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SIMPLE PROOFS OF SOME THEOREMS ON HIGH DEGREES OF UNSOLVABILITY

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If $\boldsymbol{a}$ is a degree of unsolvability, $\boldsymbol{a}$ is called *high* if $\boldsymbol{a} \leqq \mathbf{0}'$ and $\boldsymbol{a}' = \mathbf{0}''$. In [**1**], S. B. Cooper showed that if $\boldsymbol{a}$ is high, then (i) $\boldsymbol{a}$ is not a minimal degree, and (ii) there is a minimal degree $\boldsymbol{b} < \boldsymbol{a}$. We give new proofs of these results which avoid the intricate priority and recursive approximation arguments of [**1**] in favor of "oracle" constructions using the recursion theorem. Also our constructions apply to degrees $\boldsymbol{a}$ which are not below $\mathbf{0}'$. Call a degree $\boldsymbol{a}$ *generalized high* if $\boldsymbol{a}' = (\boldsymbol{a} \cup \mathbf{0}')'$. Among the degrees $\leqq \mathbf{0}'$, the generalized high degrees obviously coincide with the high degrees. We show that if $\boldsymbol{a}$ is generalized high, then i′) there is a nonzero degree $\boldsymbol{b} < \boldsymbol{a}$ such that $\boldsymbol{b}' = \boldsymbol{b} \cup \mathbf{0}'$, and ii′) there is a minimal degree $\boldsymbol{b} < \boldsymbol{a}$. The main point of the present paper is to give simple proofs for the cited results of Cooper rather than to extend them from high to generalized high degrees. However, this extension is of some interest for the following reasons pointed out by D. Posner:

(a) Cooper's result [**1**; **2**] that all degrees $\geqq \mathbf{0}'$ are jumps of minimal degrees seems to present a barrier to extending his result that high degrees are not minimal to degrees which are not necessarily below $\mathbf{0}'$ but satisfy some condition involving the jump operation. For instance it shows that the condition $\boldsymbol{a}' \geqq \mathbf{0}''$ is *not* a suitable extension of the notion of "high", at least for the purposes at hand. However, the notion of "generalized high" is suitable for extending many results about high degrees, and the class of generalized high degrees is a reasonably rich class of degrees as explained in (b).

(b) The generalized high degrees "generate" the set of all degrees in the sense that every degree is the greatest lower bound of a pair of generalized high degrees. To see this, relativize the construction of a minimal pair of high degrees to an arbitrary degree $\boldsymbol{c}$ to obtain degrees $\boldsymbol{a}_1, \boldsymbol{a}_2$ having greatest lower bound $\boldsymbol{c}$ such that $\boldsymbol{c} \leqq \boldsymbol{a}_i'$ and $\boldsymbol{c}' = \boldsymbol{a}_i''$ for $i = 1, 2$. The degrees $\boldsymbol{a}_1, \boldsymbol{a}_2$ are clearly generalized high. (A minimal pair of high r.e. degrees is constructed in [**6**, Theorem 2] but a minimal pair of high degrees may be obtained much more easily as mentioned in [**2**, p. 130].)

Very recently Posner and the author have shown that the conclusion of (i′) follows from the weaker hypothesis $\boldsymbol{a}'' = (\boldsymbol{a} \cup \mathbf{0}')'$. This and related results will appear in a future joint paper. The proof is a simple "oracle argument" but use of the recursion theorem is supplanted by a rudimentary priority

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argument. Although result (i′) is rendered obsolete by this development, we include its proof here anyway as an optimally simple illustration of the method which is used to prove (ii′) and has been used by Posner [**7**; **8**] to obtain a number of other results about high degrees. (Some of these results do not seem amenable to the full approximation methods of Cooper, or indeed to any full approximation methods.) We do not know whether (ii′) follows from the weaker hypothesis that $\mathbf{a}'' > (\mathbf{a} \cup \mathbf{0}')'$, but we conjecture that it does not.

We are grateful to Posner for helpful discussions and information on the subject of this paper.

Our notation and terminology are standard. In particular, we use the letters $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ for degrees and A, B, C for subsets of $\omega = \{0, 1, 2, \ldots\}$. We write $\leqq_T$, $'$, $\oplus$ for Turing reducibility, jump, and join respectively on subsets of ω, and $\leqq$, $'$, $\cup$ for the induced ordering and operations on the degrees. Subsets of ω are identified with their characteristic functions, so $B(x) = 1$ if and only if $x \in B$. *Strings* are functions from finite initial segments of ω into $\{0, 1\}$. The letters δ, σ, τ always denote strings. A string σ is a *beginning* of a set B if σ is extended by the characteristic function of B. When we write $\sigma \subseteq \tau$, $\bigcup_s \tau_s$, we are viewing strings as sets of ordered pairs. We assume strings are Gödel-numbered and sometimes identify them with their Gödel numbers. The notation $C = \lim_s C_s$ means that for each n there exists a number $s(n)$ such that $C(n) = C_s(n)$ for all $s \geqq s(n)$. The Limit Lemma [**12**, p. 29] asserts that $C \leqq_T A'$ if and only if there is a sequence of sets $\{C_s\}$ which are uniformly recursive in A such that $C = \lim_s C_s$. We write $\{e\}^\sigma(x) = y$ if the eth Turing reduction procedure, given argument x and oracle information σ, gives output y. Of course $\{e\}^B(x) = y$ means that $\{e\}^\sigma(x) = y$ for some beginning σ of B. Thus $B' = \{e : \{e\}^B(e) \text{ is defined}\}$. Let $\langle . , . \rangle$ be a 1—1 recursive map from ω^2 onto ω.

THEOREM 1. *If* $\mathbf{a}$ *is generalized high, then there is a non-zero degree* $\mathbf{b} < \mathbf{a}$ *such that* $\mathbf{b}' = \mathbf{b} \cup \mathbf{0}'$.

Proof. The argument is a primitive forcing argument in the style of the proof of the Friedberg completeness criterion [**12**, Ch. 10]. Given $e, x \in \omega$ and a string σ, we say that $\{e\}^\sigma(x)$ is *strongly undefined* if $\{e\}^\tau(x)$ is undefined for all strings $\tau \supseteq \sigma$. Given $B \subseteq \omega$ we say $\{e\}^B(x)$ is *strongly undefined* if $\{e\}^\sigma(x)$ is strongly undefined for some beginning σ of B. Define B^* to be the set of $e \in \omega$ such that $\{e\}^B(e)$ is strongly undefined. Clearly $B^* \cap B' = \emptyset$ for all $B \subseteq \omega$. Observe that B^* is r.e. in $B \oplus 0'$ (since $\{(e, \sigma) : \{e\}^\sigma(e) \text{ is strongly undefined}\}$ is recursive in $0'$). The set B is called 1-*generic* if $B^* \cup B' = \omega$, i.e. $B^* = \omega - B'$. (A set is 1-generic just in case it is generic for 1-quantifier arithmetical statements with respect to forcing with finite conditions, cf [**4**]). The following lemma is well-known.

LEMMA 2. *If* B *is a* 1-*generic set, then* $B' \leqq_T B \oplus 0'$ *and* B *is not recursive.*

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Proof. The sets B', B^* are each r.e. in $B \oplus 0'$ and are complementary by assumption. Hence $B' \leqq_T B \oplus 0'$.

To show that B is not recursive, we consider an arbitrary recursive function f and show that $f \neq B$. First, let e be a number such that for any set C, $\{e\}^C(e)$ is defined if and only if there exists a number k such that $f(k) \neq C(k)$. Such an e exists since f is recursive. Then $e \notin B^*$, since no finite amount of information about B can guarantee that $f = B$. Hence $e \in B'$ by 1-genericity, so $f \neq B$ as required.

We remark that 1-genericity has numerous consequences in addition to those mentioned in Lemma 2, and these may be used to strengthen Theorem 1. For instance, if B is 1-generic, then no non-recursive r.e. set is recursive in B [**3**]. Also if B is 1-generic, then every countable partially ordered set may be embedded in the degrees below the degree of B. To show the latter, it suffices by the proof of [**10**, § 4, Corollary 3] to find a recursively independent sequence of sets $B_0, B_1, \ldots$ which are uniformly recursive in B. To do this, let $B_i = \{j : \langle i, j\rangle \in B\}$. The sequence of B_i's is recursively independent since, whenever D is a finite join of sets B_j with $j \neq i$ and $\{e\}^D$ is total, no finite amount of information about B can force $\{e\}^D$ to be B_i.

Thus to prove Theorem 1 it suffices to show that for any set A satisfying $(A \oplus 0')' \leqq_T A'$, there is a 1-generic set $B \leqq_T A$. The "classical" construction of a 1-generic set B is to obtain the characteristic function of B as $\bigcup_e \sigma_e$ where $\{\sigma_e\}$ is an inductively defined, $\subseteq$-ascending, sequence of strings such that $\{e\}^{\sigma_{e+1}}(e)$ is either defined or strongly undefined. We follow this idea in constructing our set B, but in order to arrange that $B \leqq_T A$, we make infinitely many "appropriately bounded" attacks on the requirement that $\{e\}^B(e)$ be defined or strongly undefined. The attacks are arranged so that for any given e, all sufficiently late attacks are successful. Recall that B^* is r.e. in $B \oplus 0'$, so $B^* \leqq_T (B \oplus 0')'$. If A is a set such that $(A \oplus 0')' \leqq_T A'$ and we construct $B \leqq_T A$, it follows that $B^* \leqq_T (B \oplus 0')' \leqq_T (A \oplus 0')' \leqq_T A'$. Therefore by the Limit Lemma [**12**, p. 29] there exists a sequence of sets B_s^*, uniformly recursive in A, such that $\lim_s B_s^* = B^*$. Although the B^*_s depend on B, the recursion theorem will justify their use in the construction of B. (This will be explained further after the construction of B.) We obtain B as $\bigcup_e \sigma_e$ where $\sigma_0 \subseteq \sigma_1 \subseteq \ldots$ are strings, Let σ_0 be the empty string. Suppose inductively that σ_s has been defined. Let $s = \langle e, n\rangle$. We attempt to arrange that σ_{s+1} is a string σ such that $\{e\}^\sigma(e)$ is defined or strongly undefined, thus guaranteeing $e \in B' \cup B^*$. To obtain σ_{s+1} search simultaneously for $t \geqq s$ such that $e \in B_t^*$ and a string $\sigma \supsetneqq \sigma_s$ such that $\{e\}^\sigma(e)$ is defined. If the search first yields a $t \geqq s$ such that $e \in B_t^*$, let σ_{s+1} be any (effectively chosen) proper extension of σ_s. If the search first yields $\sigma \supsetneqq \sigma_s$ with $\{e\}^\sigma(e)$ defined, let σ_{s+1} be that σ. Note that the search must terminate roughly because if no $\sigma \supsetneqq \sigma_s$ with $\{e\}^\sigma(e)$ defined exists, then $e \in B^*$ so $e \in B_t^*$ for all sufficiently large t. Also if $e \notin B^*$, then for all sufficiently large n the search at stage $\langle e, n\rangle$

cannot find a t and hence must find a σ, so $e \in B'$. Hence B is 1-generic. Since the construction of B is carried out recursively in the sequence B_s^* which is uniformly recursive in A, we have $B \leqq_T A$ as required.

It remains now to justify using B_s^* in the definition of B. The argument is roughly that the recursion theorem (relativized to A) allows use of a number i such that $B = \{i\}^A$ in the construction of B, and an index j such that $\{\langle e, t\rangle : e \in B_t^*\} = \{j\}^A$ may be effectively calculated from i. However, a precise argument requires allowing for the possibility that some search in the construction never teminates and so the attempt to construct the characteristic function of B yields only a string. If ψ is a partial function, let $\psi^* = \{e : (\exists\ \sigma)[\psi \supseteq \sigma$ and $\{e\}^\sigma(e)$ is strongly undefined$]\}$. (If $\psi = C$, clearly $\psi^* = C^*$.) Observe that $(\{i\}^A)^*$ is r.e. in $A \oplus 0'$, uniformly in i. Hence $(\{i\}^A)^* \leqq_T (A \oplus 0') \leqq_T A'$ uniformly in i, so there is a double sequence of sets $B_s^{*,i}$ uniformly recursive in A, such that $\lim_s B_s^{*,i} = (\{i\}^A)^*$ for all i. By the uniformity of the construction there is a recursive function h such that, for all i, $\{h(i)\}^A$ is the union of the strings σ_s obtained from the construction when B_s^* is replaced by $B_s^{*,i}$. By the recursion theorem (relativized to A) there is a number z such that $\{z\}^A = \{h(z)\}^A$. For this z, the argument that all searches terminate in the construction using $B_s^{*,z}$ for B_s^* is easily made precise, so this construction yields a *total* function $\{h(z)\}^A$ which is the characteristic function of a 1-generic set $B \leqq_T A$.

COROLLARY 3 (Cooper). *If* $\boldsymbol{a}$ *is high, then* $\boldsymbol{a}$ *has a nonzero predecessor* $\boldsymbol{b}$ *satisfying* $\boldsymbol{b}' = \boldsymbol{0}'$, *so in particular* $\boldsymbol{a}$ *is not minimal.*

In [**1**, Theorem 2] Cooper actually asserted only that high degrees are not minimal. However, it was known to Cooper that Corollary 3 could be obtained by his methods and in fact Posner has recently observed that his construction (as it stands) produces a degree $\boldsymbol{b}$ as in Corollary 3.

THEOREM 4. *If* $\boldsymbol{a}$ *is generalized high, then there is a minimal degree* $\boldsymbol{b} < \boldsymbol{a}$.

Proof. The idea of the proof is to combine the technique of the Sacks construction of a minimal degree $< \boldsymbol{0}'$ [9] (as simplified by Shoenfield [**12**]) with the method of Theorem 2 for replacing an oracle for $\boldsymbol{0}'$ by one for a set of degree $\boldsymbol{a}$. The construction which emerges is a priority argument in which the number of injuries to each requirement is finite but not apparently recursively bounded. By contrast, in the Sacks construction there is a recursive bound to the number of times a given requirement can be injured while the proof of [**1**, Theorem 3] is an infinite injury priority argument. (A thorough exposition of the full approximation method used to prove [**1**, Theorem 3] is given in [**2**].)

We assume the reader to be familiar with some construction of a minimal degree below $\boldsymbol{0}'$. We now specify our terminology, which is essentially from [**12**, Ch. 11]. A *tree* is a partial recursive function from the set of strings to the set of strings such that, for any string σ, if one of $T(\sigma*0)$ and $T(\sigma*1)$ is defined, then all of $T(\sigma)$, $T(\sigma*0)$, and $T(\sigma*1)$ are defined, and $T(\sigma*0)$, $T(\sigma*1)$ are incompatible extensions of $T(\sigma)$. A string is *on* a tree T if it is in the range of T. A set A is a *branch* of a tree T if infinitely many beginnings of A are on T. A tree T' is a *subtree* of a tree T if every string on T' is on T.

Two strings σ, τ are called *e-split* if $\{e\}^{\sigma}(x)$ and $\{e\}^{\tau}(x)$ are defined and unequal for some x. A tree T is called an *e-splitting* tree if $T(\sigma^*o)$, $T(\sigma*1)$ are e-split whenever $T(\sigma*0)$ is defined. A string σ on T is said to be *e-splittable* on T if it has a pair of e-split extensions on T.

Suppose that all trees are reasonably Gödel-numbered, and let Z_i be the tree with Gödel number i.

If T' is a subtree of T and σ is on T, T' is called *an e-splitting subtree of T for σ* if (i) $T'(\emptyset) = \sigma$, (ii) T' is e-splitting, and (iii) every string on T' which is e-splittable on T is e-splittable on T' (necessarily by its two immediate successors on T').

For every tree T, string σ on T and number e, there exists a T' as above. (Of course T' may have many terminal nodes or even be finite.) Furthermore an index for T' may be effectively found from e, σ, and an index of T.

In the limit our construction will produce a sequence of trees $\{T_i\}$ and a sequence of strings $\{\delta_s\}$ such that

I. T_{i+1} is a subtree of T_i for all i,
II. $\delta_s \subsetneq \delta_{s+1}$ for all s,
III. $\bigcup_s \delta_s$ is a branch of T_i for all i, and
IV. for all e, either (a) T_{e+1} is an e-splitting subtree of T_e above some δ_s, or (b) $T_{e+1} = T_e$ and some δ_s is not e-splittable on T_e.

Let B be the set whose characteristic function is $\bigcup_s \delta_s$. We say B is *e-minimal* if $\{e\}^B$ is either recursive, non-total, or of the same degree as B. Standard lemmas [**12**, Ch. 11] show that B is e-minimal for all e. Specifically if $\{e\}^B$ is total, then IV(a) implies that $B \leqq_T \{e\}^B$ and IV(b) implies that $\{e\}^B$ is recursive.

Each tree T_e is obtained from a sequence of trees T_e^s such that $T_e^s = T_e$ for all sufficiently large s. The idea of the construction is that once T_e^s has "settled down" to T_e we can use an oracle for the given degree $\boldsymbol{a}$ to tell us correctly "in the limit" whether some beginning of B fails to be e-splittable on T_e. With the guidance of the oracle we eventually make either IV(a) or IV(b) hold.

To compare this with the Sacks-Shoenfield construction, observe that the question Q_e of whether every δ_s is e-splittable on T_e is a Π_2^0 question in the context of that construction. Hence in that construction the answer to Q_e can be stagewise approximated recursively in $\mathbf{0}'$. The approximation to the answer to Q_e changes at most once (after T_e^s has stabilized to T_e) since Q_e is a co-r.e. question relative to $\mathbf{0}'$. In the present construction there is no obvious recursive bound on the number of times the approximation changes, but since the approximations are eventually correct the additional changes do not complicate the proof that the construction works.

One difficulty which arises here but not in the Sacks-Shoenfield construction is that the approximation may indicate that every δ_s is e-splittable on T_e when in fact not even the current δ_s is e-splittable on T_e. The potential pitfall in this situation is that the construction could bog down in an endless vain search for a proper extension δ of δ_s on the appropriate e-splitting subtree of T_e. To avoid this pitfall, the search for such an extension of δ_s is dovetailed with a search for a number $t > s$ such that at stage t the approximation indicates that the answer to Q_e is negative.

In the construction we work with indices t_e^s for the trees T_e^s. There will be a function $k(s)$ such that t_e^s is defined exactly for $e \leqq k(s)$.

For any set C, let $C^- = \{\langle e, k\rangle$: some beginning of C is on Z_k but is not e-splittable on $Z_k\}$. (The set C^- is analogous to C^* in Theorem 2.) Observe that C^- is r.e. in $C \oplus 0'$. Let A be a set of the given degree $\boldsymbol{a}$ satisfying $\boldsymbol{a}' = (\boldsymbol{a} \cup \boldsymbol{0}')'$. If we construct $B \leqq_T A$, we will have $B^- \leqq_T (B \oplus 0')' \leqq_T (A \oplus 0')' \leqq_T A'$ so there will be a sequence of sets B_s^-, uniformly recursive in A, such that $\lim_s B_s^- = B^-$.

As in Theorem 2, our construction will be sufficiently uniform that use of B_s^- in the construction of B is justified by the recursion theorem. (A few comments on this justification will be made after the proof of Lemma 5.) At stage 0, let $k(0) = 0$, and let t_0^0 be an index of the identity tree, i.e. $T_0^0(\sigma) = \sigma$ for all σ.

Assume inductively now that stage s has been completed and that $k(s)$ and $t_e^s(e \leqq k(s))$ have been defined so that $t_0^s = t_0^0$ and for $e < k(s)$, either $t_{e+1}^s = t_e^s$ or T_{e+1}^s is an e-splitting subtree of T_e^s above some δ_t with $t \leqq s$. (Here T_e^s is the tree with index t_e^s.) For $e \leqq k(s)$, let y_e^s be the index of an effectively chosen e-splitting subtree Y_e^s of T_e^s above δ_s.

At stage $s + 1$, search in an exhaustive A-recursive manner for numbers k, t with $k \leqq k(s)$ and $t \geqq s$ and a string δ properly extending δ_s such that

(i) for all $e < k$,

$$t_{e+1}^s = t_e^s \quad \Leftrightarrow \quad \langle e, t_e^s\rangle \in B_t^-, \quad \text{and}$$

(ii) either (a) $t_{k+1}^s = t_k^s$, $\langle k, t_k^s\rangle \notin B_t^-$, and δ is on Y_k^s or (b) $t_{k+1}^s \neq t_k^s$, $\langle k, t_k^s\rangle \in B_t^-$ and δ is on T_k^s.

(N.B. For $k = k(s)$, t_{k+1}^s is undefined and the meaningless statements $t_{k+1}^s = t_k^s$, $t_{k+1}^s \neq t_k^s$ should simply be ignored.)

If no such t, k, δ are ever found, we say the construction *bogs down* at stage $s + 1$. (It will be shown that this cannot happen.) Otherwise, let (t, k, δ) be the first such triple which is found. Let $\delta_{s+1} = \delta$ and $k(s + 1) = k + 1$. Define $t_e^{s+1} = t_e^s$ for all $e \leqq k$. Finally, if case (ii) (a) applies let $t_{k+1}^{s+1} = y_k^s$, and otherwise let $t_{k+1}^{s+1} = t_k^s$. (Observe that δ_{s+1} is on T_{k+1}^{s+1} in either case).

LEMMA 5. *The construction does not bog down at any stage.*

Proof. Assume for a contradiction that the construction bogs down at stage $s + 1$, so the construction produces only the string δ_s instead of the total func-

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tion B. Extending the $^-$ operator to partial functions δ_s, one has $\langle e, j\rangle \in \delta_s^-$ if and only if for some $\tau \subseteq \delta^s$, τ is on Z_j but is not e-splittable on Z_j. Now let k be the largest number $\leqq k(s)$ such that for all sufficiently large t,

$$(1)\quad t_{e+1}^s = t_e^s \quad\Leftrightarrow\quad (e, t_e^s) \in B_t^- \quad\Leftrightarrow\quad (e, t_e^s) \in \delta_s^- \quad \text{for all } e < k.$$

Such a k exists since (1) holds vacuously for $k = 0$. By the maximality of k, and the fact that $\lim_t B_t^- = \delta_s^-$, one also has for all sufficiently large t that

$$(2)\quad t_{k+1}^s \neq t_k^s \quad\Leftrightarrow\quad \langle k, t_k^s\rangle \in B_t^- \quad\Leftrightarrow\quad \langle k, t_k^s\rangle \in \delta_s^-.$$

(Ignore the first clause if $k = k_s$).

Fix $t \geqq s$ satisfying (1) and (2). It is clear that all parts of (i) and (ii) of the construction not referring to δ are satisfied by this k and t so it remains to produce an appropriate δ. To do this we first show by induction on e that there is a proper extension of δ_s on T_e^s for $e \leqq k$. This is clear for $e = 0$ since every string is on the identity tree. Also the induction step is immediate if $t_{e+1}^s = t_e^s$. So assume $e < k$ and $t_{e+1}^s \neq t_e^s$. Then by (1), $\langle e, t_e^s\rangle \notin \delta_s^-$, so δ_s is e-splittable on T_e^s. Since $t_{e+1}^s \neq t_e^s$, T_{e+1}^s is an e-splitting subtree of T_e^s containing δ_s and so δ_s is e-splittable on T_{e+1}^s. Therefore there is a proper extension of δ_s on T_{e+1}^s as required. In particular there is a proper extension $\hat{\delta}$ of δ_s on T_k^s.

If $\langle k, t_k^s\rangle \in B_t^-$, then $t_{k+1}^s \neq t_k^s$ (or $k = k_s$) by (2) and so we may satisfy (ii)(b) in the construction by letting $\delta = \hat{\delta}$. Otherwise $\langle k, t_k^s\rangle \notin B_t^-$ and so by (2) it follows that $\langle k, t_k^s\rangle \notin \delta_s^-$ so δ_s is k-splittable on T_k^s. Thus there is a $\delta \supsetneqq \delta_s$ on Y_k^s, so (ii)(a) is satisfied for this δ. This completes the proof of Lemma 5.

At this point we remark that the recursion theorem (relativized to A) may be used to justify the use of B_s^- in the definition of B in essentially the same way it was used in Theorem 1 to justify the use of B_s^* in the definition of B. Of course the proof of Lemma 5 is now used to show that $\{z\}^A$ is total, where z is the "fixed point" obtained as before from the recursion theorem.

By Lemma 5, $\cup_s \delta_s$ is the characteristic function of a set B. Since the construction may be carried out recursively in A, B is recursive in A. The proof that B is e-minimal for all e is almost identical to the corresponding proof in the construction of a minimal degree below $\mathbf{0}'$. Specifically, one shows by induction on e that t_e^s is defined and equal to a limiting value t_e for all sufficiently large s. This is clear for $e = 0$. Assume inductively that $t_e^s = t_e$ for $s \geqq s_0$. Choose $s_1 \geqq s_0$ so that $\langle e, t_e\rangle \in B^- \Leftrightarrow \langle e, t_1\rangle \in B_t^-$ for all $t \geqq s_1$. If $t_{e+1}^{s_1}$ is defined, then $t_{e+1}^{s_1} = t_e^s$ for $s \geqq s_1$. Otherwise $k(s_1) = e$, so $t_{e+1}^{s_1+1}$ is defined and $t_{e+1}^{s+1} = t_e^s$ for all $s \geqq s_1$. Let T_e be the tree with index t_e. It is easy to see that statements $I - IV$ at the beginning of the proof hold, and B is e-minimal for the reasons outlined there.

Let $\boldsymbol{b}$ be the degree of B. To see that $\boldsymbol{b}$ is minimal it remains to show that B is nonrecursive. The nonrecursiveness of B may be easily arranged by choosing functions $h_0, h_1, \ldots,$ uniformly recursive in A and including all recursive functions, and modifying the construction so that δ_{s+1} is incompatible with h_s for all s. (Such a sequence $h_0, h_1 \ldots$ exists by [**5**] since $\boldsymbol{a}' \geqq \mathbf{0}''$.) However, the following lemma, due to Posner, shows that this modification is unnecessary.

LEMMA 6 [**3**]. *Suppose for each n, B is a branch of a tree T_{n+1} such that either T_{n+1} is an n-splitting tree or B has a beginning on T_{n+1} which is not n-splittable on T_n. Then B is not recursive.*

Proof. Suppose B were recursive. Choose e so that, for all C,

$$\{e\}^C(x) = \begin{cases} C(x) & \text{if } (\exists y)\ C(y) \neq B(y) \\ \text{undefined otherwise} \end{cases}$$

If $\sigma \subseteq B$, then $\{e\}^\sigma(x)$ is undefined for all x, and so σ is not part of any e-splitting pair on T_{e+1}. By assumption then there is a string σ on T_{e+1} such that σ is not e-splittable on T_{e+1}. We may assume also that $\{e\}^\sigma = \sigma$ whenever $\sigma \nsubseteq B$.

Let δ_1 and δ_2 be two strings on T_{e+1} which extend δ and are incompatible with each other and with B. (Such strings may be obtained by choosing two distinct beginnings of B on T_{e+1} each extending δ, say μ_1, μ_2, and then choosing δ_i to be the immediate successor of μ_i on T_{e+1} which is incompatible with B.) Then δ_1 and δ_2 witness that δ is e-splittable on T_{e+1}. This is a contradiction.

We close with some remarks about requirements which may be imposed on the jump and double jump of a minimal degree $\boldsymbol{b}$ which is constructed below a given generalized high degree $\boldsymbol{a}$. It follows from the recent result cited at the beginning of the paper that every minimal degree $\boldsymbol{b}$ satisfies $\boldsymbol{b}'' = (\boldsymbol{b} \cup \mathbf{0}')'$. Thus if $\boldsymbol{a}$, $\boldsymbol{b}$ are as above, one has $\boldsymbol{b}'' = (\boldsymbol{b} \cup \mathbf{0}')' \leqq (\boldsymbol{a} \cup \mathbf{0}')' = \boldsymbol{a}'$.

One may also require that $\boldsymbol{b}' \neq \boldsymbol{b} \cup \mathbf{0}'$ in Theorem 4. In fact one may require that $\boldsymbol{b}' \nleqq \boldsymbol{c}$ for any fixed degree $\boldsymbol{c}$ satisfying $\boldsymbol{c}' \leqq \boldsymbol{a}'$. The proof is based on the idea of Sasso's proof [**11**] that there is a minimal degree $\boldsymbol{b}$ satisfying $\boldsymbol{b}' \neq \boldsymbol{b} \cup \mathbf{0}'$ together with the refinements of Sasso, Epstein, and Cooper used to push $\boldsymbol{b}$ below $\mathbf{0}'$ (cf. [**11**] or [**13**]). However, additional technical complications of no great interest arise, and we omit the proof.

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