Proceedings of the Edinburgh Mathematical Society (1980), **23**, 1–7 ©.

LIMITS AND COLIMITS IN CATEGORIES OF D.G. NEAR-RINGS

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(Received 10th May 1978)

In this paper, using the idea of upper and lower faithful d.g. near-rings, introduced in **(6)**, we show that the category $\mathscr{B}$ of all faithful d.g. near-rings is a reflective as well as coreflective subcategory of the category $\mathscr{A}$ of all d.g. near-rings. We also prove that both $\mathscr{A}$ and $\mathscr{B}$ are complete and cocomplete categories.

1. Preliminaries

A left near-ring is a set R together with two operations, namely addition + and multiplication $\cdot$, (but we normally omit the symbol $\cdot$) such that $(R,+)$ is a group, not necessarily abelian, $(R,\cdot)$ is a semigroup, and R satisfies the left distributive law

$$x(y+z)=xy+xz \quad \text{for all } x, y, z \in R.$$

The additive identity is denoted by 0.

An element $r \in R$ is called distributive if $(x+y)r = xr + yr$ for all $x, y \in R$. The set of all distributive elements of R forms a multiplicative semigroup. R is called a distributively generated near-ring, usually written as 'a d.g. near-ring', if

$$(R,+) = gp\{S : S \text{ is a multiplicative semigroup of distributive elements}\}$$

where S need not be the set of all distributive elements. Since S is important this d.g. near-ring is denoted by (R, S).

Let G be a group (we will write all groups additively). The set of all mappings of G into itself with pointwise addition and multiplication as composition of maps forms a near-ring. The set $\mathrm{End}(G)$ of endomorphisms of G forms a semigroup of distributive elements of this near-ring. $\mathrm{End}(G)$ generates a d.g. near-ring denoted by $(E(G), \mathrm{End}(G))$.

A near-ring homomorphism θ from a near-ring R to a near-ring T is a group homomorphism from $(R,+)$ to $(T,+)$ and a semigroup homomorphism from $(R,\cdot)$ to $(T,\cdot)$. A d.g. near-ring homomorphism $\theta:(R,S)\to(T,U)$ is a near-ring homomorphism from R to T such that $S\theta \subseteq U$. The following result about d.g. near-ring homomorphisms, proved in **(6)** will be used.

Theorem 1.1. *Let (R,S) and (T,U) be two d.g. near-rings. If θ is a group homomorphism from $(R,+)$ to $(T,+)$ which is also a semigroup homomorphism $(S,\cdot)\to(U,\cdot)$ then it is a d.g. near-ring homomorphism.*

Let G be a group and (R, S) a d.g. near-ring. A d.g. near-ring homomorphism $\theta:(R, S)\to(E(G), \mathrm{End}(G))$ is called a representation of (R, S) on G and G is called an (R, S) group. We often omit the map θ and write gr for $g(r\theta)$, where $g\in G$, $r\in R$. We call a representation θ faithful if $\mathrm{Ker}\ \theta=\{0\}$, and a d.g. near-ring which has a faithful representation is called a faithful d.g. near-ring.

G is called an S-group if there is a semigroup homomorphism $\theta: S\to \mathrm{End}(G)$. Again we write gs for $g(s\theta)$, $g\in G$, $s\in S$. A homomorphism ϕ from an (R, S) group (S-group) G to another (R, S) group (S-group) H is an (R, S) homomorphism (S homomorphism) if $(gr)\phi=(g\phi)r$ for all $r\in R$ ($(gs)\phi=(g\phi)s$ for all $s\in S$). Fröhlich proved the following result (2.1.1. of (**2**)).

Theorem 1.2. *If G and H are (R, S) groups, then a homomorphism $\phi: G\to G$ is an (R, S) homomorphism if and only if it is an S-homomorphism.*

In §2 of (**6**) it is proved that for each multiplicative semigroup S we have a free d.g. near-ring $(\mathrm{Fr}(S), S)$ on S, where $(\mathrm{Fr}(S), +)$ is the free group on S, and $(\mathrm{Fr}(S), S)$ for each S is determined uniquely up to isomorphism. Clearly any d.g. near-ring (R, S) is a homomorphic image of $(\mathrm{Fr}(S), S)$, the free d.g. near-ring on S. It is also shown there that for any d.g. near-ring (R, S) and any set X there exists a group $\mathrm{Fr}(X, R, S)$ the free (R, S) group on X. If (R, S) is a faithful d.g. near-ring then it has a faithful representation on $\mathrm{Fr}(x, R, S)$, the free (R, S) group on one generator x (**6**, Lemma 3.1).

Now we give some definitions and results of the theory of categories.

Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a family of objects in a category $\mathscr{A}$. A product (coproduct) for the family is a family of morphisms $\{p_\lambda: A\to A_\lambda\}_{\lambda\in\Lambda}$ ($\{\alpha_\lambda: A_\lambda\to A\}_{\lambda\in\Lambda}$) with the property that for any family $\{f_\lambda: B\to A_\lambda\}_{\lambda\in\Lambda}$ ($\{f_\lambda: A_\lambda\to B\})_{\lambda\in\Lambda}$ there is a unique morphism $\phi: B\to A$ ($\phi: A\to B$) such that $\phi p_\lambda=f_\lambda$ ($\alpha_\lambda\phi=f_\lambda$) for each $\lambda\in\Lambda$.

A diagram scheme is a triple (Λ, M, d), where Λ is a set whose elements are called vertices, M is a set whose elements are called arrows, and d is a function from M to $\Lambda\times\Lambda$, i.e. for each $m\in M$, $md=(\lambda,\mu)$ for some $\lambda,\mu\in\Lambda$, λ is called the origin and μ the extremity of m. A diagram D in a category $\mathscr{A}$ over the scheme (Λ, M, d) is a function which assigns to each vertex $\lambda\in\Lambda$ an object $A_\lambda\in A$ and to each arrow $m\in M$ with origin λ and extremity μ a morphism $m_{\lambda\mu}$ from A_λ to A_μ.

If D is a diagram in $\mathscr{A}$ over a scheme (Λ, M, d), a family $\{p_\lambda: A\to A_\lambda\}_{\lambda\in\Lambda}$ of morphisms is said to be compatible for D if for every arrow $m\in M$, $p_\lambda m_{\lambda\mu}=p_\mu$. The family $\{p_\lambda: A\to A_\lambda\}_{\lambda\in\Lambda}$ of morphisms is called a limit for D if it is compatible, and if for every compatible family $\{f_\lambda: B\to A_\lambda\}_{\lambda\in\Lambda}$ for D, there exists a unique morphism $\phi: B\to A$ such that $\phi p_\lambda=f_\lambda$ for each $\lambda\in\Lambda$. The concepts of cocompatible family and colimit are defined dually.

Let $\mathscr{B}$ be a subcategory of a category $\mathscr{A}$ and A be an object of $\mathscr{A}$. A reflection for A in $\mathscr{B}$ is an object $AT\in\mathscr{B}$ together with a morphism $\theta_A: AT\to A$ such that for each object $B\in\mathscr{B}$ and each morphism $f: B\to A$ there exists a unique morphism $\phi: B\to AT$ in $\mathscr{B}$ such that $\phi\theta_A=f$. Dually we can define a coreflection in $\mathscr{B}$ of an object $A\in\mathscr{A}$.

If each object in $\mathscr{A}$ has a reflection (coreflection) in $\mathscr{B}$, then $\mathscr{B}$ is called a reflective (coreflective) subcategory of $\mathscr{A}$. If $\mathscr{B}$ is reflective (coreflective) subcategory of $\mathscr{A}$ then T becomes a covariant functor $\mathscr{A}\to\mathscr{B}$ called the reflector (coreflector) of $\mathscr{A}$ in $\mathscr{B}$.

Finally we state two results, the duals of which are proved in Section 5, Chapter 5 of (**8**).

Theorem 1.3. *Let $\mathcal{B}$ be a full, reflective subcategory of a Category $\mathcal{A}$. If a diagram D in $\mathcal{B}$ has a colimit in $\mathcal{A}$, then it has a colimit in $\mathcal{B}$ and it is the same.*

Theorem 1.4. *Let $\mathcal{B}$ be a full, reflective subcategory of a category $\mathcal{A}$. If a diagram D in $\mathcal{B}$ has a limit $\{p_\lambda : A \to A_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{A}$ then it has a limit $\{\theta p_\lambda : AT \to A_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{B}$, where T is the reflector functor and $\theta : AT \to A$ is the reflection of A in B.*

2. The upper and lower faithful d.g. near-rings

In this section we generalise the notions of upper and lower faithful d.g. near-rings given in (**6**), and make a correction in the case of lower faithful d.g. near-rings.

Theorem 2.1. *Let (R, S) be a d.g. near-ring. Then there exists a faithful d.g. near-ring $(\underline{R}, \underline{S})$ and a d.g. near-ring homomorphism $\theta : (R, S) \to (\underline{R}, \underline{S})$ such that*

(i) *$S\theta = \underline{S}$,*

(ii) *if (T, U) is a faithful d.g. near-ring and ψ is a d.g. near-ring-homomorphism $: (R, S) \to (T, U)$ then there exists a unique d.g. near-ring homomorphism $\phi : (\underline{R}, \underline{S}) \to (T, U)$ such that $\theta\phi = \psi$.*

Proof. As in (**6**) let $G = \mathrm{Fr}(x, R, S)$ be the free (R, S) group on one generator x and let $A = \{r \in R : Gr = 0\}$. Then A is an ideal of (R, S) and we get the quotient d.g. near-ring $(R, S)/A = (\underline{R}, \underline{S})$ with $\underline{S} = \{s + A : s \in S\}$ and the natural homomorphism $\theta : (R, S) \to (\underline{R}, \underline{S})$, $A = \ker \theta$ and $S\theta = \underline{S}$. Note that θ/S is not necessarily the identity. $(\underline{R}, \underline{S})$ is faithful, having a faithful representation on G.

We can prove (ii) on similar lines to Theorem 4.3 of (**6**).

$(\underline{R}, \underline{S})$ defined above is called the lower faithful d.g. near-ring for (R, S).

Theorem 2.2. *Let (R, S) be a d.g. near-ring. Then there exists a faithful d.g. near-ring $(\bar{R}, S)$ with a d.g. near-ring epimorphism $\theta : (\bar{R}, S) \to (R, S)$ such that*

(i) *$\theta/S = $ identity,*

(ii) *if (T, U) is a faithful d.g. near-ring with a d.g. near-ring homomorphism $\psi : (T, U) \to (R, S)$ then there exists a unique d.g. near-ring homomorphism $\phi : (T, U) \to (\bar{R}, S)$ such that $\phi\theta = \psi$.*

This result can be proved in a similar way to Theorem 4.6 of (**6**), taking note that $\psi : (T, U) \to (R, S)$ is not necessarily an epimorphism and ψ maps U into S.

We call $(\bar{R}, S)$ the upper faithful d.g. near-ring for (R, S).

Now let $\mathcal{A}$ denote the category of all d.g. near-rings and $\mathcal{B}$ the category of all faithful d.g. near-rings. Then $\mathcal{B}$ is a full subcategory of $\mathcal{A}$. From Theorem 2.1 [Theorem 2.2] we see that each object $(R, S) \in \mathcal{A}$ has a coreflection $(\underline{R}, \underline{S})$ [a reflection $(\bar{R}, S)$] in $\mathcal{B}$, so that $\mathcal{B}$ is a coreflective [reflective] subcategory of $\mathcal{A}$. Denoting $(\underline{R}, \underline{S})$ by $(R, S)F$ and $(\bar{R}, S)$ by $(R, S)G$ we get covariant functors F and G from $\mathcal{A}$ to $\mathcal{B}$

called the coreflector and reflector respectively. F is a coadjoint and G is an adjoint of the inclusion functor I from $\mathscr{B}$ to $\mathscr{A}$. (Section 5 of Chapter V, (**8**).)

3. Coproducts and colimits

In this section we show that coproducts and general colimits exist in the category $\mathscr{A}$. Then by Theorem 1.3 we get coproducts and general colimits in $\mathscr{B}$.

Coproducts in $\mathscr{A}$. Let $\{(R_\lambda, S_\lambda) : \lambda \in \Lambda\}$ be a family of d.g. near-rings in $\mathscr{A}$. Then for each $\lambda \in \Lambda$ we have $F_\lambda = (\mathrm{Fr}(S_\lambda), S_\lambda)$, the free d.g. near-ring on S_λ with $(R_\lambda, S_\lambda) \cong F_\lambda / A_\lambda$, for some ideal A_λ of F_λ. Let S^* be the free product of the family $\{S_\lambda : \lambda \in \Lambda\}$ of semigroups, and $F = (\mathrm{Fr}(S^*), S^*)$, the free d.g. near-ring on S^*. For each $\lambda \in \Lambda$ the semigroup inclusion map $U_\lambda : S_\lambda \to S^*$ extends to a group homomorphism from $(F_\lambda, +)$ to $(F, +)$ and hence is a d.g. near-ring homomorphism from $(\mathrm{Fr}(S_\lambda), S_\lambda)$ to $(\mathrm{Fr}(S^*), S^*)$ (Theorem 1.1). As in §1 of (**7**), for each $\lambda \in \Lambda$, we consider $(F_\lambda, +)$ as a subgroup of $(F, +)$ and hence as a sub d.g. near-ring of the d.g. near-ring F. Let A be the ideal of F generated by $\{A_\lambda : \lambda \in \Lambda\}$. Then we can prove the following result on similar lines to Theorem 1.1 of (**7**).

Theorem 3.1. *$(F/A, S^* + A)$ is the free product in $\mathscr{A}$ of $\{(R_\lambda, S_\lambda) : \lambda \in \Lambda\}$.*

General colimits in $\mathscr{A}$. Let D be a diagram in $\mathscr{A}$ over a scheme (Λ, M, d). Let $\{(R_\lambda, S_\lambda) : \lambda \in \Lambda\}$ be the family of d.g. near-rings involved in D. By Theorem 3.1 the free product $\underset{\lambda\in\Lambda}{*}(R_\lambda, S_\lambda)$ of the above family exists in $\mathscr{A}$. Let $\alpha_\lambda : (R_\lambda, S_\lambda) \to \underset{\lambda\in\Lambda}{*}(R_\lambda, S_\lambda)$ be the d.g. near-ring inclusion map for each $\lambda \in \Lambda$. Let K be the ideal of $\underset{\lambda\in\Lambda}{*}(R_\lambda, S_\lambda)$ generated by $\bigcup_{m\in M} \mathrm{Image}(\alpha_\lambda - m_{\lambda\mu}\alpha_\mu)$. Then we get the factor d.g. near-ring $\underset{\lambda\in\Lambda}{*}(R_\lambda, S_\lambda)/K$ together with natural homomorphism $\pi : \underset{\lambda\in\Lambda}{*}(R_\lambda, S_\lambda) \to \underset{\lambda\in\Lambda}{*}(R_\lambda, S_\lambda)/K$ and we prove the following theorem.

Theorem 3.2. *$\{\alpha_\lambda\pi : (R_\lambda, S_\lambda) \to \underset{\lambda\in\Lambda}{*}(R_\lambda, S_\lambda)/K\}_{\lambda\in\Lambda}$ is a colimit for the diagram D in $\mathscr{A}$ over a scheme (Λ, M, d).*

Proof. It is easy to see that the family

$$\{\alpha_\lambda\pi : (R_\lambda, S_\lambda) \to \underset{\lambda\in\Lambda}{*}(R_\lambda, S_\lambda)/K\}_{\lambda\in\Lambda}$$

of d.g. near-ring homomorphism is cocompatible, i.e. we can easily show that the diagram is commutative for all $m \in M$.

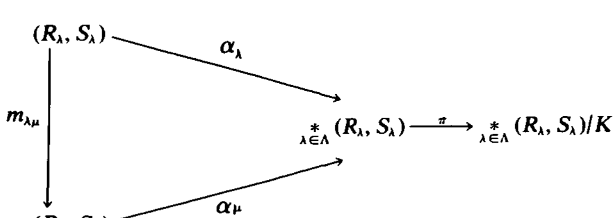

Fig. 1

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Now let (R, T) be a d.g. near-ring in $\mathscr{A}$ together with a cocompatible family

$$\{f_\lambda : (R_\lambda, S_\lambda) \to (R, T)\}_{\lambda \in \Lambda}$$

of d.g. near-ring homomorphisms. Then by the property of free products there exists a unique d.g. near-ring homomorphism

$$\phi : \underset{\lambda \in \Lambda}{*} (R_\lambda, S_\lambda) \to (R, T)$$

such that $\alpha_\lambda \phi = f_\lambda$ for each $\lambda \in \Lambda$. Now for $m \in M$ we have

$$\begin{aligned}[\text{Image}(\alpha_\lambda - m_{\lambda\mu}\alpha_\mu)]\phi &= \text{Image}(\alpha_\lambda\phi - m_{\lambda\mu}\alpha_\mu\phi).\\ &= \text{Image}(f_\lambda - m_{\lambda\mu}f_\mu)\\ &= 0 \text{ in } (R, T).\end{aligned}$$

This shows that for $m \in M$, $\text{Image}(\alpha_\lambda - m_{\lambda\mu}\alpha_\mu) \subseteq \text{Ker}\,\phi$. Therefore $\bigcup_{m \in M} \text{Image}(\alpha_\lambda - m_{\lambda\mu}\alpha_\mu)$ and hence K, is contained in $\text{Ker}\,\phi$. So there exists a unique d.g. near-ring homomorphism $\psi : \underset{\lambda \in \Lambda}{*} (R_\lambda, S_\lambda)/K \to (R, T)$ such that $\pi\psi = \phi$. Therefore we have

$$(\alpha_\lambda \pi)\psi = \alpha_\lambda(\pi\psi) = \alpha_\lambda \phi = f_\lambda.$$

Since ψ is unique with the property that $\pi\psi = \phi$ and ϕ is unique with the property that $\alpha_\lambda\phi = f_\lambda$ then ψ is unique with the property $(\alpha_\lambda\pi)\psi = f'_\lambda$.

Colimits in $\mathscr{B}$. Let D be a diagram in $\mathscr{B}$ over a scheme (Λ, M, d). Then a colimit for D exists in $\mathscr{A}$, by the above theorem. Since $\mathscr{B}$ is a reflective subcategory of $\mathscr{A}$, by Theorem 1.3 D has a colimit in $\mathscr{B}$ which is the same as in $\mathscr{A}$. Thus we have proved the following theorem.

Theorem 3.3. *If D is a diagram in $\mathscr{B}$ over a scheme (Λ, M, d), then $\{\alpha_\lambda\pi : (R_\lambda, S_\lambda) \to \underset{\lambda \in \Lambda}{*} (R_\lambda, S_\lambda)/K\}_{\lambda \in \Lambda}$ is a colimit for D in $\mathscr{B}$, where $\{\alpha_\lambda : (R_\lambda, S_\lambda) \to \underset{\lambda \in \Lambda}{*} (R_\lambda, S_\lambda)\}_{\lambda \in \Lambda}$ is the coproduct of $(R_\lambda, S_\lambda) : \lambda \in \Lambda\}$.*

4. Products and limits

Let $\{(R_\lambda, S_\lambda) : \lambda \in \Lambda\}$ be a family of d.g. near-rings in $\mathscr{A}$. Then $Q = \underset{\lambda \in \Lambda}{\pi} R_\lambda$, the Cartesian product of the family $\{R_\lambda : \lambda \in \Lambda\}$ of near-rings, is a near-ring which is not necessarily a d.g. near-ring. But it can be easily seen that $S = \underset{\lambda \in \Lambda}{\pi} S_\lambda$, the Cartesian product of the family $\{S_\lambda : \lambda \in \Lambda\}$ of semigroups, is a distributive subsemigroup of Q. Then S generates a sub d.g. near-ring (R, S) of Q. Now we prove the following result.

Theorem 4.1. *(R, S) is the product in $\mathscr{A}$ of the family $\{(R_\lambda, S_\lambda) : \lambda \in \Lambda\}$ of d.g. near-rings in $\mathscr{A}$.*

Proof. Let $p_\lambda : Q \to R_\lambda$ be the near-ring projection map for each $\lambda \in \Lambda$. It is easy to see that p_λ, for each $\lambda \in \Lambda$, maps $S \subseteq Q$ onto S_λ contained in R_λ and hence maps R onto R_λ. So, for each $\lambda \in \Lambda$, $p_\lambda | R$ is a d.g. near-ring homomorphism (Theorem 1.1.)

Let $q_\lambda = p_\lambda | R$, for $\lambda \in \Lambda$. Then $q_\lambda : (R, S) \to (R_\lambda, S_\lambda)$ is a d.g. near-ring epimorphism for each $\lambda \in \Lambda$. Let (T, U) be a d.g. near-ring in $\mathscr{A}$ together with a family $\{\psi_\lambda : (T, U) \to (R_\lambda, S_\lambda)\}_{\lambda \in \Lambda}$ of d.g. near-ring homomorphisms. We consider $\psi_\lambda : T \to R_\lambda, \lambda \in \Lambda$, as near-ring homomorphisms. By the property of products we get a unique near-ring homomorphism $\phi : T \to Q$ such that $\phi p_\lambda = \psi_\lambda$ for each $\lambda \in \Lambda$. We have $t\phi = (t\psi_\lambda)_{\lambda \in \Lambda}$ for all $t \in T$. Since ψ_λ, for each $\lambda \in \Lambda$, is a d.g. near-ring homomorphism, it maps $U \subseteq T$ into $S_\lambda \subseteq R_\lambda$. Therefore ϕ maps $U \subseteq T$ into $S \subseteq Q$ and so T into R. Hence ϕ is a d.g. near-ring homomorphism $(T, U) \to (R, S)$. Moreover $\phi q_\lambda = \psi_\lambda$, $\lambda \in \Lambda$ as d.g. near-ring homomorphisms. Since ϕ is unique as a semigroup homomorphism with the property $\phi q_\lambda = \psi_\lambda$, $\lambda \in \Lambda$, we get the uniqueness of ϕ with the property $\phi q_\lambda = \psi_\lambda$, $\lambda \in \Lambda$, as d.g. near-ring homomorphisms. This completes the proof.

General Limits in $\mathscr{A}$. Let D be a diagram in $\mathscr{A}$ over a scheme (Λ, M, d), and let $\{(R_\lambda, S_\lambda) : \lambda \in \Lambda\}$ be the family of d.g. near-rings involved in D. Let (R, S) be the product of $\{(R_\lambda, S_\lambda) : \lambda \in \Lambda\}$ in $\mathscr{A}$ together with d.g. near-ring epimorphisms $q_\lambda : (R, S) \to (R_\lambda, S_\lambda) : \lambda \in \Lambda$, as defined above.

Without loss of generality, we can assume that $0 \in S_\lambda$ for each $\lambda \in \Lambda$. Let $\mathscr{S}$ be the category of all pointed semigroups with zero element and if α is a morphism in $\mathscr{S}$, then it preserves zeros. Consider the corresponding diagram D in $\mathscr{S}$ involving $\{S_\lambda : \lambda \in \Lambda\}$. Let $S' = \bigcap_{m \in M} \operatorname{equ}(q_\lambda, m_{\lambda\mu} q_\mu) \subseteq S$, where $\operatorname{equ}(q_\lambda, m_{\lambda\mu} q_\mu)$ is the equalizer of q_λ and $m_{\lambda\mu} q_\mu$. Then S' is not empty as $0 \in S'$. Let $q'_\lambda = q_\lambda | S'$. Then by Theorem 2.9 of Chapter II of (8), $\{q'_\lambda : S' \to S_\lambda\}_{\lambda \in \Lambda}$ is a limit in $\mathscr{S}$ for the diagram D. Let (R', S') be the sub d.g. near-ring of (R, S) generated by S'. For each $\lambda \in \Lambda$, q'_λ, being the restriction of q_λ, is extended to a d.g. near-ring homomorphism from (R', S') to (R_λ, S_λ), also called q'_λ. Then we prove the following.

Theorem 4.2. *$\{q'_\lambda : (R', S') \subseteq (R, S) \to (R_\lambda, S_\lambda)\}_{\lambda \in \Lambda}$ is a limit in $\mathscr{A}$ for D.*

Proof. First of all we show that $\{q'_\lambda : (R', S') \to (R_\lambda, S_\lambda)\}_{\lambda \in \Lambda}$ is a compatible family, i.e., the following diagram is commutative, for all $m \in M$.

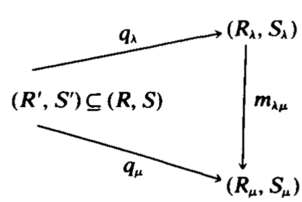

Fig. 2.

For $r' \in (R', S')$, $r' = \epsilon_1 s'_1 + \cdots + \epsilon_n s'_n$, $s'_1 \in S'$

$$r' q'_\lambda m_{\lambda\mu} = (\epsilon_1 s'_1 + \cdots + \epsilon_n s'_n) q'_\lambda m_{\lambda\mu}, \text{ where } \epsilon_i = +1 \text{ or } -1.$$

Then
$$
\begin{aligned}
&= \epsilon_1 s'_1 q'_\lambda m_{\lambda\mu} + \cdots + \epsilon_n s'_n q'_\lambda m_{\lambda\mu} \\
&= \epsilon_1 s'_1 q'_\mu + \cdots + \epsilon_n s'_n q'_\mu \text{ as } \{q'_\lambda : S' \to S_\lambda\}_{\lambda \in \Lambda} \text{ is a limit for } D \text{ in } \mathscr{S} \\
&= (\epsilon_1 s'_1 + \cdots + \epsilon_n s'_n) q'_\mu. \\
&= r' q'_\mu.
\end{aligned}
$$

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Now let (T, U) be a d.g. near-ring in $\mathscr{A}$, together with a compatible family $\{\psi_\lambda : (T, U) \to (R_\lambda, S_\lambda)\}_{\lambda \in \Lambda}$ of d.g. near-ring homomorphisms. Then by the property of products there exists a unique d.g. near-ring homomorphism $\phi : (T, U) \to (R, S)$ such that $\phi q_\lambda = \psi_\lambda$ for each $\lambda \in \Lambda$. Considering ϕ as a semigroup homomorphism from U to S we see that it factors uniquely through S' (by the property of limits in $\mathscr{S}$). Hence ϕ maps (T, U) into (R', S'). Thus ϕ is a unique d.g. near-ring homomorphism from (T, U) to (R', S') such that $\phi q'_\lambda = \psi_\lambda$, for each $\lambda \in \Lambda$. Hence $\{q'_\lambda : (R', S') \subseteq (R, S) \to (R_\lambda, S_\lambda)\}_{\lambda \in \Lambda}$ is a limit in $\mathscr{A}$ for D.

Limits in $\mathscr{B}$. Let D be a diagram in $\mathscr{B}$ over a scheme (Λ, M, d) then, by the above result, a limit for D exists in $\mathscr{A}$. Since $\mathscr{B}$ is full reflective subcategory of $\mathscr{A}$, by Theorem 1.4 D has a limit in $\mathscr{B}$. Thus we have proved the following result.

Theorem 4.3. *If D is a diagram in $\mathscr{B}$ over a scheme (Λ, M, d), then $\{\theta p_\lambda : (R, S)T \to (R_\lambda, S_\lambda)\}_{\lambda \in \Lambda}$ is a limit in $\mathscr{B}$ for D, where $\{p_\lambda : (R, S) \to (R_\lambda, S_\lambda)\}_{\lambda \in \Lambda}$ is a limit in $\mathscr{A}$ for D, and $\theta : (R, S)T \to (R, S)$ is the reflection in $\mathscr{B}$ of (R, S).*

Acknowledgement. I am grateful to my Supervisor, Dr. J. D. P. Meldrum, for suggesting the problem and his encouragement, help and guidance throughout the course of this work, which will form a part of my Ph.D. thesis.

I also wish to thank Dr. L. O'Carroll for the help and valuable suggestions I received from him.

Thanks are also due to the Government of Pakistan in helping with finance in working for my Ph.D.

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