where $\{\beta_1, \beta_2, \ldots, \beta_s\}$ is a basis of $S$ and $s = \dim(S)$. But Equation (3.1) is true over $F$. This is in contradiction to the above reasoning. Hence $R(A) \cap S = \{0\}$.

The remainder of the proof is obtained from [13, Theorem 2]. \qed

**Remark 1.** A module over the field of complex numbers is a vector space. So when $R$ is the field of complex numbers, the above theorem ensures that Theorem 3.1 extends [2, Corollary 2.10].
4. Explicit expressions for $A_{T,S}^{(1,2)}$

We now consider some explicit expressions for $A_{T,S}^{(1,2)}$, which reduce to the group inverse or $\{1\}$ inverses. Firstly we shall prove the following lemma. In the proof, we use the following fact.

**Proposition 4.1.** If $e$ is idempotent in a ring $R$ with identity $1$ and $x, y \in eRe$ then $xy = e$ if and only if $(x + 1 - e)(y + 1 - e) = 1$.

**Lemma 4.2.** Let $A$ be an $m \times n$ von Neumann regular matrix over $R$ and $G$ an $n \times m$ matrix over $R$. Then $U = AGA^{(1)} + I_m - AA^{(1)}$ is invertible if and only if $V = A^{(1)}AGA + I_n - A^{(1)}A$ is invertible.

**Proof.** If $U$ is invertible then there exists an $X$ such that $UX = XU = I_m$. That is,

$$(AGAA^{(1)} + I_m - AA^{(1)})X = I_m \quad \text{and} \quad X(AGAA^{(1)} + I_m - AA^{(1)}) = I_m.$$ 

Multiplying on the left by $A^{(1)}AA^{(1)}$ and the right by $A$ and, since $A = AA^{(1)}A$, we have

$$(A^{(1)}AGA)(A^{(1)}AA^{(1)}XA) = A^{(1)}A \quad \text{and} \quad (A^{(1)}AA^{(1)}XA)(A^{(1)}AGA) = A^{(1)}A.$$

Since $A^{(1)}AGA = A^{(1)}A(GA)A^{(1)}A$ and $A^{(1)}AA^{(1)}XA = A^{(1)}A(A^{(1)}XA)A^{(1)}A$, we know that $A^{(1)}AGA$ has the inverse matrix $A^{(1)}AA^{(1)}XA$ in $A^{(1)}AM_n(R)A^{(1)}A$. Thus $V = A^{(1)}AGA + I_n - A^{(1)}A$ has the inverse matrix

$$A^{(1)}A\left(A^{(1)}AA^{(1)}XA\right)A^{(1)}A + I_n - A^{(1)}A \quad \text{in} \quad M_n(R).$$

The proof of the converse is analogous. \qed

Next we shall show the main result of this section. The following theorem not only shows some explicit expressions for $A_{T,S}^{(1,2)}$ which reduce to the group inverse or $\{1\}$ inverses, but also gives some equivalent conditions for the existence of $A_{T,S}^{(1,2)}$.

**Theorem 4.3.** Let $A$ be an $m \times n$ matrix over $R$ and $G$ an $n \times m$ matrix over $R$. Then the following conditions are equivalent.

(i) $A$ is von Neumann regular, $U = AGAA^{(1)} + I_m - AA^{(1)}$ is invertible and $N(A) \cap R(G) = \{0\}$.

(ii) $A$ is von Neumann regular, $V = A^{(1)}AGA + I_n - A^{(1)}A$ is invertible and $N(A) \cap R(G) = \{0\}$.

(iii) $A_{R(G),N(G)}^{(1,2)}$ exists.
When these conditions are satisfied we have

\begin{align}
A^{(1,2)}_{R(G),N(G)} &= G(AG)\_g = (GA)_g G \\
\end{align}

(4.2) 
(4.3) 
(4.4) 

\begin{align}
G(AG)\_G &= G(G)G = (1)G \\
G(AG)\_G &= G(G(AG))\_G \\
GU^{-2}AG &= GU^{-1}AV^{-1}G = GAV^{-2}G.
\end{align}

Proof. (i) and (ii) are equivalent by Lemma 4.2.

To show that (ii) implies (iii), set \( B = AV^{-2}G \). Using \( VA = AGA = AV \), we have \( B = (AG)\_g \) because

\begin{align}
B(AG) &= AV^{-2}GAG = U^{-2}AGAG = U^{-1}AG = AV^{-1}G = AGAV^{-2}G \\
\quad &= (AG)B, \\
B(AG)B &= U^{-1}AG(AG)G = AV^{-2}G = B, \\
(AG)B(AG) &= (AG)AV^{-1}G = AG.
\end{align}

Analogously, we deduce that \((GA)\_G\) exists and \((GA)\_G = GU^{-2}A\). Let \( X = G(AG)\_G \). It is obvious that

\begin{align}
XAX = X.
\end{align}

Since

\begin{align}
AG &= (AG)^{2}(AG)\_G = AGAX,
\end{align}

we have \( A(G - GAX) = 0 \) and then

\( R(G - GAX) = R(G(I - AX)) \subset N(A) \cap R(G) = \{0\} \).

Therefore

\begin{align}
G &= GAX \\
\quad &= GA(G(AG)\_G) = G(AG)\_G AG \\
\quad &= XAG.
\end{align}

Using (4.6) and (4.7), we have

\begin{align}
R(X) &= R(G) \quad \text{and} \quad N(X) = N(G).
\end{align}

Since \( AV = AGA \), we get

\begin{align}
A &= AGAV^{-1} = AG(AG)\_G AGAV^{-1} = AXXA.
\end{align}

Using the equation above, together with (4.5) and (4.8), we deduce that \( A^{(1,2)}_{R(G),N(G)} \) exists and \( A^{(1,2)}_{R(G),N(G)} = X = G(AG)\_G \) by Theorem 3.1.
The generalized inverse $A_{T,S}^{(i)}$

To show that (iii) implies (i), we use Theorem 2.4 to obtain

\[(AGAA^{(i)})(AGW^2AA^{(i)}) = AGAGW^2AA^{(i)} = AGWAA^{(i)}\]
\[= AA_{R(G),N(G)}AA^{(i)} = AA^{(i)}.
\]

(4.9)

Therefore,

\[(AGAA^{(i)})(AGW^2AA^{(i)})(AGAA^{(i)}) = AA^{(i)}(AGAA^{(i)}) = AGAA^{(i)}\]

and then

\[AG\left((AGW^2AA^{(i)})(AGAA^{(i)}) - AA^{(i)}\right) = 0.
\]

By Theorem 3.1, $R(A) \cap N(G) = \{0\}$ and $N(A) \cap R(G) = \{0\}$ and so

\[R\left(G\left((AGW^2AA^{(i)})(AGAA^{(i)}) - AA^{(i)}\right)\right) \subset R(G) \cap N(A) = \{0\}.
\]

Thus

\[G\left((AGW^2AA^{(i)})(AGAA^{(i)}) - AA^{(i)}\right) = 0.
\]

From this, we have

\[R\left((AGW^2AA^{(i)})(AGAA^{(i)}) - AA^{(i)}\right) \subset R(A) \cap N(G) = \{0\},
\]

and then

\[(4.10) \quad (AGW^2AA^{(i)})(AGAA^{(i)}) = AA^{(i)}.
\]

By (4.9) and (4.10), $AGAA^{(i)}$ is invertible in $AA^{(i)}M_m(R)AA^{(i)}$ and so is $U$ in $M_m(R)$. Also, obviously, $A$ is von Neumann regular.

Now we shall prove that (4.1) $\sim$ (4.3). Since

\[G(AG)_g = G(AV^{-2}G) = GU^{-1}AV^{-1}G = (GU^{-2}A)G = (GA)_gG,
\]

we have $A_{R(G),N(G)}^{(i)} = (GA)_gG$ and (4.4).

Next we will prove (4.2). Since

\[GAG = GAG (GA)_g^2 AGAG,
\]

$GAG$ is von Neumann regular and then

\[AG = U^{-1}UAG = U^{-1}AGAG = (U^{-1}A)GAG(GAG)^{(i)}GAG\]
\[= AG(GAG)^{(i)}GAG.
\]
Therefore
\[ A \left( G - G (GAG)^{(1)} GAG \right) = 0. \]
Thus
\[ R \left( G - G (GAG)^{(1)} GAG \right) \subset N(A) \cap R(G) = \{0\}. \]
So we obtain
\begin{equation}
(4.11) \quad G = G (GAG)^{(1)} GAG
\end{equation}
Since \( A_{R(G),N(G)}^{(1,2)} \) exists, using (2.4) and (4.11), it follows that
\begin{equation}
G = GAGW = GAG (GAG)^{(1)} GAGW
= GAG (GAG)^{(1)} G.
\end{equation}

Let \( Z = G(GAG)^{(1)} G \). Using (4.11) and (4.12), it easily follows that \( ZAZ = Z, AZA = A, R(Z) = R(G) \) and \( N(Z) = N(G) \). By Theorem 3.1 we have that \( A_{R(G),N(G)}^{(1,2)} = Z = G(GAG)^{(1)} G \).

Finally, we will verify (4.3). It is obvious that \( AG \) and \( GA \) are von Neumann regular. By Proposition 4.1 and the invertibility of \( V \) there exists a matrix \( P \in A^{(1)} A M_n(R) A^{(1)} A \) such that \( P(A^{(1)} AGA) = A^{(1)} A \). Thus
\begin{equation}
(4.13) \quad A = A \left( PA^{(1)} AGA \right) = APA^{(1)} A \left( GA(GA)^{(1)} GA \right) = A(GA)^{(1)} GA.
\end{equation}
Using (4.13), we deduce that \( (AG)^{(1)} A(GA)^{(1)} \) is a \{1\} inverse of \( GAG \). Therefore, using (4.2), we obtain (4.3).

**Remark 2.** By (4.4), we can compute \( A_{R(G),N(G)}^{(1,2)} \) using \( U \) or \( V \).

**Remark 3.** If \( G = A \) where \( A \) is such that \( V = A^{(1)} A^2 + I_n = A^{(1)} A \) is invertible, then \( N(A) \cap R(A) = \{0\} \). Indeed, let \( x \in N(A) \cap R(A) \). Then there exists a \( y \in R^n \) such that \( x = Ay \) and so \( A^2 y = 0 \). Since \( V \) is invertible, there exists a matrix \( P \) such that \( PV = I_n \). Thus \( PA^{(1)} A^3 = A^{(1)} A \) and then
\[ 0 = PA^{(1)} A^3 y = A^{(1)} Ay. \]
Hence \( Ay = AA^{(1)} y = 0. \) Consequently, \( x = Ay = 0. \)

Similarly, if we take \( G = A^* \), where \( * \) is an involution on the matrices over \( R \) such that \( U = AA^* A^{(1)} A + I_m = AA^{(1)} \) is invertible, then \( N(A) \cap R(A^*) = \{0\} \). Indeed, let \( x \in N(A) \cap R(A^*) \). Then there exists a \( y \in R^n \) such that \( x = A^* y \) and so \( AA^* y = 0 \). Since \( U \) is invertible, there exists a matrix \( Q \) such that \( AA^* AA^{(1)} Q = AA^{(1)} \) and thus
\[ 0 = Q^* (A^{(1)})^* A^* AA^* y = (A^{(1)})^* A^* y. \]
So \( x = A^* y = A^* (A^{(1)})^* A^* y = 0. \)
When $G$ takes the value $A$ (respectively $A^*$) in the theorem above, we find that $A_{R(G), N(G)}^{(1,2)}$ is $A_g$ (respectively $A^*$).

**THEOREM 4.4.** Let $A$ be an $m \times n$ matrix over $R$. Then

(i) $A_{R(A), N(A)}^{(1,2)}$ exists if and only if $A_g$ exists. Moreover, $A_{R(A), N(A)}^{(1,2)} = A_g$.

(ii) If $*$ is an involution on the matrices over $R$ then $A_{R(A^*), N(A^*)}^{(1,2)}$ exists if and only if $A^*$ exists. Moreover, $A_{R(A^*), N(A^*)}^{(1,2)} = A^*$.

**PROOF.** To show the existence of $A_{R(A), N(A)}^{(1,2)}$ implies existence of $A_g$ in (i), take $G = A$ in (4.1). Then $A_{R(A), N(A)}^{(1,2)} = A(A^2)_g = (A^2)_g A$ and then $AA_{R(A), N(A)}^{(1,2)} = A_{R(A), N(A)}^{(1,2)} A$. Hence $A_{R(A^*), N(A^*)}^{(1,2)}$ is the group inverse of $A$.

To show that existence of $A_{R(A^*), N(A^*)}^{(1,2)}$ implies existence of $A^*$ in (ii), take $G = A^*$ in (4.1). Then $A_{R(A^*), N(A^*)}^{(1,2)} = A^*(AA^*)_g = (A^*A)_g A^*$ and then

$$(AA_{R(A^*), N(A^*)}^{(1,2)} N(A^*)^*) = AA_{R(A^*), N(A^*)}^{(1,2)} N(A^*)$$

and

$$(A_{R(A^*), N(A^*)}^{(1,2)} A) = A_{R(A^*), N(A^*)}^{(1,2)} A.$$

Hence $A_{R(A^*), N(A^*)}^{(1,2)}$ is the Moore-Penrose inverse of $A$.

The converses follow from Theorem 2.5. ✷

By Theorems 4.3 and 4.4 and Remark 3, we can obtain the following two corollaries, in which the first is equivalent to [8, Corollary 2] and the second is almost the same as [6, Theorem 1]:

**COROLLARY 4.5.** Let $A \in R^{n \times n}$. The following conditions are equivalent.

(i) $A$ is von Neumann regular and $U = A^3 A^{(1)} + I_n - A A^{(1)}$ is invertible.

(ii) $A$ is von Neumann regular and $V = A^{(1)} A^3 + I_n - A^{(1)} A$ is invertible.

(iii) $A_g$ exists.

Moreover,

$$A_g = A(A^2)_g = (A^2)_g A$$

$$= A(A^3)^{(1)} A$$

$$= A(A^2)^{(1)} A(A^2)^{(1)} A.$$  \hspace{1cm} (4.14) \hspace{1cm} (4.15) \hspace{1cm} (4.16)

$$= AU^{-2} A^2 = AU^{-1} AV^{-1} A = A^2 V^{-2} A.$$  \hspace{1cm} (4.17)

**REMARK 4.** The above corollary is unlike [8, Corollary 2], but they are equivalent. This is because $V$ is invertible if and only if $T = A^{(1)} A^2 + I_n - A^{(1)} A$ is invertible. Indeed, if $V$ is invertible, then there exists a matrix $P \in M_n(R)$ such that $PV = VP = I_n$. From this and $V = T^2$, we get $(PT)T = T(TP) = I_n$. Hence $T$ is invertible in $M_n(R)$. The converse is obvious from $V = T^2$.  

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COROLLARY 4.6. Let $A$ be an $m \times n$ matrix over $R$ and let $*$ be an involution on the matrices over $R$. The following conditions are equivalent.

(i) $A$ is von Neumann regular and $U = AA^* AA + I_n - AA^* A$ is invertible.

(ii) $A$ is von Neumann regular and $V = A^* AA + I_n - A^* A$ is invertible.

(iii) $A^\dagger$ exists.

Moreover,

$$A^\dagger = A^*(AA^*)^\dagger \equiv (A^* A)^\dagger A^* = A^*(AA^*)^\dagger A^* = A^*(AA^*(AA^*)^\dagger A^*)^\dagger A^* = A^* U^{-2} AA^* = A^* U^{-1} A V^{-1} A = A^* A V^{-2} A^*.$$ 

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References


The generalized inverse $A_{7.5}^{(2)}$