A NETWORK-FLOW FEASIBILITY THEOREM AND COMBINATORIAL APPLICATIONS

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1. Introduction. There are a number of interesting theorems, relative to capacitated networks, that give necessary and sufficient conditions for the existence of flows satisfying constraints of various kinds. Typical of these are the supply-demand theorem due to Gale **(4)**, which states a condition for the existence of a flow satisfying demands at certain nodes from supplies at other nodes, and the Hoffman circulation theorem (received by the present author in private communication), which states a condition for the existence of a circulatory flow in a network in which each arc has associated with it not only an upper bound for the arc flow, but a lower bound as well. If the constraints on flows are integral (for example, if the bounds on arc flows for the circulation theorem are integers), it is also true that integral flows meeting the requirements exist provided any flow does so. This fact has been used by Gale **(4)**, and by Ford and Fulkerson **(3)**, in the solution of several combinatorial problems. For example, Gale has shown how the supply-demand theorem, together with the existence of integral flows, can be used to derive simple conditions for the existence of a matrix of zeros and ones having prescribed row and column sums, a problem that was also solved independently by Ryser **(9)** by means of purely combinatorial methods.

The present paper adds some results along the lines we have described. We first establish a feasibility theorem, which may be described informally as follows. Suppose there is given a capacitated network with certain of the nodes designated as sources, others as sinks, and assume that each source is required to send, and each sink to receive, an amount that lies between prescribed bounds. Under what conditions is this possible? The theorem asserts that if (a) there is a flow that sends out of each source an amount at least as great as the lower bound for the source, and into each sink no more than the upper bound for the sink, and if (b) there is a flow that sends out of each source no more than the upper bound for the source, and into each sink at least as much as the lower bound for the sink, then there is a flow that meets all the requirements simultaneously. We do not give a direct proof of this theorem, but rather use the max-flow min-cut theorem **(1; 2)** to find a pair of conditions that are necessary and sufficient for the existence of the required flow, and then observe that one of the conditions is equivalent to (a) above, the other to (b).

Received July 28, 1958.

Our first combinatorial application (§ 5) of the feasibility theorem is to generalize the Gale-Ryser theorem on incidence matrices having prescribed row and column sums, to the extent of allowing these sums to vary within designated bounds.

Our second application (§ 6) concerns the subgraph problem for directed graphs: to find necessary and sufficient conditions that a finite directed graph G have a subgraph H possessing specified local degrees. A solution to this problem has been given by Ore **(6)**. Here, in keeping with the feasibility theorem, we extend the problem by permitting the number of arcs of H that enter or leave each node of G to vary within bounds, and then show that the conditions obtained for this latter problem reduce to Ore's conditions for the subgraph problem.

The similar problem for undirected graphs, which has been solved by Tutte **(12)**, and also by Ore **(8)**, is, so far as we known, amenable to network-flow methods only in the special case that G is an even graph, and this because the problem then is, in essence, a directed one.

Our final application (§ 7) deals with a problem involving set representatives: to find necessary and sufficient conditions for the existence of a system of distinct representatives having the further property that the intersection of the system with each member of a given partition of the fundamental set has a cardinality lying between assigned bounds. This problem was first posed and solved by Hoffman and Kuhn **(5)**; it is shown here that the conditions established in **(5)** are deducible from the feasibility theorem.

2. Definitions, notation, and prior results. Let G be a finite directed network or linear graph consisting of a set N of nodes, $x, y, \ldots$, and directed arcs joining pairs of nodes, the arc from x to y being denoted (x, y), and suppose that each arc (x, y) has associated with it a *capacity* $c(x, y)$, where $c(x, y)$ is either a non-negative real number or plus infinity. Let the set N of nodes be partitioned into three subsets: S (the set of *sources*), T (the set of *sinks*), and R (the set of *intermediate nodes*). We call a real-valued function f defined on the arcs of G a *flow from S to T* provided that

$$\sum_{y \in A(x)} f(x, y) = \sum_{y \in B(x)} f(y, x), \qquad x \in R, \tag{1}$$

$$0 \leqslant f(x, y) \leqslant c(x, y), \qquad \text{all } (x, y), \tag{2}$$

where $A(x)$ ("after" x) is the set of nodes y such that (x, y) is an arc, and $B(x)$ ("before" x) consists of those nodes y such that (y, x) is an arc. Thus (1) states that the flow out of an intermediate node is equal to the flow in, and (2) that the flow in each arc does not exceed its capacity.

We will be interested in flows from S to T that satisfy bounds on the net flow leaving each $x \in S$, and entering each $x \in T$. Thus, for $x \in S$, let $\alpha(x)$ and $\beta(x)$ be real-valued functions with

$$0 \leqslant \alpha(x) \leqslant \beta(x);$$

https://doi.org/10.4153/CJM-1959-045-1 Published online by Cambridge University Press

similarly, associate with each $x \in T$ two real numbers $a(x)$ and $b(x)$, where

$$0 \leqslant a(x) \leqslant b(x).$$

The additional constraints

$$\text{(3a)} \qquad \alpha(x) \leqslant \sum_{y \in A(x)} f(x, y) - \sum_{y \in B(x)} f(y, x) \leqslant \beta(x), \qquad x \in S,$$

$$\text{(3b)} \qquad a(x) \leqslant \sum_{y \in B(x)} f(y, x) - \sum_{y \in A(x)} f(x, y) \leqslant b(x), \qquad x \in T,$$

will be termed *feasible* provided there is a flow f from S to T satisfying them. In this case, f will also be called a *feasible flow.*

To simplify the notation, we adopt the following conventions. If X and Y are subsets of N, denote by (X, Y) the set of arcs leading from X to Y; and for any fuction f defined on the arcs, let

$$\sum_{(x,y) \in (X,Y)} f(x, y) = f(X, Y).$$

Similarly, if α is defined on a subset X of N, let

$$\sum_{x \in X} \alpha(x) = \alpha(X).$$

We shall also use $A(X)$ to denote the set of all nodes y such that (x, y), for some $x \in X$, is an arc of G, and similarly for $B(X)$.

The *value* $v(f)$ of a flow f from S to T is the net flow leaving the sources, which, in the notation just introduced, is given by

$$\text{(4)} \qquad v(f) = f(S, A(S)) - f(B(S), S).$$

In view of (1), $v(f)$ may also be expressed as the net flow entering the sinks:

$$\text{(5)} \qquad v(f) = f(B(T), T) - f(T, A(T)).$$

Let $X, \bar{X}$ be a partition of N with $S \subset X$, $T \subset \bar{X}$. The set of arcs $(X, \bar{X})$ is a *cut* in G (separating S and T), and $c(X, \bar{X})$ is the *cut capacity.*

A fundamental theorem concerning flows from S to T in a network G asserts that the maximal flow value is equal to the minimal cut capacity **(1, 2)**. A second theorem, important for combinatorial applications, is that if the capacity function c assumes only integral vlaues, then there exists a maximal flow f that is likewise integral **(2, 3)**.

Gale **(4)** has used the max-flow min-cut theorem to prove that if $\alpha(x) = 0$, $b(x) = \infty$ in (3), then a feasible flow (that is, a flow satisfying the "demands" $a(x)$ at the sinks from the "supplies" $\beta(x)$ at the sources) exists if and only if, for every partition $X, \bar{X}$ of N, we have

$$\text{(6)} \qquad a(T \cdot \bar{X}) \leqslant c(X, \bar{X}) + \beta(S \cdot \bar{X}),$$

where $X \cdot Y$ denotes the intersection of the sets X and Y.

3. Feasibility theorems. In this section, we develop a generalization of the supply-demand feasibility theorem by finding conditions under which the full set of constraints (3) is feasible.

We begin by adjoining to the given network G four new nodes, s, t, u, v, and several sets of arcs, as follows:

$$(s, S),\ (u, S),\ (T, t),\ (T, v),\ (u, t),\ (s, v),\ (t, s).$$

Next, we extend the capacity function c defined on arcs of G to the new network G^* by

$$\begin{aligned} c(s, x) &= \beta(x) - \alpha(x), && x \in S,\\ c(u, x) &= \alpha(x), && x \in S,\\ c(x, t) &= b(x) - a(x), && x \in T,\\ c(x, v) &= a(x), && x \in T,\\ c(u, t) &= a(T), &&\\ c(s, v) &= \alpha(S), &&\\ c(t, s) &= \infty. && \end{aligned}$$

We assert that a feasible flow exists in G if, and only if, the value of a maximal flow from u to v in G^* is $\alpha(S) + a(T)$. Suppose first that f is feasible in G; extend f to f^*, defined on the arcs of G^*, as follows:

$$\begin{aligned} f^*(s, x) &= f(x, A(x)) - f(B(x), x) - \alpha(x), && x \in S,\\ f^*(u, x) &= \alpha(x), && x \in S,\\ f^*(x, t) &= f(B(x), x) - f(x, A(x)) - a(x), && x \in T,\\ f^*(x, v) &= a(x), && x \in T,\\ f^*(u, t) &= a(T), &&\\ f^*(s, v) &= \alpha(S), &&\\ f^*(t, s) &= f(S, A(S)) - f(B(S), S), &&\\ f^*(x, y) &= f(x, y), && \text{for arcs } (x, y) \text{ of } G. \end{aligned}$$

It is a routine matter to check that f^* is a flow from u to v in G^*. Clearly, f^* has value

$$v(f^*) = \alpha(S) + a(T).$$

Conversely, let f^* be a flow from u to v in G^*, of value $\alpha(S) + a(T)$. Then

$$\begin{aligned} f^*(u, x) &= \alpha(x), && x \in S,\\ f^*(x, v) &= a(x), && x \in T. \end{aligned}$$

Let f be f^* restricted to G. Then f is a flow from S to T in G, and it remains only to show that f is feasible. Consider any $x \in S$. From (1) applied to x, we have

$$f^*(u, x) + f^*(s, x) = f(x, A(x)) - f(B(x), x),$$

or

$$\alpha(x) + f^*(s, x) = f(x, A(x)) - f(B(x), x);$$

https://doi.org/10.4153/CJM-1959-045-1 Published online by Cambridge University Press

and, since

$$0 \leqslant f^*(s, x) \leqslant \beta(x) - \alpha(x),$$

we get

$$\alpha(x) \leqslant f(x, A(x)) - f(B(x), x) \leqslant \beta(x),$$

which is (3a). Inequalities (3b) are similarly proved. This completes the proof of the assertion.

We may, therefore, in searching for feasibility criteria, rephrase the question as follows. Under what conditions does there exist a flow f^* from u to v in G^* having value $v(f^*) = \alpha(S) + a(T)$—that is, saturating all source and sink arcs? The max-flow min-cut theorem can now be used to provide an answer to this question by insisting that the capacitities of all cuts separating u and v be at least as great as $\alpha(S) + a(T)$.

Thus, let $(X^*, \bar{X}^*)$ be a cut in G^*, and consider cases.

Case 1. $s \in X^*$, $t \in \bar{X}^*$. Partition $X^*, \bar{X}^*$ as follows: $X^* = u + s + X$, $\bar{X}^* = v + t + \bar{X}$. Then

$$\begin{aligned} c(X^*, \bar{X}^*) &= c(u, t) + c(u, \bar{X}), + c(s, v) + c(s, \bar{X}) \\ &\quad + c(X, v) + c(X, t) + c(X, \bar{X}) \\ &= a(T) + \alpha(S\cdot\bar{X}) + \alpha(S) + \beta(S\cdot\bar{X}) - \alpha(S\cdot\bar{X}) \\ &\quad + a(T\cdot X) + b(T\cdot X) - a(T\cdot X) + c(X, \bar{X}). \end{aligned}$$

Hence, in this case, we always have $c(X^*, \bar{X}^*) \geqslant \alpha(S) + a(T)$.

Case 2. $s \in \bar{X}^*$, $t \in X^*$. Then $c(X, \bar{X}^*)$ is infinite. Hence again no condition is obtained.

Case 3. $s \in X^*$, $t \in X^*$. Letting $X^* = s + t + u + X$, $\bar{X}^* = v + \bar{X}$, we have

$$\begin{aligned} c(X^*, \bar{X}^*) &= c(s, v) + c(s, \bar{X}) + c(u, \bar{X}) + c(X, v) + c(X, \bar{X}) \\ &= \alpha(S) + \beta(S\cdot\bar{X}) - \alpha(S\cdot\bar{X}) + \alpha(S\cdot\bar{X}) \\ &\quad + a(T\cdot X) + c(X, \bar{X}). \end{aligned}$$

Thus $c(X^*, \bar{X}^*) \geqslant \alpha(S) + a(T)$ if, and only if,

$$\beta(S\cdot\bar{X}) + c(X, \bar{X}) \geqslant a(T\cdot\bar{X}). \tag{7}$$

Case 4. $s \in \bar{X}^*$, $t \in \bar{X}^*$. Let $X^* = u + X$, $\bar{X}^* = s + t + v + \bar{X}$. Then

$$\begin{aligned} c(X^*, \bar{X}^*) &= c(u, t) + c(u, \bar{X}) + c(X, t) + c(X, v) + c(X, \bar{X}) \\ &= a(T) + \alpha(S\cdot\bar{X}) + b(T\cdot X) - a(T\cdot X) \\ &\quad + a(T\cdot X) + c(X, \bar{X}), \end{aligned}$$

and we obtain the condition

$$b(T\cdot X) + c(X, \bar{X}) \geqslant \alpha(S\cdot X). \tag{8}$$

We may therefore state the following result.

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THEOREM 1. *The constraints* (3) *are feasible if and only if* (7) *and* (8) *hold for all partitions* $X, \bar{X}$ *of* N.

Notice that (7) is precisely condition (6) for the supply-demand case; that is, if $\alpha(x) = 0$ for $x \in S$, and $b(x) = \infty$ for $x \in T$, then Theorem 1 reduces to the supply-demand theorem of **(4)**. Condition (8) may be interpreted as follows. If we interchange sources and sinks in G, reverse all arc directions, and think of α as the demand function at the set S of sinks, b as the supply function at the set T of sources, then (8) is a necessary and sufficient condition for feasibility of the supplies and demands in the reversed network. Thus Theorem 1 may be restated as follows.

THEOREM 2. *The constraints* (3a) *and* (3b) *are jointly feasible if, and only if, the constraints*

$$(9) \qquad \begin{cases} \alpha(x) \leqslant f(x, A(x)) - f(B(x), x), & x \in S, \\ f(B(x), x) - f(x, A(x)) \leqslant b(x), & x \in T, \end{cases}$$

and

$$(10) \qquad \begin{cases} f(x, A(x)) - f(B(x), x) \leqslant \beta(x), & x \in S, \\ a(x) \leqslant f(B(x), x) - f(x, A(x)), & x \in T, \end{cases}$$

are separately feasible.

Theorem 2 is the formulation described verbally in the Introduction. One suspects that there should be a simple method of constructing a flow satisfying all the constraints from the two separate flows, but we have not found such a method.

We note one other fact for the combinatorial applications. Namely, if the functions α, β, a, b, and c are integral-valued, and if the constraints (3) are feasible, then there is an integral feasible flow f. This follows directly from the proof of Theorem 1 and the existence of integral maximal flows in networks having integral capacities.

4. Application to matrices. When the network G is suitably specialized, Theorem 2 (or Theorem 1) provides criteria for the existence of a non-negative matrix whose row and column sums lie between designated limits, or, more generally, for the existence of a matrix with this property and the further property that the elements of the matrix are bounded above by specified numbers. We state the criteria provided by Theorem 2 explicitly as follows:

THEOREM 3. *Let* $0 \leqslant \alpha_i \leqslant \beta_i$, $i = 1, \ldots, m$, $0 \leqslant a_j \leqslant b_j$, $j = 1, \ldots, n$, *and* $c_{ij} \geqslant 0$ *be given constants. If there are matrices* $f^1{}_{ij}$, $f^2{}_{ij}$ *satisfying*

$$(11) \qquad \alpha_i \leqslant \sum_j f^1_{ij}, \qquad \sum_i f^1_{ij} \leqslant b_j, \qquad 0 \leqslant f^1_{ij} \leqslant c_{ij},$$

$$(12) \qquad \sum_j f^2_{ij} \leqslant \beta_i, \qquad a_j \leqslant \sum_i f^2_{ij}, \qquad 0 \leqslant f^2_{ij} \leqslant c_{ij},$$

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then there is a matrix f_{ij} *satisfying*

(13) $$\alpha_i \leqslant \sum_j f_{ij} \leqslant \beta_i, \qquad a_j \leqslant \sum_i f_{ij} \leqslant b_j, \qquad 0 \leqslant f_{ij} \leqslant c_{ij}.$$

To prove Theorem 3, take G to be the network consisting of nodes x_i $(i = 1, \ldots, m)$, y_j $(j = 1, \ldots, n)$, and arcs (x_i, y_j) of capacity c_{ij}. Let $S = \{x_1, \ldots, x_m\}$, $T = \{y_1, \ldots, y_n\}$, so that R is vacuous. Associate with each source x_i the bounds α_i, β_i, and with each sink y_j the bounds a_j, b_j. Then a flow from S to T is a matrix f_{ij} satisfying $0 \leqslant f_{ij} \leqslant c_{ij}$; a feasible flow satisfies, in addition, the first two inequalities of (13). Thus Theorem 3 is a direct consequence of Theorem 2.

5. Incidence matrices.

Gale **(4)** and Ryser **(9)** have found simple conditions for the existence of a matrix of zeros and ones having prescribed row and column sums—or, what is the same thing, for the existence of an incidence matrix whose row sums are bounded below by given integers and whose column sums are bounded above by given integers.

The following is one interpretation of their problem. Suppose there is given a finite set $E = \{e_1, \ldots, e_m\}$. Under what conditions on the sets of integers $\{\alpha_1, \ldots, \alpha_m\}$ and $\{b_1, \ldots, b_n\}$ is it possible to construct n subsets $E_1, \ldots, E_n$ of E such that (a) the number of sets E_j that contain the element e_i is at least α_i, and (b) the set E_j contains at most b_j elements?

The conditions are surprisingly simple. Arrange the α's in decreasing order,

$$\alpha_{i_1} \geqslant \alpha_{i_2} \geqslant \ldots \geqslant \alpha_{i_m},$$

and define σ_k to be the number of integers in the set of b's that are greater than or equal to k. Then the required incidence matrix exists if, and only if, we have

(14) $$\sum_{k=1}^{l} \alpha_{i_k} \leqslant \sum_{k=1}^{l} \sigma_k, \qquad l = 1, 2, \ldots,$$

where we take $\alpha_{i_k} = 0$ for $k > m$.

As a corollary of the Gale-Ryser condition (14), Theorem 3 with all $c_{ij} = 1$, and the remark at the end of § 3, we have the following result:

THEOREM 4. *There exists a matrix of zeros and ones for which the* ith *row sum lies between given non-negative integers* α_i *and* β_i, *and the* jth *column sum lies between given non-negative integers* a_j *and* b_j, *where* $\alpha_i \leqslant \beta_i$, $a_j \leqslant b_j$, *if, and only if,*

(15) $$\sum_{k=1}^{l} \alpha_{i_k} \leqslant \sum_{k=1}^{l} \sigma_k, \qquad l = 1, 2, \ldots,$$

(16) $$\sum_{k=1}^{l} a_{j_k} \leqslant \sum_{k=1}^{l} \tau_k \qquad l = 1, 2, \ldots,$$

where

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$$\alpha_{i_1} \geqslant \alpha_{i_2} \geqslant \ldots \geqslant \alpha_{i_m}, \qquad a_{j_1} \geqslant a_{j_2} \geqslant \ldots \geqslant a_{j_n},$$

and σ_k is the number of b's, and τ_k the number of β's, that are greater than or equal to k.

6. The subgraph problem. Let G be a finite directed graph, and let $e(x)$ and $i(x)$ be, respectively, the number of arcs entering and the number of arcs issuing from node x. Then the (local) degree of G at x is defined to be the pair $e(x), i(x)$.

The subgraph problem is the problem of determining conditions under which G has a subgraph H having prescribed local degrees. We consider the following generalization of this problem. Associate with each node $x \in N$ four integers $a(x)$, $b(x)$, $\alpha(x)$, $\beta(x)$, satisfying

$$0 \leqslant a(x) \leqslant b(x), \tag{17a}$$

$$0 \leqslant \alpha(x) \leqslant \beta(x), \tag{17b}$$

and determine conditions under which G has a subgraph H with local degrees $e_H(x)$, $i_H(x)$ satisfying

$$a(x) \leqslant e_H(x) \leqslant b(x), \tag{18a}$$

$$\alpha(x) \leqslant i_H(x) \leqslant \beta(x). \tag{18b}$$

To find such conditions, we convert the problem to a flow problem and apply Theorem 1. First construct from G a new directed graph G' having twice as many nodes as G but the same number of arcs: to each node x of G correspond two nodes x', x'' of G'; if (x, y) is an arc of G, then (x', y'') is an arc of G' and these are all the arcs of G'. Assign unit capacity to each arc of G'. In G', let S and T be the set of primed and double primed nodes, respectively. Next impose, for each $x' \in S$, the condition (3a) that the flow out of x' lie between $\alpha(x)$ and $\beta(x)$; similarly, for $x'' \in T$, insist that the flow into x'' lie between $a(x)$ and $b(x)$.

It is clear that an integral feasible flow f from S to T in G' singles out a subgraph H of G satisfying (18) simply by putting (x, y) in H if and only if $f(x', y'') = 1$. Conversely, of course, a subgraph H satisfying (18) produces an integral feasible flow in G'. Hence, if we let U, V be arbitrary subsets of S, T, respectively, and denote their respective complements in S, T by $\bar{U}$, $\bar{V}$, it follows from Theorem 1 and the existence of integral feasible flows that H exists if, and only if,

$$\beta(\bar{U}) + |(U, \bar{V})| \geqslant a(\bar{V}), \tag{19a}$$

$$b(V) + |(U,\bar{V})| \geqslant \alpha(U), \qquad \text{all } U \subset S,\ V \subset T, \tag{19b}$$

where $|\ |$ denotes cardinality.

Before proceeding further, let us consider inequalities (19) in the special case for which $a(x) = b(x)$, $\alpha(x) = \beta(x)$—that is, in the case for which the

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local degrees of H are specified exactly. Then a necessary condition for H to exist is that $\alpha(N) = b(N)$, or, in G',

(20) $$\alpha(S) = b(T).$$

On the other hand, (20) and (19b) now imply (19a), since

$$\begin{aligned}\alpha(\bar{U}) + |(U, \bar{V})| &\geqslant \alpha(\bar{U}) + \alpha(U) - b(V) = \alpha(S) - b(V) \\ &\geqslant b(T) - b(V) = b(\bar{V}),\end{aligned}$$

which is (19a) with $\alpha = \beta$, $a = b$.

Thus, (20) and (19b) are necessary and sufficient for the existence of a subgraph H having local degrees $e_H(x) = b(x)$, $i_H(x) = \alpha(x)$.

Each of the conditions (19a), (19b) is stated in terms of selections of pairs of sets. Each can, however, be simplified to a condition involving the choice of but one set. Consider (19b), for example. For given $U \subset S$, let

$$V = \{y'' \in T \mid b(y'') < |(U, y'')|\}.$$

For this pair U, V, the left-hand side of (19b) may be written as

$$\sum_{y'' \epsilon A(U)} \min\,[b(y''), |(U, y'')|].$$

On the other hand, for fixed $U \subset S$, this sum clearly minimizes $b(V) + |(U,\bar{V})|$ over all $V \subset T$. Thus inequalities (19b) are equivalent to the inequalities

(21) $$\sum_{y'' \epsilon A(U)} \min\,[b(y''), |(U, y'')|] \geqslant \alpha(U), \qquad \text{all } U \subset S.$$

Similarly, (19a) reduces to

(22) $$\sum_{y' \epsilon B(\bar{V})} \min\,[\beta(y'), |(y', \bar{V})|] \geqslant a(\bar{V}), \qquad \text{all } \bar{V} \subset T.$$

Thus, translating (21) and (22) to conditions stated in terms of the given graph G, we have the following theorems:

THEOREM 5. *Let G be a finite directed graph with node set N, and suppose that, corresponding to each $x \in N$, there are integers $a(x)$, $b(x)$, $\alpha(x)$, $\beta(x)$ with*

$$0 \leqslant a(x) \leqslant b(x),$$
$$0 \leqslant \alpha(x) \leqslant \beta(x).$$

Then G has a subgraph H whose local degrees $e_H(x)$, $i_H(x)$ satisfy

$$a(x) \leqslant e_H(x) \leqslant b(x),$$
$$\alpha(x) \leqslant i_H(x) \leqslant \beta(x),$$

if, and only if, for all $X \subset N$, we have

(23) $$\alpha(X) \leqslant \sum_{y \epsilon A(X)} \min\,[b(y), |(X, y)|],$$

(24) $$a(X) \leqslant \sum_{y \epsilon B(X)} \min\,[\beta(y), |(y, X)|].$$

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THEOREM 6 (Ore). *The finite directed graph G has a subgraph H with local degrees*

$$e_H(x) = b(x) \geqslant 0,$$
$$i_H(x) = \alpha(x) \geqslant 0,$$

if, and only if,

$$\alpha(N) = b(N) \tag{25}$$

and, for all $X \subset N$,

$$\alpha(X) \leqslant \sum_{y \in A(X)} \min\,[b(y),\, |(X, y)|]. \tag{26}$$

As a consequence of Theorem 2, we may also state the following result:

THEOREM 7. *If the finite directed graph G has subgraphs* H_1, H_2, *such that*

$$a(x) \leqslant e_{H_1}(x),\ i_{H_1}(x) \leqslant \beta(x),$$
$$e_{H_2}(x) \leqslant b(x),\ \alpha(x) \leqslant i_{H_2}(x),$$

where $0 \leqslant a(x) \leqslant b(x)$, $0 \leqslant \alpha(x) \leqslant \beta(x)$, *then G has a subgraph H such that*

$$a(x) \leqslant e_H(x) \leqslant b(x), \qquad \alpha(x) \leqslant i_H(x) \leqslant \beta(x).$$

For undirected graphs G, the (local) degree of G at x is the number of arcs incident with x, and the subgraph problem is to determine conditions under which G has a subgraph H with prescribed local degrees. In case G has only even cycles, so that the nodes of G can be partitioned into two sets S, T such that all arcs join nodes of S to those of T, the subgraph problem can be stated as a flow problem in G, and hence Theorem 1 can be applied. We know of no way, however, to make use of flow theory in the general case.

7. Systems of representatives. In our applications of the feasibility theorem thus far, the set R of intermediate nodes has been vacuous. We conclude with an application, suggested to us by Gale **(4)**, in which this will not be the case.

Let $E_1, \ldots, E_n$ be subsets of a given set $E = \{e_1, \ldots, e_m\}$. A list

$$D = \{e_{i_1}, \ldots, e_{i_n}\}$$

of n distinct elements of E, such that $e_{ij} \in E_j$, is a system of distinct representatives for $E_1, \ldots, E_n$, in which e_{i_j} represents E_j. (A well-known theorem of P. Hall gives necessary and sufficient conditions for the existence of a system of distinct representatives.) Suppose, in addition, that $P_1, \ldots, P_p$ is a partition of E, and that it is desired to establish existence conditions for a D such that the intersection of D with each P_k has cardinality between prescribed bounds. Hoffman and Kuhn **(5)** have used the duality theorem of linear-equality theory, applied to a linear-programming problem of transportation type, to prove the following theorem:

https://doi.org/10.4153/CJM-1959-045-1 Published online by Cambridge University Press

THEOREM 8 (Hoffman-Kuhn).* *Let* α_k *and* β_k, $k = 1, 2, \ldots, p$, *satisfying* $0 \leqslant \alpha_k \leqslant \beta_k$, *be integers associated with a partition* $P_1, \ldots, P_p$ *of a given set* $E = \{e_1, \ldots, e_m\}$. *The subsets* $E_1, \ldots, E_n$ *of* E *have a system of distinct representatives* D *satisfying* $\alpha_k \leqslant |D \cdot P_k| \leqslant \beta_k$, $k = 1, \ldots, p$, *if, and only if,*

$$\left| \left(\sum_{k \epsilon U} P_k \right) \cdot \left(\sum_{j \epsilon V} E_j \right) \right| \geqslant |V| - \sum_{k \epsilon \bar{U}} \beta_k, \tag{27}$$

$$\left| \left(\sum_{k \epsilon U} P_k \right) \cdot \left(\sum_{j \epsilon V} E_j \right) \right| \geqslant |V| - n + \sum_{k \epsilon U} \alpha_k, \tag{28}$$

hold for all subsets $U \subset \{1, \ldots, p\}$ *and* $V \subset \{1, \ldots, n\}$.

To establish (27) and (28) as necessary and sufficient conditions for the existence of the required system of distinct representatives, we set up the following feasibility problem. Let

$$S = \{x_1, \ldots, x_p\},$$
$$R = \{y_1, \ldots, y_m\},$$
$$T = \{z_1, \ldots, z_n\},$$

be the nodes of a network G, and define arcs in G as follows:

(x_k, y_i) is an arc if, and only if, $e_i \in P_k$,
(y_i, z_j) is an arc if, and only if, $e_i \in E_j$.

The capacity function is taken to be

$$c(x_k, y_i) = 1,$$
$$c(y_i, z_j) = \infty.$$

With each $x_k \in S$, associate the bounds α_k, β_k on the flow leaving x_k, and similarly require that the flow into $z_j \in T$ be precisely unity ($a(z_j) = b(z_j) = 1$).

From the definition of the capacity function and the assumption that $P_1, \ldots, P_p$ is a partition of E, it follows that the amount of flow through each node $y_i \in R$ is at most one. Thus an integral feasible flow f from S to T picks out a set D fulfilling the hypotheses of the theorem:

$$D = \{e_i | f(S, y_i) = f(y_i, T) = 1\}.$$

Conversely, given a D satisfying the assumptions of the theorem, we can define an integral feasible flow f by

$$f(x_k, y_i) = \begin{cases} 1 & \text{if } e_i \in D \cdot P_k, \\ 0 & \text{otherwise;} \end{cases}$$

$$f(y_i, z_j) = \begin{cases} 1 & \text{if } e_i \text{ represents } E_j, \\ 0 & \text{otherwise.} \end{cases}$$

*It is also stated in (5) that the authors have not been able to prove this result without using the duality theorem. However, Gale has recently shown that Theorem 8 is a consequence of the circulation theorem. It is therefore not surprising that the result can be deduced from our Theorem 1.

https://doi.org/10.4153/CJM-1959-045-1 Published online by Cambridge University Press

Thus the feasibility problem in G is equivalent to the existence of a D meeting the requirements of the theorem, and we may consequently apply Theorem 1. Let $X, \bar{X}$ be a partition of the nodes of G, and set

$$S \cdot X = U, \quad R \cdot X = W, \quad T \cdot X = \bar{V},$$
$$S \cdot \bar{X} = \bar{U}, \quad R \cdot \bar{X} = \bar{W}, \quad T \cdot \bar{X} = V.$$

Then (7) and (8) become

$$\beta(\bar{U}) + c(X, \bar{X}) \geqslant |V|, \tag{29}$$

$$|V| + c(X, \bar{X}) \geqslant \alpha(U), \tag{30}$$

respectively. Since $c(y_i, z_j) = \infty$, these conditions hold automatically unless $(X, \bar{X})$ contains no arcs from R to T. Thus we may restrict attention to partitions $X, \bar{X}$ such that $B(V) \subset \bar{W}$, so that $c(X, \bar{X}) = c(U, \bar{W})$. But since the right-hand sides of (29) and (30) are independent of W, it suffices to select $\bar{W} = B(V)$. Then we have

$$c(X, \bar{X}) = c(U, B(V)) = |A(U) \cdot B(V)|.$$

Consequently, a feasible flow from S to T exists if, and only if,

$$|A(U) \cdot B(V)| \geqslant |V| - \beta(\bar{U}), \tag{31}$$

$$|A(U) \cdot B(V)| \geqslant \alpha(U) - |\bar{V}|, \quad \text{all } U \subset S, \quad V \subset T. \tag{32}$$

Replacing $|\bar{V}|$ by $n - |V|$ in (32) and translating (31) and (32) into set-theoretic statements yield (27) and (28), respectively. Thus (27) is a necessary and sufficient condition that there be a system of distinct representatives D such that $|D \cdot P_k| \leqslant \beta_k$, whereas (28) is a necessary and sufficient condition that there be a system of distinct representatives D with $|D \cdot P_k| \geqslant \alpha_k$.

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