In 1959, Professor N. A. Court [2] generated synthetically a twisted cubic $C$ circumscribing a tetrahedron $T$ as the poles for $T$ of the planes of a coaxal family whose axis is called the Lemoine axis of $C$ for $T$. Here is an analytic attempt to relate a normal rational curve $r^n$ of order $n$, whose natural home is an $n$-space $[n]$, with its Lemoine $[n—2]$ $L$ such that the first polars of points in $L$ for a simplex $S$ inscribed to $r^n$ pass through $r^n$ and the last polars of points on $r^n$ for $S$ pass through $L$. Incidentally we come across a pair of mutually inscribed or Moebius simplexes but as a privilege of odd spaces only. In contrast, what happens in even spaces also presents a case, not less interesting, as considered here.

1. Polarity for a simplex

(a) If $P$ be a point $(p_0, p_1, \ldots, p_n)$ referred to a simplex $S = A_0A_1\cdots A_n$, the first polar of $P$ for $S$ is the primal $(P) \equiv \sum(p_i|x_i) = 0$ of order $n$, and the last or $n$th polar is the prime $p \equiv \sum(x_i|p_i) = 0$ (i = 0, 1, \ldots, n) as a well known fact. Thus: If the polar prime $q \equiv \sum(x_i|q_i) = 0$ of a point $Q(q_1)$ for $S$ pass through $P$; i.e., $(p_i|q_i) = 0$, $(P)$ passes through $Q$. Or, $(P)$ is the locus of the poles for $S$ of the primes through $P$.

(b) Let the secant through $P$ to an edge $A_iA_j$ of $S$ and its opposite $[n—2]$ $a^i$ meet the edge in a point $P_{ij}$, and $Q_{ij}$ be the point on this edge as the harmonic conjugate of $P_{ij}$ w.r.t. the pair of the vertices $A_i, A_j$. That is, $H(A_iA_j, P_{ij}Q_{ij})$ or $(A_iP_{ij}A_jQ_{ij}) = -1$. The $\binom{n+1}{2}$ points $Q_{ij}$ then all lie in the polar prime $p$ of $P$ for $S$ [4; 7—11]. Conversely, if a prime $p$ cuts $A_iA_j$ in $Q_{ij}$ and $P_{ij}$ be such that $H(A_iA_j, P_{ij}Q_{ij})$, the $\binom{n+1}{2}$ primes $a^i|p_{ij}$ concur at the pole $P$ of $p$ for $S$.

Hence, if $p$ pass through $A_i, Q_{ij}$ and therefore $P_{ij}$ both coincide at $A_i$ which then becomes the pole of $p$ for $S$. Or, the pole of a prime through a vertex of $S$ for $S$ lies at this vertex.

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2. Normal rational curve

(a) The normal rational curve (n.r.c.) \( r^n \) is generated by the corresponding primes of \( n \) related pencils whose \( n \) vertices \([n-2]\)'s form its chords [14]. As the prime \( p \) in 1(b) varies in a pencil cutting the \( n \) edges \( A_iA_j \) of the simplex \( S \) through its vertex \( A_i \) in the \( n \) points \( Q_{ij} \), the \( n \) corresponding primes \( a^iP_{ij} \) of the \( n \) pencils with vertices as the \([n-2]\)'s \( a^i \) of the prime \( a^i \) of \( S \) opposite \( A_i \) generate \( r^n \) as the locus of the poles of primes \( p \) of the given pencil for \( S \). From the symmetry of the result follows the following:

**Theorem 1.** The locus of the poles of the primes of pencil for a simplex \( S \) in \([n]\) is an n.r.c. \( r^n \) through its vertices.

(b) Conversely we may have the following:

**Theorem 2.** The polar primes of the points of an n.r.c. \( r^n \) circumscribing a simplex \( S \) for \( S \) form a coaxal family.

**Proof 1.** Following Court [2], we can prove synthetically the proposition by induction. For it is true in plane \((n = 2)\) and solid \((n = 3)\).

**Proof 2.** Let \( r^n \) be represented parametrically by the \( n+1 \) coordinates \( x_i = 1/(k-u_i) \) of a point \( P \) on \( r^n \), \( k \) being the parameter [14; p. 220]. The polar prime \( p \) of \( P \) for \( S \) by 1(a) is

\[
\sum (k-u_i)x_i = 0, \text{ or } k \sum x_i - \sum u_ix_i = 0.
\]

This equation shows that \( p \) passes through the \([n-2]\) \( L \) common to the 2 primes: \( \sum x_i = 0, \sum u_ix_i = 0 \), thus proving the proposition.

**Remark 1.** Theorem 1 could be proved by taking the vertex \([n-2]\) of the pencil as \( L \) above and deduce the parametric equations \( x_i = 1/(k-u_i) \) of the \( r^n \).

**Definition.** \( L \) is said to be the Lemoine \([n-2]\] of \( r^n \) for the simplex \( S \).

**Theorem 3.** Any \( n+3 \) general points in \([n]\) determine an n.r.c. \( r^n \) in \( \binom{n+3}{2} \) ways by choosing any \( n+1 \) of them to form a simplex inscribed to it thus giving us \( \binom{n+3}{2} \) Lemoine \([n-2]\)'s, one for each simplex.

**Proof.** Theorem 2 tells us that an \( r^n \) is determined by \( n+3 \) points, \( n+1 \) forming a simplex \( S \) and the other two points being the poles for \( S \) of a couple of primes through the Lemoine \([n-2]\) of \( r^n \) for \( S \).

3. Polar and Cevian quadrics

The polar quadric of a point \( P \) on an \( r^n \) circumscribing a simplex \( S \) with coordinates \( x_i = 1/(k-u_i) \) for \( S \) is
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(ii) \[ \sum (k-u_i)(k-u_j)x_ix_j = 0 \]

or

\[ k^2 \sum x_ix_j - k \sum (u_i+u_j)x_ix_j + \sum u_iu_jx_ix_j = 0, \]

showing that it belongs to a special net [5] determined by the 3 quadrics:

\[ \sum x_ix_j = 0, \sum (u_i+u_j)x_ix_j = 0, \sum u_iu_jx_ix_j = 0. \]

The cevian quadric [10] of \( P \) for \( S \) touching the edges of \( S \) at the feet thereat of its bicevians through \( P \) is

\[ \sum (k-u_i)^2x_i^2 - 2 \sum (k-u_i)(k-u_j)x_ix_j = 0, \]

or,

\[ 4 \sum (k-u_i)(k-u_j)x_ix_j - (\sum k-u_i)^2 = 0 \]

showing that it too belongs to a special net, and has ring contact with the corresponding quadric of the net (ii) along the polar prime \( p \) (i) of \( P \) for \( S \). Thus we have

**Theorem 4.** The polar as well as cevian quadrics of the points of an n.r.c. \( r^n \) circumscribing a simplex \( S \) for \( S \) belong respectively to two special nets such that the pair of quadrics corresponding to a point \( P \) on \( r^n \) have ring contact along the polar prime \( p \) of \( P \) for \( S \).

### 4. Lemoine axes

**Theorem 5.** The Lemoine \([q-2]\)'s of the n.r. curves in the \([q]\)'s of a simplex \( S \) in \([n]\), which are projections therein of an n.r.c. \( r^n \) circumscribing \( S \) from the opposite \([n-q-1]\)'s, all lie in the Lemoine \([n-2]\) \( L \) of \( r^n \). In particular, the Lemoine axes of the cubic projections of \( r^n \) in the solids of \( S \) from the opposite \([n-4]\)'s and the Lemoine points of the conic projections of \( r^n \) in the planes of \( S \) from the opposite \([n-3]\)'s lie in \( L \).

**Proof.** The polar prime \( p \) of a point \( P \) for simplex \( S \) in \([n]\) passes through the polar \([q-1]\) \( p_q \) of the projection \( P_q \) of \( P \) in a \([q]\) of \( S \) from its opposite \([n-q-1]\) for its \( q \)-simplex in this \([q]\). If \( p \) varies in a pencil through an \([n-2]\) \( L \), \( p_q \) too varies in a pencil through the \([q-2]\) \( L_q \) which is a section of \( L \) by the \([q]\). Thus \( P_q \) traces an n.r.c. \( r^q \), as a projection of \( r^n \) traced by \( P \) from the chordal \([n-q-1]\), having Lemoine \([q-2]\) as \( L_q \).

Conversely we have

**Theorem 6.** If the Lemoine \([q-2]\)'s of certain n.r.c.s. in the \([q]\)'s of a simplex \( S \) in \([n]\) all lie in an \([n-2]\) \( L \), every such \( r_q \) is then the projection of an \( r^n \) circumscribing \( S \) from its \([n-q-1]\) opposite its \([q]\) of the \( r^q \).
5. First polars

THEOREM 7. The $n-1$ first polars for a simplex $S$ in $[n]$ of any $n-1$ independent points determining an $[n-2]$ $L$ determine or have an n.r.c. $r^n$ common such that the first polar of any point of $L$ for $S$ passes through $r^n$.

PROOF. The first polar of a point for a simplex in $[n]$ is a primal of order $n$ and dimension $n-1$, and contains the $\binom{n+1}{2}$ $[n-2]$'s of $S$ once, the $\binom{n+1}{3}$ $[n-3]$'s twice, \ldots, the $\binom{n+1}{r}$ $[n-r]$'s $(r-1)$-times, \ldots and $\binom{n+1}{n-2}$ edges of $S$ $(n-1)$-times. Thus the intersection of the first polars of 2 points for $S$ is of dimension $n-2$ but order $n^2-\binom{n+1}{2}=\binom{n}{2}$, that of 3 independent points is of dimension $n-3$ but order $n\binom{n}{2}-2\binom{n+1}{3}=\binom{n}{3}$, \ldots, that of $r$ independent points is of dimension $n-r$ but order $n\binom{n}{r-1}-(r-1)\binom{n+1}{r}=\binom{n}{r}$, \ldots and that of $n-1$ independent points is of dimension 1 but order $\binom{n}{n-1}=n$.

THEOREM 8. $L$ of the preceding theorem is the Lemoine $[n-2]$ of the $r^n$ for the simplex $S$.

PROOF. Let us take $L$ to be the $[n-2]$ given by the pair of linear equations: $\sum x_i=0$, $\sum u_i x_i=0$, and $P$ be a point $(p_0, p_1, \ldots, p_n)$ in $L$ such that $\sum p_i=0=\sum u_i p_i$. Now the first polar of $P$ is $(P)\equiv \sum (p_i x_i)=0$ which obviously passes through the $r^n$ given by the coordinates $x_i=1/(k-u_i)$ of any point on it because of the two conditions satisfied by $P$. Hence, by the definition of the Lemoine $[n-2]$ of an $r^n$, follows the theorem.

6. Tangents

THEOREM 9. The meets of the primes $a^i$ of a simplex $S$ in $[n]$ with the tangents, at its opposite vertices $A_i$, of an n.r.c. $r^n$ circumscribing $S$ are the poles of the $[n-2]$ projections therein, of the Lemoine $[n-2]$ $L$ of $r^n$ for $S$ from $A_i$, for the respective $(n-1)$-simplexes of $S$.

PROOF. The equations of the tangent line of an n.r.c. $r^n$ at any point with coordinates $x_i=(k-u_i)^{-1}$ on it are given by
\[
0 = \begin{pmatrix}
x_0 & \cdots & x_i & \cdots & x_n \\
(k-u_0)^{-1} \cdots (k-u_i)^{-1} \cdots (k-u_n)^{-1} \\
(k-u_0)^{-2} \cdots (k-u_i)^{-2} \cdots (k-u_n)^{-2}
\end{pmatrix}
\]
following the notations of Professor T. G. Room [14]. To find the tangents at the vertices of the simplex $S$ of reference, we may write (iv) as
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\[ \begin{pmatrix} x_0(k-u_0)^2 & \cdots & x_i(k-u_i)^2 & \cdots & x_n(k-u_n)^2 \\ (k-u_0) & \cdots & (k-u_i) & \cdots & (k-u_n) \\ 1 & \cdots & 1 & \cdots & 1 \end{pmatrix} = 0 \]

and put \( k = u_i \) in (v) to find one at the vertex \( A_i \) of \( S \). Thus the tangent of \( r^n \) at \( A_i \) is given by the equations

\[ x_0(u_i-u_0) = \cdots = x_{i-1}(u_i-u_{i-1}) = x_{i+1}(u_i-u_{i+1}) = \cdots = x_n(u_i-u_n) \]

meeting the opposite prime \( x_i = 0 \) of \( S \) in the point \( A'_{i} \) whose \( n \) coordinates other than \( x_i \) are then \( x_j = (u_i-u_i)^{-1} \).

The equation of the \([n-2]\) projection in the prime \( x_i = 0 \) of \( S \), of the Lemoine \([n-2]\) of the \( r^n \) for \( S \) from the opposite vertex \( A_i \) is found to be \( \sum_{j \neq i} (u_i-u_j)x_j = 0 \) showing it to be the last polar \((1a)\) of \( A'_i \) for the \((n-1)\)-simplex of \( S \) in the prime under consideration.

**Remark 2.** \( r^n \) being the locus (Theorem 1) of the poles, for \( S \), of the primes through \( L \), \( A_i \) being the pole of the prime \( LA_i \) for \( S \) \((1b)\) and the tangent of \( r^n \) at \( A_i \) being the limit of the chords of \( r^n \) through \( A_i \), the Theorem 9 follows immediately from the definition of the pole and polar for a simplex \((2; 4; 7-11)\).

**Theorem 10.** The \( n \) tangents of the \( n \) \( r^n_{n-1} \) projections of an \( n.r.c. \) \( r^n \) circumscribing a simplex \( S \) in \([n]\), in its \( n \) primes through a vertex \( A_i \) of \( S \) from the opposite vertices, at their common point \( A_i \) meet its \( n \) opposite \([n-2]\)'s in the \( n \) points \( A'_{ij} \) which form a Cevian \((n-1)\)-simplex of the \((n-1)\)-simplex of \( S \) opposite \( A_i \) for the meet \( A'_{i} \) of its prime \( a^i \) with the tangent of \( r^n \) at \( A_i \) \([10]\).

**Proof.** The tangent of the n.r.c. \( r^{n-1} \) projection of \( r^n \), in the prime \( x_i = 0 \) of \( S \) from the opposite vertex \( A_j \), at the vertex \( A_i \) meets the opposite \([n-2]\) \( a^{ij} \) \((1b)\) in the point \( A'_{ij} \) whose coordinates referred to \( S \) are \( x_i = 0 = x_j, \ x_k = 1/(u_i-u_k) \) for all values of \( k \) other than \( i, j \) \((7a)\). Thus \( A_j, A'_{i}, A'_{ij} (\neq A'_{i}) \) are collinear.

**Remark 3.** In view of Remark 2, Theorem 10 can also be deduced from the definition of the pole and polar for a simplex \([2]\).

### 7. Even spaces

If we put down the \( n+1 \) coordinates \((6a)\) of the meet \( A'_{i} \) of a prime \( a^i \) of the simplex \( S \) of reference with the tangent of an n.r.c. \( r^n \) circumscribing \( S \) at its opposite vertex \( A_i \) as the \( i \)th row of a matrix \( M \) \((i = 0, \cdots, n)\), we find \( M \) to be skew symmetric such that its determinant \(|M| = 0\), thus showing that the \( n+1 \) points \( A'_{i} \) are co-primal if \( n \) is even. Hence follows the following:
Theorem 11. The $2m+1$ meets of the $2m+1$ primes of a simplex $S$ in $[2m]$ with the tangents of an n.r.c. $r^{2m}$ circumscribing $S$ at its opposite vertices all lie in a prime which coincides with the Lemoine axis of a triangle for a conic circumscribing it when $m = 1$ [11].

8. Odd spaces

Theorem 12. The $2m$ meets of the $2m$ primes of a simplex $S$ in $[2m−1]$ with the tangents of an n.r.c. $r^{2m−1}$ circumscribing $S$ at its opposite vertices form another simplex $S'$ Moebius or mutually inscribed with $S$ [1—3; 6; 12].

Proof. The first minor of a skew symmetric matrix obtained by crossing its $i^{th}$ row and $i^{th}$ column is also skew symmetric. Hence if we substitute the $n+1$ coordinates $x_i = 1, x_j = 0$ (for all $j \neq i$) of a vertex $A_i$ of a simplex $S$ in the $i$th row of the matrix $M$ of the preceding section, we find $|M| = 0$ thus showing that $A_i$ lies in the prime determined by the $n$ points $A'_i$ if $n$ is odd.

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References


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