

## ON THE MELLIN TRANSFORM OF A PRODUCT OF HYPERGEOMETRIC FUNCTIONS

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### Abstract

We obtain representations for the Mellin transform of the product of generalized hypergeometric functions  ${}_0F_1[-a^2x^2]{}_1F_2[-b^2x^2]$  for  $a, b > 0$ . The later transform is a generalization of the discontinuous integral of Weber and Schafheitlin; in addition to reducing to other known integrals (for example, integrals involving products of powers, Bessel and Lommel functions), it contains numerous integrals of interest that are not readily available in the mathematical literature. As a by-product of the present investigation, we deduce the second fundamental relation for  ${}_3F_2[1]$ . Furthermore, we give the sine and cosine transforms of  ${}_1F_2[-b^2x^2]$ .

### 1. Introduction

Although definite integrals of products of two generalized hypergeometric functions have numerous applications in pure and applied mathematics (see, for example, [3]), not all such integrals have been collected in tables or are readily available in the mathematical literature. In what follows, we shall consider for  $a > 0$  and  $b > 0$  one such rather general improper integral, namely the Mellin transform:

$$F(s) := \int_0^\infty x^{s-1} {}_0F_1 \left[ \begin{matrix} \phantom{\alpha} \\ 1 + \mu \end{matrix}; -a^2x^2 \right] {}_1F_2 \left[ \begin{matrix} \alpha \\ \beta, 1 + \nu \end{matrix}; -b^2x^2 \right] dx. \quad (1.1)$$

Convergence of this integral will be discussed in Section 2.

If  $\alpha = \beta$ , the above Mellin transform reduces to the hypergeometric formulation of the discontinuous integral of Weber and Schafheitlin (*cf.* [11, p. 398] and [6, Equations

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(2.3) to (2.5)) which, since we shall need it later, we record below:

$$\int_0^\infty x^{s-1} {}_0F_1 \left[ \frac{\quad}{1+\mu}; -a^2x^2 \right] {}_0F_1 \left[ \frac{\quad}{1+\nu}; -b^2x^2 \right] dx$$

$$= \frac{1}{2} b^{-s} \frac{\Gamma(1+\nu)\Gamma(\frac{s}{2})}{\Gamma(1+\nu-\frac{s}{2})} {}_2F_1 \left[ \frac{s}{2}, \frac{s}{2}-\nu; \frac{a^2}{b^2} \right] \tag{1.2a}$$

$(0 < a < b; 0 < \Re(s) < \Re(2 + \mu + \nu))$

$$= \frac{1}{2} a^{-s} \frac{\Gamma(\frac{s}{2})\Gamma(1+\mu)\Gamma(1+\nu)\Gamma(1+\mu+\nu-s)}{\Gamma(1+\mu-\frac{s}{2})\Gamma(1+\nu-\frac{s}{2})\Gamma(1+\mu+\nu-\frac{s}{2})} \tag{1.2b}$$

$(b = a; 0 < \Re(s) < \Re(1 + \mu + \nu))$

$$= \frac{1}{2} a^{-s} \frac{\Gamma(1+\mu)\Gamma(\frac{s}{2})}{\Gamma(1+\mu-\frac{s}{2})} {}_2F_1 \left[ \frac{s}{2}, \frac{s}{2}-\mu; \frac{b^2}{a^2} \right] \tag{1.2c}$$

$(0 < b < a; 0 < \Re(s) < \Re(2 + \mu + \nu)).$

The above integral is, in fact, continuous at  $a = b$ . It is called “discontinuous” because the representations on the right-hand sides of Equations (1.2a) and (1.2c) are not analytic continuations of each other. Thus “discontinuous” refers to a discontinuity in representation across  $a = b$ . It should be noted also that when  $a = b$ , if  $\mu - \nu$  is an odd positive integer, then Equation (1.2b) actually holds true for  $0 < \Re(s) < \Re(2 + \mu + \nu)$  (see Watson’s treatise [11, p. 403] for further details).

In addition, we record for later use the well-known Mellin transform (see, for example, [7, Section 8.4.47.(1), p. 717])

$$\int_0^\infty x^{s-1} {}_0F_1 \left[ \frac{\quad}{1+\mu}; -a^2x^2 \right] dx = \frac{1}{2} a^{-s} \frac{\Gamma(1+\mu)\Gamma(\frac{s}{2})}{\Gamma(1+\mu-\frac{s}{2})}, \tag{1.3}$$

where  $a > 0$  and  $0 < \Re(s) < \Re(\frac{3}{2} + \mu)$ .

Since we may write Bessel, Struve and Lommel functions, respectively, as (see, for example, [10, p. 44])

$$J_\mu(z) = \frac{(\frac{z}{2})^\mu}{\Gamma(1+\mu)} {}_0F_1 \left[ \frac{\quad}{1+\mu}; -\frac{z^2}{4} \right], \tag{1.4}$$

$$H_\mu(z) = \frac{(\frac{z}{2})^{1+\mu}}{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2}+\mu)} {}_1F_2 \left[ \frac{1}{2}, \frac{3}{2}+\mu; -\frac{z^2}{4} \right],$$

$$s_{\mu,\nu}(z) = \frac{z^{1+\mu}}{(1+\mu-\nu)(1+\mu+\nu)} {}_1F_2 \left[ \frac{1}{2}(3+\mu-\nu), \frac{1}{2}(3+\mu+\nu); -\frac{z^2}{4} \right],$$

once a representation in terms of generalized hypergeometric functions is deduced for the Mellin transform  $F(s)$  (which will be done in Section 5), not only will we be able

to obtain immediately all the known special cases (for example, integrals containing the product of two Bessel functions or the integral of Weber and Schafheitlin [11, pp. 401–403], products of Bessel and Lommel functions [7, Section 2.9.5.(1), p. 110], and products of Bessel and Struve functions [7, Section 2.7.14.(1), p. 88]), but we shall also be able to specialize, for example, to products of Bessel functions and the Fresnel sine and cosine integrals

$$S(z) = \frac{1}{3} \sqrt{\frac{2}{\pi}} z^{3/2} {}_1F_2 \left[ \begin{matrix} \frac{3}{4}; \\ \frac{3}{2}, \frac{7}{4}; \end{matrix} -\frac{z^2}{4} \right]$$

and

$$C(z) = \sqrt{\frac{2}{\pi}} z^{1/2} {}_1F_2 \left[ \begin{matrix} \frac{1}{4}; \\ \frac{1}{2}, \frac{5}{4}; \end{matrix} -\frac{z^2}{4} \right],$$

and every other special case of  ${}_1F_2$  that is of interest.

Recently, M. A. Chaudhry [2] reconsidered  $F(s)$  specialized to Bessel and Lommel functions  $s_{\mu,\nu}(z)$  and discussed the importance of these integrals in many applications. Unfortunately, Chaudhry's results [2, Equation (4)] holds true for the Lommel functions  $S_{\mu,\nu}(z)$  and not for  $s_{\mu,\nu}(z)$  and the right side of [2, Equation (6)] is in error by a factor of  $\frac{1}{2}$ . Aside from numerous typographical and other oversights in [2], the latter two mentioned equations produced further erroneous results. Prudnikov *et al.* [7, Section 2.22.4.(1), p. 337] records the Mellin transform of products of Bessel and generalized hypergeometric functions  ${}_pF_q[z]$ , but the result is obviously not correct when  $p = q - 1$ , since the integral must be discontinuous.

## 2. Convergence of the Integral (1.1)

We shall need the following asymptotic results: for  $|x| \rightarrow \infty$  and  $|\arg(x)| < \frac{\pi}{2}$ ,

$${}_0F_1 \left[ \begin{matrix} \text{---}; \\ 1 + \mu; \end{matrix} -x^2 \right] = \frac{\Gamma(1 + \mu)}{\Gamma(\frac{1}{2})} x^{-\frac{1}{2} - \mu} \cos \left[ 2x - \frac{\pi}{2} \left( \frac{1}{2} + \mu \right) \right] \quad (2.1)$$

and

$${}_1F_2 \left[ \begin{matrix} \alpha; \\ \beta, 1 + \nu; \end{matrix} -x^2 \right] = \frac{\Gamma(\beta)\Gamma(1 + \nu)}{\Gamma(\frac{1}{2})\Gamma(\alpha)} x^{-\frac{1}{2} - \nu + \alpha - \beta} \cos \left[ 2x - \frac{\pi}{2} \left( \frac{1}{2} + \nu + \beta - \alpha \right) \right] \\ + \frac{\Gamma(\beta)\Gamma(1 + \nu)}{\Gamma(\beta - \alpha)\Gamma(1 + \nu - \alpha)} x^{-2\alpha}. \quad (2.2)$$

The above results are further simplifications of more precise asymptotic formulas which we shall record in Section 4 (see Equations (4.1) and (4.2)). Note that Equation (2.2) reduces to Equation (2.1) if we set  $\alpha = \beta$ .

Replacing  $x$  by  $bx$  ( $b > 0$ ) in Equation (2.2) and using Equation (2.1), it is easy to see that we may also write for  $|x| \rightarrow \infty$  and  $|\arg(x)| < \frac{\pi}{2}$

$$\begin{aligned}
 {}_1F_2 \left[ \begin{matrix} \alpha; \\ \beta, 1 + \nu; \end{matrix} -b^2x^2 \right] &= \frac{\Gamma(\beta)\Gamma(1 + \nu)}{\Gamma(\alpha)\Gamma(1 + \nu + \beta - \alpha)} {}_0F_1 \left[ \begin{matrix} \text{---} \\ 1 + \nu + \beta - \alpha; \end{matrix} -b^2x^2 \right] \\
 &+ \frac{\Gamma(\beta)\Gamma(1 + \nu)}{\Gamma(\beta - \alpha)\Gamma(1 + \nu - \alpha)} (bx)^{-2\alpha}, \tag{2.3}
 \end{aligned}$$

which now yields an identity when  $a = \beta$ .

Next, by multiplying Equation (2.3) by

$$x^{s-1} {}_0F_1 \left[ \text{---}; 1 + \mu; -a^2x^2 \right],$$

and, for  $z > 0$  sufficiently large, integrating the result over  $(z, \infty)$ , we have for  $a, b > 0$  the asymptotic result

$$\begin{aligned}
 &\int_z^\infty x^{s-1} {}_0F_1 \left[ \begin{matrix} \text{---} \\ 1 + \mu; \end{matrix} -a^2x^2 \right] {}_1F_2 \left[ \begin{matrix} \alpha; \\ \beta, 1 + \mu; \end{matrix} -b^2x^2 \right] dx \\
 &= \frac{\Gamma(\beta)\Gamma(1 + \nu)}{\Gamma(\alpha)\Gamma(1 + \nu + \beta - \alpha)} \int_z^\infty x^{s-1} {}_0F_1 \left[ \begin{matrix} \text{---} \\ 1 + \mu; \end{matrix} -a^2x^2 \right] \\
 &\quad \cdot {}_0F_1 \left[ \begin{matrix} \text{---} \\ 1 + \nu + \beta - \alpha; \end{matrix} -b^2x^2 \right] dx \\
 &\quad + \frac{\Gamma(\beta)\Gamma(1 + \nu)b^{-2\alpha}}{\Gamma(\beta - \alpha)\Gamma(1 + \nu - \alpha)} \int_z^\infty x^{s-2\alpha-1} {}_0F_1 \left[ \begin{matrix} \text{---} \\ 1 + \mu; \end{matrix} -a^2x^2 \right] dx. \tag{2.4}
 \end{aligned}$$

It is obvious that, in order for the Mellin transform  $F(s)$  defined by equation (1.1) to converge at its lower limit of integration, we must have  $\Re(s) > 0$ . Further, from the latter asymptotic result and the convergence criteria of equations (1.2) and (1.3), we see that, for  $b \neq a$ , convergence at the upper limit of integration is attained, provided that

$$\Re(s) < \Re(2 + \mu + \nu + \beta - \alpha) \quad \text{and} \quad \Re(s) < \Re\left(\frac{3}{2} + 2\alpha + \mu\right).$$

If  $b = a$ ,  $F(s)$  converges, provided that

$$0 < \Re(s) < \Re(1 + \mu + \nu + \beta - \alpha) \quad \text{and} \quad 0 < \Re(s) < \Re\left(\frac{3}{2} + 2\alpha + \mu\right);$$

but if  $b = a$  and  $\mu - \nu + \alpha - \beta$  is an odd positive integer, then  $F(s)$  converges, provided that

$$0 < \Re(s) < \Re(2 + \mu + \nu + \beta - \alpha) \quad \text{and} \quad 0 < \Re(s) < \Re\left(\frac{3}{2} + 2\alpha + \mu\right).$$

We note that if  $\alpha = \beta$ , the condition  $\Re(s) < \Re(\frac{3}{2} + 2\alpha + \mu)$  becomes superfluous, since the second term in equation (2.4) is no longer present. This evidently completes the analysis of the convergence of  $F(s)$ .

### 3. The Incomplete Mellin Transform

If the upper limit of integration in equation (1.1) is replaced by  $z$  such that  $|z| < \infty$ , the integral is said to be incomplete and we define the Euler-type integral

$$F(s; z) := \int_0^z x^{s-1} {}_0F_1 \left[ \begin{matrix} \text{---} \\ 1 + \mu; \end{matrix} ; -a^2x^2 \right] {}_1F_2 \left[ \begin{matrix} \alpha; \\ \beta, 1 + \nu; \end{matrix} ; -b^2x^2 \right] dx, \quad (3.1)$$

where, for convergence at the lower limit of integration,  $\Re(s) > 0$ . But now the parameters  $a$  and  $b$  may be arbitrary complex numbers.

The incomplete Mellin transform  $F(s; z)$  is easily evaluated by expressing the functions  ${}_0F_1$  and  ${}_1F_2$  as hypergeometric sums, interchanging the resulting double sum and integral and then performing the required term-by-term integrations. Or the integral in equation (3.1) may be evaluated by using a tabulated result in Exton’s handbook [3, Equation A(1.2.5), p. 172]. Either way we obtain  $F(s; z)$  in terms of a single Kampé de Fériet function which is everywhere convergent in its two independent variables:

$$F(s; z) = \frac{z^2}{s} F_{1:1;2}^{1:0;1} \left[ \begin{matrix} \frac{s}{2}; \text{---} \\ 1 + \frac{s}{2}; 1 + \mu; \beta, 1 + \nu; \end{matrix} ; \alpha; -a^2z^2, -b^2z^2 \right], \quad (3.2)$$

where  $\Re(s) > 0$ . Regions of convergence for Kampé de Fériet and other generalized hypergeometric functions in two variables may be determined by using Horn’s theorem for double series (see Srivastava and Karlsson [9, p. 57]).

Further, it is easy to show that the incomplete transform  $F(s; z)$  in equation (3.2) may be written in the following two different ways for  $\Re(s) > 0$ :

$$F(s; z) = \frac{z^2}{s} \sum_{n=0}^{\infty} \frac{\left(\frac{s}{2}\right)_n (\alpha)_n}{\left(1 + \frac{s}{2}\right)_n (\beta)_n (1 + \nu)_n} \frac{(-b^2z^2)^n}{n!} \cdot {}_1F_2 \left[ \begin{matrix} \frac{s}{2} + n; \\ 1 + \frac{s}{2} + n, 1 + \mu; \end{matrix} ; -a^2z^2 \right] \quad (3.3)$$

and

$$F(s; z) = \frac{z^2}{s} \sum_{n=0}^{\infty} \frac{\left(\frac{s}{2}\right)_n}{\left(1 + \frac{s}{2}\right)_n (1 + \mu)_n} \frac{(-a^2z^2)^n}{n!} \cdot {}_2F_3 \left[ \begin{matrix} \frac{s}{2} + n, \alpha; \\ 1 + \frac{s}{2} + n, \beta, 1 + \nu; \end{matrix} ; -b^2z^2 \right]. \quad (3.4)$$

### 4. Asymptotic Formulas for ${}_pF_{p+1}[-z^2]$

We shall want to evaluate  $F(s; z)$  given by equations (3.3) and (3.4) as  $z \rightarrow \infty$ , thereby obtaining the Mellin transform  $F(s)$  defined by equation (1.1). To this end, we record below somewhat simplified asymptotic formulas for the generalized hypergeometric function

$${}_pF_{p+1}[(a_p); (b_{p+1}); -z^2], \quad p = 0, 1, 2.$$

These results are special cases of a general formula due to C. S. Meijer (circa 1946) given by Luke [4, p. 203, Eq. (4)] for  $|z| \rightarrow \infty$  and  $|\arg(z)| < \pi/2$ :

$${}_0F_1 \left[ \begin{matrix} - \\ a; \end{matrix} -z^2 \right] = \frac{\Gamma(a)}{\Gamma(\frac{1}{2})} \left( \frac{1}{z^2} \right)^\eta \left[ 1 + O \left( \frac{1}{z^2} \right) \right] \cos \xi(z), \tag{4.1}$$

where

$$\eta = \frac{1}{2} \left( a - \frac{1}{2} \right) \quad \text{and} \quad \xi(z) = 2z - \pi\eta + O \left( \frac{1}{z} \right);$$

$$\begin{aligned} {}_1F_2 \left[ \begin{matrix} a; \\ b, c; \end{matrix} -z^2 \right] &= \frac{\Gamma(b)\Gamma(c)}{\Gamma(b-a)\Gamma(c-a)} \left( \frac{1}{z^2} \right)^a \\ &\quad \cdot {}_3F_0 \left[ \begin{matrix} a, 1+a-b, 1+a-c; \\ \text{---}; \end{matrix} -\frac{1}{z^2} \right] \\ &\quad + \frac{\Gamma(b)\Gamma(c)}{\Gamma(\frac{1}{2})\Gamma(a)} \left( \frac{1}{z^2} \right)^\eta \left[ 1 + O \left( \frac{1}{z^2} \right) \right] \cos \xi(z), \end{aligned} \tag{4.2}$$

where

$$\eta = \frac{1}{2} (b + c - a - \frac{1}{2}) \quad \text{and} \quad \xi(z) = 2z - \pi\eta + O \left( \frac{1}{z} \right);$$

and

$$\begin{aligned} {}_2F_3 \left[ \begin{matrix} a, b; \\ c, d, e; \end{matrix} -z^2 \right] &= \frac{\Gamma(b-a)}{\Gamma(b)} \frac{\Gamma(c)\Gamma(d)\Gamma(e)}{\Gamma(c-a)\Gamma(d-a)\Gamma(e-a)} \left( \frac{1}{z^2} \right)^a \\ &\quad \cdot {}_4F_1 \left[ \begin{matrix} a, 1+a-c, 1+a-d, 1+a-e; \\ 1+a-b; \end{matrix} -\frac{1}{z^2} \right] \\ &\quad + \frac{\Gamma(a-b)}{\Gamma(a)} \frac{\Gamma(c)\Gamma(d)\Gamma(e)}{\Gamma(c-b)\Gamma(d-b)\Gamma(e-b)} \left( \frac{1}{z^2} \right)^b \\ &\quad \cdot {}_4F_1 \left[ \begin{matrix} b, 1+b-c, 1+b-d, 1+b-e; \\ 1+b-a; \end{matrix} -\frac{1}{z^2} \right] \\ &\quad + \frac{\Gamma(c)\Gamma(d)\Gamma(e)}{\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)} \left( \frac{1}{z^2} \right)^\eta \left[ 1 + O \left( \frac{1}{z^2} \right) \right] \cos \xi(z), \end{aligned} \tag{4.3}$$

where

$$\eta = \frac{1}{2}(c + d + e - a - b - \frac{1}{2}) \quad \text{and} \quad \xi(z) = 2z - \pi\eta + O(\frac{1}{z}).$$

Noting equation (1.4), we see that equation (4.1) is essentially the well-known asymptotic formula for Bessel functions. We remark also that the right members of equations (4.1) to (4.3) may be written in further abbreviated, yet useful, approximate forms (cf. [5, p. 146])

$${}_0F_1 \left[ \begin{matrix} - \\ a; \end{matrix} -z^2 \right] \sim Az^{\frac{1}{2}-a} \cos(2z + B), \tag{4.4}$$

$${}_1F_2 \left[ \begin{matrix} a; \\ b, c; \end{matrix} -z^2 \right] \sim Az^{-2a} + Bz^{\frac{1}{2}+a-b-c} \cos(2z + C) \tag{4.5}$$

and

$${}_2F_3 \left[ \begin{matrix} a, b; \\ c, d, e; \end{matrix} -z^2 \right] \sim Az^{-2a} + Bz^{-2b} + Cz^{\frac{1}{2}+a+b-c-d-e} \cos(2z + D),$$

where  $|z| \rightarrow \infty$ ,  $|\arg(z)| < \pi/2$ , and  $A, B, C, D$  are dependent on the parameters of the function  ${}_pF_{p+1}[-z^2]$  ( $p = 0, 1, 2$ ).

### 5. Evaluation of the Mellin Transform (1.1)

Now that we have noted and developed some preliminaries, we are ready to derive our main result for the Mellin transform  $F(s)$  defined in equation (1.1) as

$$F(s) := \int_0^\infty x^{s-1} {}_0F_1 \left[ \begin{matrix} - \\ 1 + \mu; \end{matrix} -a^2x^2 \right] {}_1F_2 \left[ \begin{matrix} \alpha; \\ \beta, 1 + \nu; \end{matrix} -b^2x^2 \right] dx,$$

where  $a > 0$  and  $b > 0$ . Thus we shall show that

$$F(s) = \frac{1}{2}a^{-s} \frac{\Gamma(\frac{s}{2})\Gamma(1 + \mu)}{\Gamma(1 + \mu - \frac{s}{2})} {}_3F_2 \left[ \begin{matrix} \alpha, \frac{s}{2}, \frac{s}{2} - \mu; \\ \beta, 1 + \nu; \end{matrix} \frac{b^2}{a^2} \right] \quad (b < a) \tag{5.1a}$$

and

$$\begin{aligned} F(s) &= \frac{1}{2}b^{-s} \frac{\Gamma(\frac{s}{2})\Gamma(\alpha - \frac{s}{2})\Gamma(\beta)\Gamma(1 + \nu)}{\Gamma(\alpha)\Gamma(\beta - \frac{s}{2})\Gamma(1 + \nu - \frac{s}{2})} {}_3F_2 \left[ \begin{matrix} \frac{s}{2}, 1 + \frac{s}{2} - \beta, \frac{s}{2} - \nu; \\ 1 + \frac{s}{2} - \alpha, 1 + \mu; \end{matrix} \frac{a^2}{b^2} \right] \\ &\quad + \frac{1}{2}a^{-s} \left( \frac{a^2}{b^2} \right)^\alpha \frac{\Gamma(1 + \mu)\Gamma(1 + \nu)\Gamma(\beta)\Gamma(\frac{s}{2} - \alpha)}{\Gamma(1 + \nu - \alpha)\Gamma(1 + \mu + \alpha - \frac{s}{2})\Gamma(\beta - \alpha)} \\ &\quad \cdot {}_3F_2 \left[ \begin{matrix} \alpha, 1 + \alpha - \beta, \alpha - \nu; \\ 1 + \alpha - \frac{s}{2}, 1 + \mu + \alpha - \frac{s}{2}; \end{matrix} \frac{a^2}{b^2} \right] \quad (a < b), \end{aligned} \tag{5.1b}$$

where, for convergence of the integral (see Section 2),

$$0 < \Re(s) < \Re(2 + \mu + \nu + \beta - \alpha)$$

and

$$0 < \Re(s) < \Re\left(\frac{3}{2} + 2\alpha + \mu\right).$$

If  $\alpha = \beta$ , the latter conditional inequality is superfluous and equations (5.1a) and (5.1b) reduce, respectively, to equations (1.2c) and (1.2a). The case  $a = b$  will be discussed in Section 6. In Section 7 we shall use equations (5.1) to derive the sine and cosine transforms of  ${}_1F_2[-b^2x^2]$ .

To derive equation (5.1a), we approximate  ${}_1F_2[-a^2z^2]$  in equation (3.3) for large positive  $z$  by using equation (4.2), thus obtaining

$$\begin{aligned} {}_1F_2\left[1 + \frac{s}{2} + n, 1 + \mu; \frac{s}{2} + n; -a^2z^2\right] &= \frac{\Gamma(1 + \frac{s}{2} + n)\Gamma(1 + \mu)}{\Gamma(1 + \mu - \frac{s}{2} - n)} \left(\frac{1}{a^2z^2}\right)^{\frac{s}{2} + n} \\ &\cdot {}_3F_0\left[\frac{s}{2} + n, 0, \frac{s}{2} - \mu + n; \text{---}; -\frac{1}{a^2z^2}\right] \\ &+ \frac{\Gamma(1 + \frac{s}{2} + n)\Gamma(1 + \mu)}{\Gamma(\frac{1}{2})\Gamma(\frac{s}{2} + n)} \left(\frac{1}{a^2z^2}\right)^{\frac{1}{2}(\frac{s}{2} + \mu)} \\ &\cdot \left[1 + O\left(\frac{1}{z^2}\right)\right] \cos\left[2az - \frac{\pi}{2}\left(\frac{s}{2} + \mu\right) + O\left(\frac{1}{z}\right)\right]. \end{aligned}$$

Noting that  ${}_3F_0[-1/(a^2z^2)] = 1$  for all integers  $n \geq 0$  and using the identity

$$\Gamma(\alpha - n) = (-1)^n \frac{\Gamma(\alpha)}{(1 - \alpha)_n} \tag{5.2}$$

we may then write

$$\begin{aligned} &{}_1F_2\left[1 + \frac{s}{2} + n, 1 + \mu; \frac{s}{2} + n; -a^2z^2\right] \\ &= \frac{\Gamma(1 + \mu)\Gamma(1 + \frac{s}{2})}{\Gamma(1 + \mu - \frac{s}{2})} \left(1 + \frac{s}{2}\right)_n \left(\frac{s}{2} - \mu\right)_n \left(\frac{1}{az}\right)^s \left(-\frac{1}{a^2z^2}\right)^n \\ &+ \frac{s}{2} \left(\frac{1}{az}\right)^{\frac{s}{2} + \mu} \frac{\Gamma(1 + \mu)}{\Gamma(\frac{1}{2})} \frac{(1 + \frac{s}{2})_n}{(\frac{s}{2})_n} \left[1 + O\left(\frac{1}{z^2}\right)\right] \cos(2az + D), \end{aligned} \tag{5.3}$$

where  $D$  is obviously dependent on  $\mu$  and  $O(\frac{1}{z})$ .



Now, substituting the result in equation (5.3) for  ${}_1F_2[-a^2z^2]$  into equation (3.3), we obtain

$$\begin{aligned}
 F(s; z) = & \frac{1}{2}a^{-s} \frac{\Gamma(\frac{s}{2})\Gamma(1 + \mu)}{\Gamma(1 + \mu - \frac{s}{2})} {}_3F_2 \left[ \begin{matrix} \alpha, \frac{s}{2}, \frac{s}{2} - \mu; \\ \beta, 1 + \nu; \end{matrix} \frac{b^2}{a^2} \right] \\
 & + \frac{1}{2}a^{-\mu-\frac{3}{2}} \frac{\Gamma(1 + \mu)}{\Gamma(\frac{1}{2})} z^{s-\mu-\frac{3}{2}} {}_1F_2 \left[ \begin{matrix} \alpha; \\ \beta, 1 + \nu; \end{matrix} -b^2z^2 \right] \\
 & \cdot \left[ 1 + O\left(\frac{1}{z^2}\right) \right] \cos(2az + D),
 \end{aligned} \tag{5.4}$$

where, for convergence of  ${}_3F_2\left[\frac{b^2}{a^2}\right]$ , we must have  $b < a$ . Next, denoting the second term in the above equation (5.4) by  $S_0(z)$  and using equation (4.5) to approximate  ${}_1F_2[-b^2z^2]$ , we have

$$\begin{aligned}
 S_0(z) = & \left[ \frac{1}{2}Aa^{-\mu-\frac{3}{2}}b^{-2\alpha} \frac{\Gamma(1 + \mu)}{\Gamma(\frac{1}{2})} z^{s-2\alpha-\mu-\frac{3}{2}} \right. \\
 & \left. + \frac{1}{2}Ba^{-\mu-\frac{3}{2}}b^{\alpha-\beta-\nu-\frac{1}{2}} \frac{\Gamma(1 + \mu)}{\Gamma(\frac{1}{2})} z^{s-\mu-\nu+\alpha-\beta-2} \cos(2bz + C) \right] \\
 & \cdot \left[ 1 + O\left(\frac{1}{z^2}\right) \right] \cos(2az + D).
 \end{aligned} \tag{5.5}$$

Finally, letting  $z \rightarrow \infty$  we see that, since

$$\Re(s - 2\alpha - \mu - \frac{3}{2}) < 0$$

and

$$\Re(s - \mu - \nu + \alpha - \beta - 2) < 0$$

(both of which conditions secure the convergence of  $F(s; z)$  to  $F(s)$ ),  $S_0(z)$  or the second term in equation (5.4) vanishes. Thus equation (5.4) yields equation (5.1a) upon letting  $z \rightarrow \infty$ .

The derivation of equation (5.1b) is similar to that of equation (5.1a), but is somewhat more complex in its details, because the asymptotic formula for  ${}_2F_3[-z^2]$  given by equation (4.3) has three terms. Thus, for large  $z$ , by using the latter together with equation (3.4) and employing equation (5.2), we obtain, for  $a < b$ ,

$$\begin{aligned}
 F(s; z) = & \frac{1}{2}b^{-s} \frac{\Gamma(\frac{s}{2})\Gamma(\alpha - \frac{s}{2})\Gamma(\beta)\Gamma(1 + \nu)}{\Gamma(\alpha)\Gamma(\beta - \frac{s}{2})\Gamma(1 + \nu + \frac{s}{2})} {}_3F_2 \left[ \begin{matrix} \frac{s}{2}, 1 + \frac{s}{2} - \beta, \frac{s}{2} - \nu; \\ 1 + \frac{s}{2} - \alpha, 1 + \mu; \end{matrix} \frac{a^2}{b^2} \right] \\
 & + S_1(z) + S_2(z),
 \end{aligned} \tag{5.6}$$

where

$$\begin{aligned}
 S_1(z) &= \frac{1}{2} b^{-2\alpha} z^{s-2\alpha} \frac{\Gamma\left(\frac{s}{2} - \alpha\right) \Gamma(\beta) \Gamma(1 + \nu)}{\Gamma\left(1 + \frac{s}{2} - \alpha\right) \Gamma(\beta - \alpha) \Gamma(1 + \nu - \alpha)} \\
 &\cdot \sum_{n=0}^{\infty} \frac{\left(\frac{s}{2} - \alpha\right)_n}{\left(1 + \frac{s}{2} - \alpha\right)_n (1 + \mu)_n} \frac{(-a^2 z^2)^n}{n!} \\
 &\cdot {}_4F_1 \left[ \begin{matrix} \alpha, 1 + \alpha - \beta, \alpha - \nu, \alpha - \frac{s}{2} - n; \\ 1 + \alpha - \frac{s}{2} - n; \end{matrix} -\frac{1}{b^2 z^2} \right],
 \end{aligned} \tag{5.7}$$

and

$$\begin{aligned}
 S_2(z) &= \frac{1}{2} z^{s+\alpha-\beta-\nu-\frac{3}{2}} \left[ 1 + O\left(\frac{1}{z^2}\right) \right] \cos \left[ 2bz - \frac{\pi}{2} \left( \beta - \alpha + \nu + \frac{3}{2} \right) + O\left(\frac{1}{z}\right) \right] \\
 &\cdot {}_0F_1 \left[ \begin{matrix} - \\ 1 + \mu; \end{matrix} -a^2 z^2 \right].
 \end{aligned} \tag{5.8}$$

Now, by using Equation (4.4) we can rewrite Equation (5.8) as

$$S_2(z) = \frac{1}{2} A z^{s+\alpha-\beta-\mu-\nu-2} \left[ 1 + O\left(\frac{1}{z^2}\right) \right] \cos(2az + B) \cos(2bz + C), \tag{5.9}$$

where the definition of  $C$  is obvious. Further, by writing  ${}_4F_1\left[-\frac{1}{b^2 z^2}\right]$  in Equation (5.7) as a hypergeometric  $m$ -summation, interchanging it with its preceding  $n$ -summation, and then employing Equation (5.2), we arrive at

$$\begin{aligned}
 S_1(z) &= \frac{1}{2} b^{-2\alpha} z^{s-2\alpha} \frac{\Gamma\left(\frac{s}{2} - \alpha\right) \Gamma(\beta) \Gamma(1 + \nu)}{\Gamma\left(1 + \frac{s}{2} - \alpha\right) \Gamma(\beta - \alpha) \Gamma(1 + \nu - \alpha)} \\
 &\cdot \sum_{m=0}^{\infty} \frac{(\alpha)_m (1 + \alpha - \beta)_m (\alpha - \nu)_m \left(\alpha - \frac{s}{2}\right)_m}{\left(1 + \alpha - \frac{s}{2}\right)_m} \frac{\left(-\frac{1}{b^2 z^2}\right)^m}{m!} \\
 &\cdot {}_1F_2 \left[ \begin{matrix} \frac{s}{2} - \alpha - m; \\ 1 + \frac{s}{2} - \alpha - m, 1 + \mu; \end{matrix} -a^2 z^2 \right].
 \end{aligned} \tag{5.10}$$

We see from Equations (4.2) and (5.2) that  ${}_1F_2[-a^2 z^2]$  in Equation (5.10) may be

written as

$$\begin{aligned}
 {}_1F_2 \left[ \begin{matrix} \frac{s}{2} - \alpha - m; \\ 1 + \frac{s}{2} - \alpha - m, 1 + \mu; \end{matrix} -a^2 z^2 \right] &= \frac{z^{2\alpha-s} \Gamma(1 + \mu) \Gamma(1 + \frac{s}{2} - \alpha)}{a^{s-2\alpha} \Gamma(1 + \mu + \alpha - \frac{s}{2})} \\
 &\cdot \frac{(-a^2 z^2)^m}{(\alpha - \frac{s}{2})_m (1 + \mu + \alpha - \frac{s}{2})_m} \\
 &\cdot {}_3F_0 \left[ \begin{matrix} \frac{s}{2} - \alpha - m, \frac{s}{2} - \alpha - \mu - m; \\ \text{---}; \end{matrix} -\frac{1}{a^2 z^2} \right] \quad (5.11) \\
 &+ \frac{z^{-\frac{3}{2}-\mu} \Gamma(1 + \mu) \Gamma(1 + \frac{s}{2} - \alpha) (1 + \alpha - \frac{s}{2})_m}{a^{\frac{3}{2}+\mu} \Gamma(\frac{1}{2}) \Gamma(\frac{s}{2} - \alpha) (\alpha - \frac{s}{2})_m} \left[ 1 + O\left(\frac{1}{z^2}\right) \right] \\
 &\cdot \cos \left[ 2az - \frac{\pi}{2} \left( \frac{3}{2} + \mu \right) + O\left(\frac{1}{z}\right) \right].
 \end{aligned}$$

Upon noting in Equations (5.11) that  ${}_3F_0[-1/a^2 z^2] = 1$  for all integers  $m \geq 0$ , we find from Equations (5.10) and (5.11) that

$$\begin{aligned}
 S_1(z) &= \frac{1}{2} a^{-s} \left( \frac{a^2}{b^2} \right)^\alpha \frac{\Gamma(1 + \mu) \Gamma(1 + \nu) \Gamma(\beta) \Gamma(\frac{s}{2} - \alpha)}{\Gamma(1 + \nu - \alpha) \Gamma(1 + \mu + \alpha - \frac{s}{2}) \Gamma(\beta - \alpha)} \\
 &\cdot \sum_{m=0}^{\infty} \frac{(\alpha)_m (1 + \alpha - \beta)_m (\alpha - \nu)_m}{(1 + \alpha - \frac{s}{2})_m (1 + \mu + \alpha - \frac{s}{2})_m} \frac{\left( \frac{a^2}{b^2} \right)^m}{m!} \\
 &+ \frac{1}{2} \frac{b^{-2\alpha}}{a^{\frac{3}{2}+\mu}} z^{s-\mu-2\alpha-\frac{3}{2}} \frac{\Gamma(1 + \mu) \Gamma(1 + \nu) \Gamma(\beta)}{\Gamma(\frac{1}{2}) \Gamma(\beta - \alpha) \Gamma(1 + \nu - \alpha)} \left[ 1 + O\left(\frac{1}{z^2}\right) \right] \\
 &\cdot \cos \left[ 2az - \frac{\pi}{2} \left( \frac{3}{2} + \mu \right) + O\left(\frac{1}{z}\right) \right] \quad (5.12) \\
 &\cdot {}_3F_0 \left[ \alpha, 1 + \alpha - \beta, \alpha - \nu; \text{---}; -\frac{1}{b^2 z^2} \right].
 \end{aligned}$$

Equation (5.12) may be written more simply as

$$\begin{aligned}
 S_1(z) &= \frac{1}{2} a^{-s} \left( \frac{a^2}{b^2} \right)^\alpha \frac{\Gamma(1 + \mu) \Gamma(1 + \nu) \Gamma(\beta) \Gamma(\frac{s}{2} - \alpha)}{\Gamma(1 + \nu - \alpha) \Gamma(1 + \mu + \alpha - \frac{s}{2}) \Gamma(\beta - \alpha)} \\
 &\cdot {}_3F_2 \left[ \begin{matrix} \alpha, 1 + \alpha - \beta, \alpha - \nu; a^2 \\ 1 + \alpha - \frac{s}{2}, 1 + \mu + \alpha - \frac{s}{2}; b^2 \end{matrix} \right] \quad (5.13) \\
 &+ \frac{1}{2} A' \left[ 1 + O\left(\frac{1}{z^2}\right) \right] z^{s-\mu-2\alpha-\frac{3}{2}} \cos(2az + B') \\
 &\cdot {}_3F_0 \left[ \alpha, 1 + \alpha - \beta, \alpha - \nu; \text{---}; -\frac{1}{b^2 z^2} \right],
 \end{aligned}$$

where  $a < b$  and the definitions of  $A'$  and  $B'$  are obvious.

Now, combining Equations (5.6), (5.9), and (5.13), we find that

$$\begin{aligned}
 F(s; z) = & \frac{1}{2}b^{-s} \frac{\Gamma(\frac{s}{2}) \Gamma(\alpha - \frac{s}{2}) \Gamma(\beta) \Gamma(1 + \nu)}{\Gamma(\alpha) \Gamma(\beta - \frac{s}{2}) \Gamma(1 + \nu - \frac{s}{2})} {}_3F_2 \left[ \begin{matrix} \frac{s}{2}, 1 + \frac{s}{2} - \beta, \frac{s}{2} - \nu; \\ 1 + \frac{s}{2} - \alpha, 1 + \mu; \end{matrix} \frac{a^2}{b^2} \right] \\
 & + \frac{1}{2}a^{-s} \left(\frac{a^2}{b^2}\right)^\alpha \frac{\Gamma(1 + \mu) \Gamma(1 + \nu) \Gamma(\beta) \Gamma(\frac{s}{2} - \alpha)}{\Gamma(1 + \nu - \alpha) \Gamma(1 + \mu + \alpha - \frac{s}{2}) \Gamma(\beta - \alpha)} \\
 & \cdot {}_3F_2 \left[ \begin{matrix} \alpha, 1 + \alpha - \beta, \alpha - \nu; \\ 1 + \alpha - \frac{s}{2}, 1 + \mu + \alpha - \frac{s}{2}; \end{matrix} \frac{a^2}{b^2} \right] \\
 & + \frac{1}{2}Az^{s+\alpha-\beta-\mu-\nu-2} \left[ 1 + O\left(\frac{1}{z^2}\right) \right] \cos(2az + B) \cos(2bz + C) \\
 & + \frac{1}{2}A'z^{s-\mu-2\alpha-\frac{3}{2}} \left[ 1 + O\left(\frac{1}{z^2}\right) \right] \cos(2az + B') \tag{5.14} \\
 & \cdot {}_3F_0 \left[ \alpha, 1 + \alpha - \beta, \alpha - \nu; \text{---}; -\frac{1}{b^2z^2} \right].
 \end{aligned}$$

Finally, recalling that

$$0 < \Re(s) < \Re(2 + \mu + \nu + \beta - \alpha) \quad \text{and} \quad 0 < \Re(s) < \Re\left(\frac{3}{2} + 2\alpha + \mu\right),$$

which secure the convergence of  $F(s; z)$  to  $F(s)$ , upon letting  $z \rightarrow \infty$ , we see that the third and fourth terms in Equation (5.14) vanish, and we are left with Equation (5.1b). This evidently completes the derivation of Equations (5.1).

### 6. The Case $a = b$

The expressions on the right-hand side of Equations (5.1a) and (5.1b) are not analytic continuations of each other. In particular, for  $\alpha = \beta$  we noted this in Section 1 and mentioned also that the “discontinuous” nature of  $F(s)$  refers to the discontinuity in its representation across  $a = b$ . However,  $F(s)$  is continuous when  $a = b$ . To see this, since the first integral on the right-hand side of Equation (2.4) is continuous when  $a = b$  (see [11, p. 403]), then so is the integral on the left-hand side. Thus also it is evident that  $F(s)$  is continuous when  $a$  passes through  $b$ .

In addition, we showed in Section 2 that when  $a = b$ , a necessary condition that the integral  $F(s)$  converges is that

$$0 < \Re(s) < \Re(1 + \mu + \nu + \beta - \alpha),$$

and *a fortiori* all three  ${}_3F_2$  functions in equations (5.1) converge absolutely when  $a = b$ , provided that the latter condition holds true. Thus, for  $a > 0$ ,

$$0 < \Re(s) < \Re(1 + \mu + \nu + \beta - \alpha) \quad \text{and} \quad 0 < \Re(s) < \Re\left(\frac{3}{2} + 2\alpha + \mu\right),$$

we deduce, respectively, from equations (5.1a) and (5.1b) that

$$\int_0^\infty x^{s-1} {}_0F_1 \left[ \begin{matrix} - \\ 1 + \mu \end{matrix}; -a^2x^2 \right] {}_1F_2 \left[ \begin{matrix} \alpha \\ \beta, 1 + \nu \end{matrix}; -a^2x^2 \right] dx \tag{6.1}$$

$$= \frac{1}{2} a^{-s} \frac{\Gamma(\frac{s}{2})\Gamma(1 + \mu)}{\Gamma(1 + \mu - \frac{s}{2})} {}_3F_2 \left[ \begin{matrix} \alpha, \frac{s}{2}, \frac{s}{2} - \mu \\ \beta, 1 + \nu \end{matrix}; 1 \right]$$

and

$$\int_0^\infty x^{s-1} {}_0F_1 \left[ \begin{matrix} - \\ 1 + \mu \end{matrix}; -a^2x^2 \right] {}_1F_2 \left[ \begin{matrix} \alpha \\ \beta, 1 + \nu \end{matrix}; -a^2x^2 \right] dx \tag{6.2}$$

$$= \frac{1}{2} a^{-s} \frac{\Gamma(\frac{s}{2})\Gamma(\alpha - \frac{s}{2})\Gamma(\beta)\Gamma(1 + \nu)}{\Gamma(\alpha)\Gamma(\beta - \frac{s}{2})\Gamma(1 + \nu - \frac{s}{2})} {}_3F_2 \left[ \begin{matrix} \frac{s}{2}, 1 + \frac{s}{2} - \beta, \frac{s}{2} - \nu \\ 1 + \frac{s}{2} - \alpha, 1 + \mu \end{matrix}; 1 \right]$$

$$+ \frac{1}{2} a^{-s} \frac{\Gamma(1 + \mu)\Gamma(1 + \nu)\Gamma(\beta)\Gamma(\frac{s}{2} - \alpha)}{\Gamma(1 + \nu - \alpha)\Gamma(1 + \mu + \alpha - \frac{s}{2})\Gamma(\beta - \alpha)}$$

$$\cdot {}_3F_2 \left[ \begin{matrix} \alpha, 1 + \alpha - \beta, \alpha - \nu \\ 1 + \alpha - \frac{s}{2}, 1 + \mu + \alpha - \frac{s}{2} \end{matrix}; 1 \right].$$

When  $\alpha = \beta$ , the second right member in equation (6.2) vanishes and it is easy to see that the right-hand sides of equations (6.1) and (6.2) reduce, respectively, via Gauss’s theorem to equation (1.2b). However, the  ${}_3F_2[1]$  functions in equations (6.1) and (6.2) are not, in general, reducible, since their parameters are not interrelated [12].

If we equate the right members of equations (6.1) and (6.2), divide the result by  $a^{-s}/2$ , and then set

$$a = \frac{s}{2} - \mu, \quad b = \frac{s}{2}, \quad c = \alpha, \quad e = \beta, \quad \text{and} \quad f = 1 + \nu, \tag{6.3}$$

we deduce the second fundamental relation for  ${}_3F_2[a, b, c; e, f; 1]$ , which is

$${}_3F_2 \left[ \begin{matrix} a, b, c \\ e, f \end{matrix}; 1 \right]$$

$$= \frac{\Gamma(1 - a)\Gamma(e)\Gamma(f)\Gamma(c - b)}{\Gamma(e - b)\Gamma(f - b)\Gamma(1 + b - a)\Gamma(c)} {}_3F_2 \left[ \begin{matrix} b, b - e + 1, b - f + 1 \\ 1 + b - c, 1 + b - a \end{matrix}; 1 \right]$$

$$+ \frac{\Gamma(1 - a)\Gamma(e)\Gamma(f)\Gamma(b - c)}{\Gamma(e - c)\Gamma(f - c)\Gamma(1 + c - a)\Gamma(b)} {}_3F_2 \left[ \begin{matrix} c, c - e + 1, c - f + 1 \\ 1 + c - b, 1 + c - a \end{matrix}; 1 \right].$$

The conditional inequality  $0 < \Re(s) < \Re(\frac{3}{2} + 2\alpha + \mu)$  with the relevant substitutions of equations (6.3) may now be waived by appealing to the principle of analytic continuation. That equation (6.4) manifests itself as a corollary is both surprising and interesting, especially since Bailey [1] derives it by considering a certain contour

integral, deforming its contour in two different ways, and then computing the integrals via residues [1, p. 15].

Finally, we recall that in Section 2 we showed, in the case  $a = b$ , that  $F(s)$  converges also when

$$0 < \Re(s) < \Re(2 + \mu + \nu + \beta - \alpha) \quad \text{and} \quad 0 < \Re(s) < \Re\left(\frac{3}{2} + 2\alpha + \mu\right),$$

provided that  $\mu - \nu + \alpha - \beta$  is an odd positive integer. It is evident that equation (6.1) does not hold true in this case since  ${}_3F_2[1]$  does not converge. It should be remarked also that, in an earlier work, Srivastava and Exton [8] considered a generalization of the Weber-Schafheitlin integral given in equation (1.2) for the product of several Bessel (or  ${}_0F_1$ ) functions.

### 7. The Sine and Cosine Transforms of ${}_1F_2[-b^2x^2]$

For brevity, we define

$$\mathcal{S}(a, b) := \int_0^\infty \sin(2ax) {}_1F_2[\alpha; \beta, \gamma; -b^2x^2] dx$$

and

$$\mathcal{C}(a, b) := \int_0^\infty \cos(2ax) {}_1F_2[\alpha; \beta, \gamma; -b^2x^2] dx,$$

which are, respectively, the sine and cosine transforms of  ${}_1F_2[-b^2x^2]$ .

Since

$$\sin z = z {}_0F_1\left[\text{---}; \frac{3}{2}; -\frac{1}{4}z^2\right]$$

and

$$\cos z = {}_0F_1\left[\text{---}; \frac{1}{2}; -\frac{1}{4}z^2\right],$$

it is easy to deduce from equations (5.1) that

$$\mathcal{S}(a, b) = \begin{cases} \frac{1}{2a} {}_3F_2\left[\frac{1}{2}, 1, \alpha; \beta, \gamma; \frac{b^2}{a^2}\right] & (0 < b < a) \\ \frac{a}{b^2} \frac{(\beta-1)(\gamma-1)}{\alpha-1} {}_3F_2\left[1, 2-\beta, 2-\gamma; \frac{a^2}{b^2}, \frac{3}{2}, 2-\alpha; \frac{a^2}{b^2}\right] + \frac{\sqrt{\pi}}{2a} \left(\frac{a^2}{b^2}\right)^\alpha & \\ \frac{\Gamma(\beta)\Gamma(\gamma)\Gamma(1-\alpha)}{\Gamma(\beta-\alpha)\Gamma(\gamma-\alpha)\Gamma(\frac{1}{2}+\alpha)} {}_2F_1\left[1+\alpha-\beta, 1+\alpha-\gamma; \frac{a^2}{b^2}, \frac{1}{2}+\alpha; \frac{a^2}{b^2}\right] & (0 < a < b), \end{cases}$$

and

$$\mathcal{C}(a, b) = \begin{cases} 0 & (0 < b < a) \\ \frac{\sqrt{\pi}}{2b} \frac{\Gamma(\alpha - \frac{1}{2})\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta - \frac{1}{2})\Gamma(\gamma - \frac{1}{2})} {}_2F_1 \left[ \begin{matrix} \frac{3}{2} - \beta, \frac{3}{2} - \gamma; \frac{a^2}{b^2} \\ \frac{3}{2} - \alpha; \frac{a^2}{b^2} \end{matrix} \right] + \frac{\sqrt{\pi}}{2a} \left( \frac{a^2}{b^2} \right)^\alpha & (0 < a < b), \\ \frac{\Gamma(\frac{1}{2} - \alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta - \alpha)\Gamma(\gamma - \alpha)} {}_2F_1 \left[ \begin{matrix} 1 + \alpha - \beta, 1 + \alpha - \gamma; \frac{a^2}{b^2} \\ \frac{1}{2} + \alpha; \frac{a^2}{b^2} \end{matrix} \right] & \end{cases}$$

where

$$0 < \Re(\alpha) < \Re(\beta + \gamma - \frac{1}{2}).$$

### 8. Concluding Remarks

It should be mentioned that, by using a general result for the Mellin transform of a product of generalized hypergeometric functions in [7, Section 2.22, p. 333],  $F(s)$  defined by equation (1.1) may be represented by Meijer’s  $G$ -function. Thus, for  $a > 0$  and  $b > 0$ , we have

$$F(s) = \frac{1}{2} a^{-s} \frac{\Gamma(1 + \mu)\Gamma(1 + \nu)\Gamma(\beta)}{\Gamma(\alpha)} G_{3,3}^{1,2} \left( \frac{b^2}{a^2} \middle| \begin{matrix} 1 - \frac{s}{2}, 1 - \alpha, 1 + \mu - \frac{s}{2} \\ 0, -\nu, 1 - \beta \end{matrix} \right), \tag{8.1}$$

where

$$0 < \Re(s) < \Re(2 + \mu + \nu + \beta - \alpha) \quad \text{and} \quad 0 < \Re(s) < \Re\left(\frac{3}{2} + 2\alpha + \mu\right).$$

Furthermore, by using formulas for reducing the  $G$ -function to generalized hypergeometric functions (see, for example, [4, Section 6.5, p. 230]), we may obtain equations (5.1) from equation (8.1). Nonetheless, the derivation of equations (5.1) presented herein is elementary in the sense that it does not require knowledge of the  $G$ -function and its properties. In addition, the results given by equations (6.1), (6.2), and (6.4) require a detailed analysis of the continuity and convergence criteria for  $F(s)$  when  $a = b$ , and, therefore, may not be deduced directly from equation (8.1).

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