THE GENERALIZED INVERSE $A^{(2)}_{T,S}$ OF A MATRIX OVER AN ASSOCIATIVE RING

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Abstract

In this paper we establish the definition of the generalized inverse $A^{(2)}_{T,S}$ which is a $[2]$ inverse of a matrix $A$ with prescribed image $T$ and kernel $S$ over an associative ring, and give necessary and sufficient conditions for the existence of the generalized inverse $A^{(1,2)}_{T,S}$ and some explicit expressions for $A^{(1,2)}_{T,S}$ of a matrix $A$ over an associative ring, which reduce to the group inverse or $[1]$ inverses. In addition, we show that for an arbitrary matrix $A$ over an associative ring, the Drazin inverse $A^d$, the group inverse $A^g$ and the Moore-Penrose inverse $A^r$, if they exist, are all the generalized inverse $A^{(2)}_{T,S}$.


1. Introduction

It is a well known that, over the field of complex numbers, the Moore-Penrose inverse, the Drazin inverse, the group inverse and so on, are all the generalized inverse $A^{(2)}_{T,S}$, which is a $[2]$ inverse of a matrix $A$ with prescribed range $T$ and null space $S$ (see [2, 10]). Y. Wei in [11] gave an explicit expression for the generalized inverse $A^{(2)}_{T,S}$ which reduces to the group inverse.

There are some results on generalized inverses of matrices, such as the Drazin inverse, the group inverse and the Moore-Penrose inverse, over an associative ring (see, for example, [3–8]). These results include necessary and sufficient conditions for the existence of these generalized inverses. In [8], Corollary 1 implies that over an associative ring, a von Neumann regular matrix $A$ has a group inverse if and only if $A^2A^{(1)} + I - AA^{(1)}$ is invertible, if and only if $A^{(1)}A^2 + I - A^{(1)}A$ is invertible. Recently,
similar results about the Moore-Penrose inverse and the Drazin inverse appeared in [6, 7]. This is a motivation for our research.

Throughout this paper, $R$ denotes an associative ring with identity 1 and $R^{m \times n}$ denotes the set of $m \times n$ matrices over $R$. In particular, we write $R^m$ for $R^{m \times 1}$ and $M_n(R)$ for $R^{n \times n}$, the ring of square $n \times n$ matrices over $R$. By a module we mean a right $R$-module. If $S$ is an $R$-submodule of an $R$-module $M$ then we write $S \subseteq M$.

Let $A \in R^{m \times n}$. We denote the image of $A$ (that is $\{Ax | x \in R^n\}$) by $R(A)$ and the kernel of $A$. (that is $\{x \in R^n | Ax = 0\}$) by $N(A)$.

An $m \times n$ matrix $A$ over $R$ is said to be von Neumann regular if there exists an $n \times m$ matrix $X$ over $R$ such that

1. $AXA = A$.

In this case $X$ is called a $1$ inverse of $A$ and is denoted by $A^{(1)}$.

An $n \times n$ matrix $A$ over $R$ is said to be Drazin invertible if for some positive integer $k$ there exists a matrix $X$ over $R$ such that

2. $A^kXA = A^k$,
3. $XAX = X$,
4. $AX = XA$.

If $X$ exists then it is unique and is called the Drazin inverse of $A$ and denoted by $A_d$.

If $k$ is the smallest positive integer such that $X$ and $A$ satisfy (2), (3) and (4), then it is called the Drazin index and denoted by $k = \text{Ind}(A)$. If $k = 1$ then $A_d$ is denoted by $A_g$ and is called the group inverse of $A$.

Let $\ast$ be an involution on the matrices over $R$. Recall that an $m \times n$ matrix $A$ over $R$ is said to be Moore-Penrose invertible (with respect to $\ast$) if there exists an $n \times m$ matrix $X$ such that (1) and (3) hold and

6. $(AX)^* = AX,$
7. $(XA)^* = XA$.

If $X$ exists then it is unique and is called the Moore-Penrose inverse of $A$ and denoted by $A^\dagger$. If a matrix $X$ satisfies condition (3) then $X$ is called a $2$ inverse of $A$.

In Section 2 we shall establish the definition of the generalized inverse $A_{T,S}^{(2)}$, which is a $2$ inverse of a matrix $A$ over an associative ring with prescribed image $T$ and kernel $S$, and show that for an arbitrary matrix $A$ over an associative ring the Drazin inverse $A_d$, the group inverse $A_g$ and the Moore-Penrose inverse $A^\dagger$, if they exist, are all the generalized inverse $A_{T,S}^{(2)}$. In Section 3, we give necessary and sufficient conditions for the existence of the generalized inverse $A_{T,S}^{(1,2)}$. In Section 4 we study some explicit expressions for $A_{T,S}^{(1,2)}$ of a matrix $A$ over an associative ring, which reduce to the group inverse or $1$ inverses, and some equivalent conditions for the existence of $A_{T,S}^{(1,2)}$. 
The generalized inverse $A_{T,S}^{(2)}$

Suppose that $L, M \subset \mathbb{R}^n$ and $L \oplus M = \mathbb{R}^n$. Then every $x \in \mathbb{R}^n$ can be uniquely written as $x = x_1 + x_2$, where $x_1 \in L$, $x_2 \in M$. Thus

$$P_{L,M}x = x_1$$

defines a homomorphism $P_{L,M} : \mathbb{R}^n \to \mathbb{R}^n$ called the projection of $\mathbb{R}^n$ on $L$ along $M$. This homomorphism can be represented by a matrix with respect to the standard basis of $\mathbb{R}^n$, since the module $\mathbb{R}^n$ is free. The symbol $P_{L,M}$ is used to denote the matrix as well.

About $P_{L,M}$, we have the following results, whose proof is analogous to that over the field of complex numbers.

**Lemma 2.1.** If $L, M \subset \mathbb{R}^n$ and $L \oplus M = \mathbb{R}^n$ then

(i) $P_{L,M}A = A$ if and only if $R(A) \subseteq L$.

(ii) $AP_{L,M} = A$ if and only if $N(A) \supseteq M$.

We now characterize the $(2)$ inverse of a matrix $A$ over $\mathbb{R}$ with prescribed image $T$ and kernel $S$. The proof of the following theorem is analogous to that of [13, Theorem 1].

**Theorem 2.2.** Let $A$ be an $m \times n$ matrix over an associative ring $R$ with identity and $T \subset \mathbb{R}^n$ and $S \subset \mathbb{R}^m$. Then the following conditions are equivalent.

(i) There exists some $X \in \mathbb{R}^{n \times m}$ such that

$$XAX = X, \quad R(X) = T, \quad N(X) = S.$$ (2.1)

(ii) $AT \oplus S = \mathbb{R}^m$ and $N(A) \cap T = \{0\}$.

If these conditions are satisfied then $X$ is unique.

**Proof.** (i)$\Rightarrow$(ii) Since $XAX = X$, $AX$ is an idempotent homomorphism from $\mathbb{R}^n$ to $\mathbb{R}^m$. So, by [1, Lemma 5.6],

$$R(AX) \oplus N(AX) = \mathbb{R}^m.$$  

It is easy to see that $R(AX) = AR(X) = AT$ and $N(AX) = N(X) = S$. Hence

$$AT \oplus S = \mathbb{R}^m.$$  

Next we will show that $N(A) \cap T = \{0\}$. Let $x \in N(A) \cap T$. Then $Ax = 0$ and there exists a $y \in \mathbb{R}^m$ such that $x = Xy$. So $x = Xy = XAXy = XAx = 0$. Therefore we have $N(A) \cap T = \{0\}$. 


(ii)⇒(i) Obviously \( A|_T \) is an epimorphism from \( T \) to \( AT \). Since \( N(A|_T) = N(A) \cap T = 0 \), \( A|_T \) is a monomorphism and so \( A|_T \) has an inverse \( (A|_T)^{-1} : AT \to T \).

From \( AT \oplus S = R^m \), we know that any \( y \in R^m \), can be uniquely written as \( y = y_1 + y_2 \), where \( y_1 \in AT, y_2 \in S \). So we define \( X : R^m \to R^n \) by \( Xy = (A|_T)^{-1}y_1 \).

Obviously \( X \) is a homomorphism and satisfies

\[
\begin{align*}
Xy &= (A|_T)^{-1}y, & \text{if } y \in AT; \\
Xy &= 0, & \text{if } y \in S.
\end{align*}
\]

Because \( R^m \) and \( R^n \) are both free modules, there exists a matrix of the homomorphism \( X \) with respect to the standard bases of \( R^m \) and \( R^n \), and we write \( X \) for the matrix as well. It is easy to see that \( R(X) = T \) and \( N(X) = S \) by \( AT \oplus S = R^m \).

For every \( y \in R^m = AT \oplus S \) we have \( y = y_1 + y_2 \) where \( y_1 \in AT, y_2 \in S \). Then

\[
XAAXy = XAAXy_1 = XA(A|_T)^{-1}y_1 = Xy_1 = XY.
\]

This implies that \( XAX = X \).

Now we prove the uniqueness. Suppose that \( X_1 \) and \( X_2 \) both satisfy (2.1). Then \( X_1A \) and \( AX_2 \) are idempotent matrices of order \( m \) and \( n \) respectively, and

\[
X_1A = P_{R(X_1),N(X_1)} = P_{R(X_1),N(X_1)} = P_{T,N(X_1)},
\]

\[
AX_2 = P_{R(AX_2),N(AX_2)} = P_{R(AX_2),N(X_2)} = P_{R(AX_2),S}.
\]

By Lemma 2.1, we deduce that

\[
X_2 = P_{T,N(X_1)}X_2 = (X_1A)X_2 = X_1(AX_2) = X_1P_{R(AX_2),S} = X_1.
\]

A matrix \( X \in R^{n \times m} \) is called the generalized inverse which is a \( \{2\} \) inverse of a matrix \( A \) over \( R \) with prescribed image \( T \) and kernel \( S \) if it satisfies the equivalent conditions in Theorem 2.2, and is denoted by \( A^{(2)}_{T,S} \).

By (2.2), we have that

\[
A^{(2)}_{T,S} = (A|_T)^{-1}P_{AT,S}.
\]

From the proof of uniqueness in the theorem above and Lemma 2.1, we have the following corollary.

**COROLLARY 2.3.** Let \( A \) and \( G \) be matrices over an associative ring \( R \). If the generalized inverse \( A^{(2)}_{T,S} \) exists, then

(i) \( A^{(2)}_{T,S}AG = G \) if and only if \( R(G) \subseteq T \);

(ii) \( GA^{(2)}_{T,S} = G \) if and only if \( N(G) \supset S \).

About the generalized inverse, we also have the following property.
Theorem 2.4. Let $A$ be a matrix over $R$. If $A_{T,S}^{(2)}$ exists and there exists a matrix $G$ over $R$ satisfying $R(G) = T$ and $N(G) = S$ then there exists a matrix $W$ over $R$ such that

\begin{align}
\text{(2.4)} & \quad GAGW = G, \\
\text{(2.5)} & \quad A_{T,S}^{(2)}AGW = A_{T,S}^{(2)}.
\end{align}

Proof. Suppose $A_{T,S}^{(2)}$ exists with $R(G) = T$ and $N(G) = S$ for a matrix $G$. Then $AR(G) \oplus N(G) = R^m$ and so there exists an epimorphism $R^m \rightarrow N(G) \rightarrow 0$. By [1, Theorem 8.1], $N(G)$ has a finite spanning set whose elements constitute a matrix, denoted by $L$. Thus $GL = 0$, and the columns of $(AG, L)$ generate $R^m$, that is, there exists a matrix $(W^T, W_1^T)^T$ such that

$$AGW + LW_1 = I_m.$$ 

If we multiply the left hand side by $G$ and $A_{T,S}^{(2)}$ respectively, then we obtain (2.4) and (2.5). \[\square\]

The following theorem shows that for an arbitrary matrix $A$ over an associative ring, $A^+, A_d$ and $A_g$, if they exist, are all the generalized inverse $A_{T,S}^{(2)}$.

Theorem 2.5. (i) Let $A$ be an $m \times n$ matrix over $R$ and let $*$ be an involution on the matrices over $R$. If $A^*$ exists, then $A^+ = A_{R(A^*), N(A^*)}^{(2)}$.

(ii) Let $A$ be an $n \times n$ matrix over $R$, and $k = \text{Ind}(A)$. If $A_d$ exists, then $A_d = A_{R(A^+), N(A^+)}^{(2)}$.

(iii) Let $A$ be an $n \times n$ matrix over $R$. If $A_g$ exists, then $A_g = A_{R(A), N(A)}^{(2)}$.

Proof. (i) Since $A^+ \in A[1, 2]$ and $A^* \in A^+[1, 2]$, we easily see that

\begin{align}
R(A^+) &= R(A^+A) = R((A^*)^*) = R(A^*A^+) = R(A^*), \\
N(A^+) &= N(AA^+) = N((A^A)^*) = N(A^*A^+) = N(A^*),
\end{align}

and $N(A) = N(A^+A)$.

Since $AA^+$ and $A^+A$ are idempotent, we have

\begin{align}
R^m &= R(AA^+) \oplus N(AA^+) = AR(A^+) \oplus N(AA^+) = AR(A^+) \oplus N(A^*), \\
N(A) \cap R(A^*) &= N(A^+A) \cap R(A^+A) = \{0\}
\end{align}

by [1, Lemma 5.6]. So, by Theorem 2.2, $A_{R(A^*), N(A^*)}^{(2)}$ exists and $A^+ = A_{R(A^*), N(A^*)}^{(2)}$. 

\[\square\]
(ii) Firstly, we shall show that
\[ R(A_d) = R(AA_d) = R(A^l) \quad \text{and} \quad N(A_d) = N(AA_d) = N(A^l) \]
for any positive integer \( l \geq k \). Since
\[ R(A_d) = R(AA_d^2) \subseteq R(AA_d) = R(A_dA) \subseteq R(A_d), \]
we have \( R(A_d) = R(AA_d) \) and so
\[ R(AA_d) = AR(A_d) = AR(AA_d) = A^2R(A_d). \]
It is easy to obtain inductively that \( R(AA_d) = A^hR(A_d) \) for any positive integer \( h \). This gives us that \( R(A_d) = R(AA_d) = R(A^l) \) for any positive integer \( l \geq k \). Also, since for any positive integer \( l \geq k \),
\[ N(A_d) \subseteq N(A^{l+1}A_d) = N(A^l) \subseteq N(A_d^lA^l) = N(A_dA) \subseteq N(A_d^2A) = N(A_d), \]
we get that \( N(A_d) = N(AA_d) = N(A^l) \).

Since \( AA_d \) is idempotent, by [1, Lemma 5.6], we have
\[ R^n = R(AA_d) \oplus N(AA_d) = AR(A^k) \oplus N(A^k) = R^n. \]
Since
\[ N(A) \cap R(A^k) \subseteq N(A^k) \cap R(A^{k+1}) = \{0\}, \]
\( A^{(2)}_{R(A^k),N(A^k)} \) exists and \( A^{(2)}_{R(A^k),N(A^k)} = A_d \) by Theorem 2.2.

(iii) Take \( k = 1 \) in (ii).

3. The generalized inverse \( A^{(1,2)}_{T,S} \)

If the generalized inverse \( A^{(2)}_{T,S} \) satisfies \( AA^{(2)}_{T,S}A = A \) then it is called the generalized inverse which is a \( (1,2) \) inverse of a matrix \( A \) over \( R \) with prescribed image \( T \) and kernel \( S \), and is denoted by \( A^{(1,2)}_{T,S} \). (Its uniqueness is guaranteed by the following theorem.)

**THEOREM 3.1.** Let \( A \) be an \( m \times n \) matrix over an associative ring \( R \) with identity and \( T \subseteq R^n \) and \( S \subseteq R^m \). Then the following conditions are equivalent.

(i) \( AT \oplus S = R^m, \ N(A) \cap T = \{0\} \) and \( N(A) \cap S = \{0\} \).

(ii) \( R(A) \oplus S = R^m, \ N(A) \oplus T = R^n. \)

(iii) There exists some \( X \in R^{n \times m} \) such that
\[ AXA = A, \quad XAX = X, \quad R(X) = T, \quad N(X) = S. \]

If these conditions are satisfied then \( X \) is unique.
The generalized inverse $A_{T,S}^{(2)}$

PROOF. (ii) $\Rightarrow$ (i) It is obvious that $R(A) \cap S = \{0\}$ and $N(A) \cap T = \{0\}$. To obtain $AT \oplus S = R^m$, it suffices to prove $AT = R(A)$.

Obviously, $AT \subset R(A)$. For any $x \in R(A)$, we have $x = Ay$, where $y \in R^n$. Since $N(A) \oplus T = R^n$, we can write $y = y_1 + y_2$, where $y_1 \in N(A), y_2 \in T$. Thus,

$$x = Ay = Ay_1 + Ay_2 = Ay_2 \in AT,$$

and therefore $R(A) \subset AT$. Consequently, $AT = R(A)$.

(i) $\Rightarrow$ (iii) By Theorem 2.2, from $AT \oplus S = R^m$ and $N(A) \cap T = \{0\}$, we know that $X = A_{T,S}^{(2)}$ exists and that $R(X) = T$ and $N(X) = S$. We shall show $AXA = A$.

Since $XAX = X$, we have $XAXA = XA$ and then $X(AXA - A) = 0$. So

$$R(AXA - A) \subset R(A) \cap N(X) = R(A) \cap S = \{0\}.$$ 

Hence $AXA = A$.

(iii) $\Rightarrow$ (ii) From (iii), we have $(AX)^2 = AX$, $(XA)^2 = XA$, and

$$N(X) \subset N(AX) \subset N(XAX) = N(X),$$

$$N(XA) \subset N(XAXA) = N(A) \subset N(XA),$$

$$R(XA) \subset R(X) = R(XAX) \subset R(XA),$$

$$R(AX) \subset R(A) = R(XAXA) \subset R(AX).$$

So

$$N(AX) = N(X) = S, \quad N(XA) = N(A),$$

$$R(XA) = R(X) = T, \quad R(AX) = R(A).$$

By [1, Lemma 5.6] and the four equations above, we reach (ii).

By Theorem 2.2, $X$ is unique. 

The next result is concerning the equivalent conditions in Theorem 3.1.

THEOREM 3.2. Let $A$ be an $m \times n$ matrix over an associative ring $R$ with identity and $T \subset R^n$ and $S \subset R^m$.

(i) If $N(A) + T = R^n$ then $AT = R(A)$.

(ii) If $AT \oplus S = R^m$ then

$$AT = R(A) \quad \text{if and only if} \quad R(A) \cap S = \{0\}.$$ 

PROOF. (i) From the proof of the theorem above (ii) implies (i).

(ii) Suppose that $R(A) \cap S = \{0\}$. Obviously, $AT \subset R(A)$. Now we will show the inclusion in reverse. For any $x \in R(A), x = x_1 + x_2 \in R^m = AT \oplus S,$
where \( x_1 \in AT, x_2 \in S \). By \( AT \subset R(A), x_1 \in R(A) \). So

\[
x_2 = x - x_1 \in R(A) \cap S = \{0\}.
\]

Therefore, \( x_2 = 0 \) and then \( x = x_1 \in AT \). Hence \( R(A) \subset AT \).

Conversely, suppose that \( AT = R(A) \). Since \( AT \oplus S = R^n \) and \( AT = R(A) \), we have \( R(A) \cap S = AT \cap S = \{0\} \). \( \square \)

We denote the maximal order of a nonvanishing minor of \( A \) over a commutative ring \( R \) by \( \rho(A) \). This is called the **determinantal rank** of \( A \). Obviously \( \rho(AB) \leq \min\{\rho(A), \rho(B)\} \) (see [9, Theorem 2.3]). When \( R \) is the complex number field, \( \rho(A) = \text{rank}(A) \).

**Theorem 3.3.** Let \( A \) be an \( m \times n \) matrix over an integral domain \( R \) and \( T \subset R^n \) and \( S \subset R^m \) be free submodules. If \( AT \oplus S = R^m \) then the following conditions are equivalent.

(i) \( N(A) \cap T = \{0\} \) and \( R(A) \cap S = \{0\} \).

(ii) \( \text{dim}(T) = \rho(A) \) and \( \text{dim}(S) = m - \text{dim}(T) \).

**Proof.** Suppose that (i) holds and let the columns of \( U \) be a basis of \( T \). From the proof of [13, Theorem 2], we have \( \text{dim}(T) = \text{dim}(AT) = \rho(AU) \leq \rho(A) \) and \( \text{dim}(S) = m - \text{dim}(T) \). By Theorem 3.2, \( AT = R(A) \). Thus there exists a matrix \( X \) over \( R \) such that \( A = AU X \). Thus \( \rho(A) \leq \rho(AU) = \text{dim}(AT) \). Therefore \( \rho(A) = \text{dim}(AT) = \text{dim}(T) \).

Conversely, suppose that (ii) holds. We have that \( \text{dim}(T) = \text{dim}(AT) \) from the proof of [13, Theorem 2]. Thus \( \rho(A) = \text{dim}(T) = \text{dim}(AT) \). By [12, Lemma 1], the maximal number of linearly independent columns of \( A \) is \( \text{dim}(AT) \). Since \( AT \subset R(A), R(A) + S = R^m \). Over the quotient field \( F \) of \( R \), \( AT = R(A) \) because \( \rho(A) = \text{dim}(AT) \), and \( R(A) \oplus S = R^m \). Therefore \( x \) and \( y \) are linear independent over \( F \) for any \( x \in R(A), y \in S \).

On the other hand, over an integral domain \( R \), suppose that \( 0 \neq z \in R(A) \cap S \). Then there exist \( r_i \in R, i = 1, \ldots, s \), such that

\[
3.1 \quad z = \sum_{i=1}^{s} \beta_i r_i,
\]

where \( \{\beta_1, \beta_2, \ldots, \beta_s\} \) is a basis of \( S \) and \( s = \text{dim}(S) \). But Equation (3.1) is true over \( F \). This is in contradiction to the above reasoning. Hence \( R(A) \cap S = \{0\} \).

The remainder of the proof is obtained from [13, Theorem 2]. \( \square \)

**Remark 1.** A module over the field of complex numbers is a vector space. So when \( R \) is the field of complex numbers, the above theorem ensures that Theorem 3.1 extends [2, Corollary 2.10].
4. Explicit expressions for $A_{T,S}^{(1,2)}$

We now consider some explicit expressions for $A_{T,S}^{(1,2)}$, which reduce to the group inverse or $\{1\}$ inverses. Firstly we shall prove the following lemma. In the proof, we use the following fact.

**Proposition 4.1.** If $e$ is idempotent in a ring $R$ with identity $1$ and $x, y \in eR$ then $xy = e$ if and only if $(x + 1 - e)(y + 1 - e) = 1$.

**Lemma 4.2.** Let $A$ be an $m \times n$ von Neumann regular matrix over $R$ and $G$ an $n \times m$ matrix over $R$. Then $U = AGAA^{(1)} + I_m - AA^{(1)}$ is invertible if and only if $V = A^{(1)}AGA + I_n - A^{(1)}A$ is invertible.

**Proof.** If $U$ is invertible then there exists an $X$ such that $UX = XU = I_m$. That is,

$$(AGAA^{(1)} + I_m - AA^{(1)})X = I_m \quad \text{and} \quad X(AGAA^{(1)} + I_m - AA^{(1)}) = I_m.$$ 

Multiplying on the left by $A^{(1)}AA^{(1)}$ and the right by $A$ and, since $A = AA^{(1)}A$, we have

$$(A^{(1)}AGA)(A^{(1)}AA^{(1)}XA) = A^{(1)}A \quad \text{and} \quad (A^{(1)}AA^{(1)}XA)(A^{(1)}AGA) = A^{(1)}A.$$ 

Since $A^{(1)}AGA = A^{(1)}A(GA)A^{(1)}A$ and $A^{(1)}AA^{(1)}XA = A^{(1)}A(A^{(1)}XA)A^{(1)}A$, we know that $A^{(1)}AGA$ has the inverse matrix $A^{(1)}AA^{(1)}XA$ in $A^{(1)}AM_n(R)A^{(1)}A$. Thus $V = A^{(1)}AGA + I_n - A^{(1)}A$ has the inverse matrix

$$A^{(1)}A(A^{(1)}AA^{(1)}XA)A^{(1)}A + I_n - A^{(1)}A \quad \text{in} \quad M_n(R).$$

The proof of the converse is analogous. □

Next we shall show the main result of this section. The following theorem not only shows some explicit expressions for $A_{T,S}^{(1,2)}$ which reduce to the group inverse or $\{1\}$ inverses, but also gives some equivalent conditions for the existence of $A_{T,S}^{(1,2)}$.

**Theorem 4.3.** Let $A$ be an $m \times n$ matrix over $R$ and $G$ an $n \times m$ matrix over $R$. Then the following conditions are equivalent.

(i) $A$ is von Neumann regular, $U = AGAA^{(1)} + I_m - AA^{(1)}$ is invertible and $N(A) \cap R(G) = \{0\}$.

(ii) $A$ is von Neumann regular, $V = A^{(1)}AGA + I_n - A^{(1)}A$ is invertible and $N(A) \cap R(G) = \{0\}$.

(iii) $A_{R(G), N(G)}^{(1,2)}$ exists.
When these conditions are satisfied we have

\[(4.1) \quad A_{R(G), N(G)}^{(1,2)} = G(AG)_g = (GA)_g G\]
\[(4.2) \quad = G(GAG)^{(1)} G\]
\[(4.3) \quad = G(AG)^{(1)} A(GA)^{(1)} G\]
\[(4.4) \quad = GU^{-2}AG = GU^{-1}AV^{-1}G = GAV^{-2}G.\]

**PROOF.** (i) and (ii) are equivalent by Lemma 4.2.

To show that (ii) implies (iii), set \(B = AV^{-2}G\). Using \(UA = AGA = AV\), we have \(B = (AG)_g\) because

\[B(AG) = AV^{-2}GAG = U^{-2}AGAG = U^{-1}AG = AV^{-1}G = AGAV^{-2}G\]
\[= (AG)B,\]
\[B(AG)B = U^{-1}AG(AV^{-2}G) = AV^{-2}G = B,\]
\[(AG)B(AG) = (AG)AV^{-1}G = AG.\]

Analogously, we deduce that \((GA)_g\) exists and \((GA)_g = GU^{-2}A\). Let \(X = G(AG)_g\).

It is obvious that

\[(4.5) \quad XAX = X.\]

Since

\[AG = (AG)^2(AG)_g = AGAX,\]
we have \(A(G - GAX) = 0\) and then

\[R(G - GAX) = R(G(I - AX)) \subset N(A) \cap R(G) = \{0\}.\]

Therefore

\[(4.6) \quad G = GAX\]
\[= GA(G(AG)_g) = G(AG)_g AG\]
\[(4.7) \quad = XAG.\]

Using (4.6) and (4.7), we have

\[(4.8) \quad R(X) = R(G) \quad \text{and} \quad N(X) = N(G).\]

Since \(AV = AGA\), we get

\[A = AGAV^{-1} = AG(AG)_g AGAV^{-1} = AXA.\]

Using the equation above, together with (4.5) and (4.8), we deduce that \(A_{R(G), N(G)}^{(1,2)}\) exists and \(A_{R(G), N(G)}^{(1,2)} = X = G(AG)_g\) by Theorem 3.1.
To show that (iii) implies (i), we use Theorem 2.4 to obtain

\[(AGAA^{(1)})(AGW^2AA^{(1)}) = AGAGW^2AA^{(1)} = AGWAA^{(1)}\]
\[= AA^{(1,2)}_{R(G),N(G)}AGWAA^{(1)} = AA^{(1,2)}_{R(G),N(G)}AA^{(1)}\]
\[(4.9) = AA^{(1)}.\]

Therefore,

\[(AGAA^{(1)})(AGW^2AA^{(1)}) (AGAA^{(1)}) = AA^{(1)}(AGAA^{(1)}) = AGAA^{(1)}\]

and then

\[AG\left((AGW^2AA^{(1)})(AGAA^{(1)}) - AA^{(1)}\right) = 0.\]

By Theorem 3.1, \(R(A) \cap N(G) = \{0\}\) and \(N(A) \cap R(G) = \{0\}\) and so

\[R\left(G\left((AGW^2AA^{(1)})(AGAA^{(1)}) - AA^{(1)}\right)\right) \subset R(G) \cap N(A) = \{0\}.\]

Thus

\[G\left((AGW^2AA^{(1)})(AGAA^{(1)}) - AA^{(1)}\right) = 0.\]

From this, we have

\[R\left((AGW^2AA^{(1)})(AGAA^{(1)}) - AA^{(1)}\right) \subset R(A) \cap N(G) = \{0\},\]

and then

\[(4.10) (AGW^2AA^{(1)})(AGAA^{(1)}) = AA^{(1)}.\]

By (4.9) and (4.10), \(AGAA^{(1)}\) is invertible in \(AA^{(1)}M_m(R)AA^{(1)}\) and so is \(U\) in \(M_m(R)\).

Also, obviously, \(A\) is von Neumann regular.

Now we shall prove that (4.1) \(\sim\) (4.3). Since

\[G(AG)_g = G(AV^{-2}G) = GU^{-1}AV^{-1}G = (GU^{-2}A)G = (GA)_gG,\]

we have \(A_{R(G),N(G)}^{(1,2)} = (GA)_gG\) and (4.4).

Next we will prove (4.2). Since

\[GAG = GAG ((AG)_g)^2 AGAG,\]

\(GAG\) is von Neumann regular and then

\[AG = U^{-1}UAG = U^{-1}AGAG = (U^{-1}A)GAG(GAG)^{(1)}GAG\]
\[= AG(GAG)^{(1)}GAG.\]
Therefore
\[ A(G - G(GAG)^{(1)}GAG) = 0. \]
Thus
\[ R(G - G(GAG)^{(1)}GAG) \subseteq N(A) \cap R(G) = \{0\}. \]
So we obtain
\[ G = G(GAG)^{(1)}GAG \]
Since \(A_{R(G),N(G)}^{(1,2)}\) exists, using (2.4) and (4.11), it follows that
\[ G = GAGW = GAG(GAG)^{(1)}GAGW \]
\[ = GAG(GAG)^{(1)}G. \]
Let \(Z = G(GAG)^{(1)}G\). Using (4.11) and (4.12), it easily follows that \(ZA = Z\), \(AZA = A\), \(R(Z) = R(G)\) and \(N(Z) = N(G)\). By Theorem 3.1 we have that \(A_{R(G),N(G)}^{(1,2)} = Z = G(GAG)^{(1)}G\).

Finally, we will verify (4.3). It is obvious that \(AG\) and \(GA\) are von Neumann regular. By Proposition 4.1 and the invertibility of \(V\) there exists a matrix \(P \in A^{(1)}A_{M_n(R)}A^{(1)}\) such that \(P(A^{(1)}AGA) = A^{(1)}A\). Thus
\[ A = A(PA^{(1)}AGA) = APA^{(1)}A(GA(GAG)^{(1)}GAG) = A(GA)^{(1)}G. \]
Using (4.13), we deduce that \((AG)^{(1)}A(GA)^{(1)}\) is a \(\{1\}\) inverse of \(GAG\). Therefore, using (4.2), we obtain (4.3). \(\square\)

Remark 2. By (4.4), we can compute \(A_{R(G),N(G)}^{(1,2)}\) using \(U\) or \(V\).

Remark 3. If \(G = A\) where \(A\) is such that \(V = A^{(1)}A^2 + I_n - A^{(1)}A\) is invertible, then \(N(A) \cap R(A) = \{0\}\). Indeed, let \(x \in N(A) \cap R(A)\). Then there exists a \(y \in R^n\) such that \(x = Ay\) and so \(A^2y = 0\). Since \(V\) is invertible, there exists a matrix \(P\) such that \(PV = I_n\). Thus \(PA^{(1)}A^3 = A^{(1)}A\) and then
\[ 0 = PA^{(1)}A^3y = A^{(1)}Ay. \]
Hence \(Ay = AA^{(1)}Ay = 0\). Consequently, \(x = Ay = 0\).

Similarly, if we take \(G = A^*\), where \(\ast\) is an involution on the matrices over \(R\) such that \(U = AA^*A^{(1)} + I_m - AA^{(1)}\) is invertible, then \(N(A) \cap R(A^*) = \{0\}\). Indeed, let \(x \in N(A) \cap R(A^*)\). Then there exists a \(y \in R^n\) such that \(x = A^*y\) and so \(AA^*y = 0\). Since \(U\) is invertible, there exists a matrix \(Q\) such that \(AA^*AA^{(1)}Q = AA^{(1)}\) and thus
\[ 0 = Q^*(A^{(1)})^*A^*AA^*y = (A^{(1)})^*A^*y. \]
So \(x = A^*y = A^*(A^{(1)})^*A^*y = 0\).
When $G$ takes the value $A$ (respectively $A^*$) in the theorem above, we find that $A_{R(G),N(G)}^{(1,2)}$ is $A_g$ (respectively $A^*$).

**THEOREM 4.4.** Let $A$ be an $m \times n$ matrix over $R$. Then

(i) $A_{R(A),N(A)}^{(1,2)}$ exists if and only if $A_g$ exists. Moreover, $A_{R(A),N(A)}^{(1,2)} = A_g$.

(ii) If $*$ is an involution on the matrices over $R$ then $A_{R(A^*),N(A^*)}^{(1,2)}$ exists if and only if $A^*$ exists. Moreover, $A_{R(A^*),N(A^*)}^{(1,2)} = A^*$.

**PROOF.** To show the existence of $A_{R(A),N(A)}^{(1,2)}$ implies existence of $A_g$ in (i), take $G = A$ in (4.1). Then $A_{R(A),N(A)}^{(1,2)} = A(A^2)_g = (A^2)_g A$ and then $A A_{R(A),N(A)}^{(1,2)} = A_{R(A),N(A)}^{(1,2)} A$. Hence $A_{R(A^*),N(A^*)}^{(1,2)}$ is the group inverse of $A$.

To show that existence of $A_{R(A^*),N(A^*)}^{(1,2)}$ implies existence of $A^*$ in (ii), take $G = A^*$ in (4.1). Then $A_{R(A^*),N(A^*)}^{(1,2)} = A^*(AA^*)_g = (A^*A)_g A^*$ and then

$$
(A A_{R(A^*),N(A^*)}^{(1,2)})^* = A A_{R(A^*),N(A^*)}^{(1,2)}
$$

and

$$
(A_{R(A^*),N(A^*)}^{(1,2)} A)^* = A_{R(A^*),N(A^*)}^{(1,2)} A.
$$

Hence $A_{R(A^*),N(A^*)}^{(1,2)}$ is the Moore-Penrose inverse of $A$.

The converses follow from Theorem 2.5. \qed

By Theorems 4.3 and 4.4 and Remark 3, we can obtain the following two corollaries, in which the first is equivalent to [8, Corollary 2] and the second is almost the same as [6, Theorem 1]:

**COROLLARY 4.5.** Let $A \in R^{n \times n}$. The following conditions are equivalent.

(i) $A$ is von Neumann regular and $U = A^2 A^{(1)} + I_n - A A^{(1)}$ is invertible.

(ii) $A$ is von Neumann regular and $V = A^{(1)} A^2 + I_n - A^{(1)} A$ is invertible.

(iii) $A_g$ exists.

Moreover,

$$
A_g = A (A^2)_g = (A^2)_g A
$$

$$
= A (A^3)^{(1)} A
$$

$$
= A (A^2)^{(1)} A (A^2)^{(1)} A.
$$

$$
= A U^{-2} A^2 = A U^{-1} A V^{-1} A = A^2 V^{-2} A.
$$

**REMARK 4.** The above corollary is unlike [8, Corollary 2], but they are equivalent. This is because $V$ is invertible if and only if $T = A^{(1)} A^2 + I_n - A^{(1)} A$ is invertible. Indeed, if $V$ is invertible, then there exists a matrix $P \in M_n(R)$ such that $PV = VP = I_n$. From this and $V = T^2$, we get $(PT)T = T(TP) = I_n$. Hence $T$ is invertible in $M_n(R)$. The converse is obvious from $V = T^2$.
COROLLARY 4.6. Let $A$ be an $m \times n$ matrix over $R$ and let $*$ be an involution on the matrices over $R$. The following conditions are equivalent.

(i) $A$ is von Neumann regular and $U = AA^*AA^{(1)} + I_n - AA^{(1)}$ is invertible.
(ii) $A$ is von Neumann regular and $V = A^{(1)}AA^* + I_n - A^{(1)}A$ is invertible.
(iii) $A^\dagger$ exists.

Moreover,

$$A^\dagger = A^*(AA^*)^\alpha = (A^*A)^\alpha A^* = A^*(A^*AA^*)^{(1)}A^* = A^*(AA^*)^{(1)}A(A^*A)^{(1)}A^* = A^*U^{-2}AA^* = A^*U^{-1}AV^{-1}A^* = A^*AV^{-2}A^*.$$

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