FORMALLY NORMAL OPERATORS HAVING NO NORMAL EXTENSIONS

EARL A. CODDINGTON

1. Introduction. The domain and null space of an operator A in a Hilbert space $\mathfrak{H}$ will be denoted by $\mathfrak{D}(A)$ and $\mathfrak{N}(A)$, respectively. A *formally normal operator* N in $\mathfrak{H}$ is a densely defined closed (linear) operator such that $\mathfrak{D}(N) \subset \mathfrak{D}(N^*)$, and $||Nf|| = ||N^*f||$ for all $f \in \mathfrak{D}(N)$. A *normal operator* in $\mathfrak{H}$ is a formally normal operator N satisfying $\mathfrak{D}(N) = \mathfrak{D}(N^*)$. A study of the possibility of extending a formally normal operator N to a normal operator in the given $\mathfrak{H}$, or in a larger Hilbert space, was made in **(1)**. Necessary and sufficient conditions for such an extension in $\mathfrak{H}$ were presented, as well as sufficient conditions for a normal extension in a larger Hilbert space. At the time of the writing of that paper it was not known to us whether or not a given formally normal N always could be extended to a normal operator, in a possibly larger Hilbert space. The main purpose of this paper is to present an example of a formally normal N in a Hilbert space $\mathfrak{H}$ which has no normal extensions in $\mathfrak{H}$ or in any larger Hilbert space. This situation thus contrasts sharply with that which obtains for symmetric operators, for every symmetric operator in $\mathfrak{H}$ may be extended, in a trivial way, to a self-adjoint operator in a larger Hilbert space.

When we mentioned to B. Fuglede our suspicion that such an example existed, he recalled his knowledge of a pair of densely defined symmetric operators S_1, S_2 in a Hilbert space $\mathfrak{H}$ which have a common invariant domain $\mathfrak{D}$ ($S_1\mathfrak{D} \subset \mathfrak{D}$, $S_2\mathfrak{D} \subset \mathfrak{D}$), $S_1S_2u = S_2S_1u$ for all $u \in \mathfrak{D}$, and the closures $\tilde{S}_1$, $\tilde{S}_2$ self-adjoint, but such that the spectral resolutions of $\tilde{S}_1$, $\tilde{S}_2$ do not commute. He then indicated to us that the closure of the operator $S_1 + iS_2$ is a formally normal operator having no normal extensions. Although Fuglede never published his interesting example, a different pair of such operators S_1, S_2 was exhibited by E. Nelson in **(3**, p. 606**)**.

Our example is of a different nature, and is of interest since it has a certain minimum character. It is an ordinary differential operator of the third order for which dim $(\mathfrak{D}(N^*)/\mathfrak{D}(N)) = 1$. Using this operator one can construct further examples of formally normal operators N having no normal extensions, such that dim $(\mathfrak{D}(N^*)/\mathfrak{D}(N))$ is any given positive integer. In our example the symmetric operators Re $N = (N + \bar{N})/2$, Im $N = (N - \bar{N})/2i$ ($\bar{N}$ being the restriction of N^* to $\mathfrak{D}(N)$) have deficiency indices $(0, 0)$ and $(0, 1)$ respectively.

Received September 1, 1964. This work was supported in part by the National Science Foundation.

We indicate that, for our example, the domains of $\bar{N}^*$, N^*, $(\mathrm{Re}\ N)^*$, $(\mathrm{Im}\ N)^*$ are not comparable. In a concluding section we show that in some situations, where these domains are comparable for a formally normal N in $\mathfrak{H}$, normal extensions exist in $\mathfrak{H}$.

2. General considerations. Let $\mathfrak{G}(T)$ denote the graph of an operator T in a Hilbert space $\mathfrak{H}$. If A, B are closed operators with dense domains, and $A \subset B$, then it is easy to verify that $\mathfrak{G}(B) \ominus \mathfrak{G}(A)$ consists of all $\{u, Bu\} \in \mathfrak{G}(B)$ such that $u \in \mathfrak{N}(I + A^*B)$, where I is the identity operator. Since

$$\mathfrak{G}(B) = \mathfrak{G}(A) \oplus [\mathfrak{G}(B) \ominus \mathfrak{G}(A)],$$

we have

$$\mathfrak{D}(B) = \mathfrak{D}(A) + \mathfrak{N}(I + A^*B), \tag{1}$$

which is a direct sum.

If N is formally normal in $\mathfrak{H}$, and $\bar{N}$ is N^* restricted to $\mathfrak{D}(N)$, then $N \subset \bar{N}^*$ since $\bar{N} \subset N^*$. The above shows that

$$\mathfrak{D}(\bar{N}^*) = \mathfrak{D}(N) + \mathfrak{M}, \qquad \mathfrak{M} = \mathfrak{N}(I + N^*\bar{N}^*),$$
$$\mathfrak{D}(N^*) = \mathfrak{D}(N) + \bar{\mathfrak{M}}, \qquad \bar{\mathfrak{M}} = \mathfrak{N}(I + \bar{N}^*N^*).$$

The example we give is an N for which

$$\dim \mathfrak{M} = \dim \bar{\mathfrak{M}} = 1, \qquad \dim(\mathfrak{M} \cap \bar{\mathfrak{M}}) = 0. \tag{2}$$

We shall now indicate that *any such N is maximal formally normal in $\mathfrak{H}$ (has no proper formally normal extensions in $\mathfrak{H}$), and has no normal extensions in any Hilbert space containing $\mathfrak{H}$ as a subspace.*

Let N be a formally normal operator in $\mathfrak{H}$ for which (2) is valid. It is not normal since $\mathfrak{D}(N) \neq \mathfrak{D}(N^*)$. Also N is a maximal formally normal operator in $\mathfrak{H}$. Indeed, the first condition in (1) will guarantee this. Suppose N_1 is a formally normal extension of N in $\mathfrak{H}$. Then we must have

$$N \subset N_1 \subset \bar{N}^*, \qquad \bar{N} \subset N_1^* \subset N^*,$$

and an application of the result (1) gives

$$\begin{aligned} \mathfrak{D}(N_1) &= \mathfrak{D}(N) + \mathfrak{M}_1, & \mathfrak{M}_1 &= \mathfrak{N}(I + N^*N_1), \\ \mathfrak{D}(N_1^*) &= \mathfrak{D}(N) + \bar{\mathfrak{M}}_1, & \bar{\mathfrak{M}}_1 &= \mathfrak{N}(I + \bar{N}^*N_1^*). \end{aligned} \tag{3}$$

It is now clear from the definitions of $\mathfrak{M}$ and $\mathfrak{M}_1$ that $\mathfrak{M}_1 \subset \mathfrak{M}$. Thus, if $\mathfrak{M}_1$ is non-empty, and $\dim \mathfrak{M} = 1$, we must have $\mathfrak{M}_1 = \mathfrak{M}$. But then $N_1 = \bar{N}^*$, which is not formally normal, since $\mathfrak{D}(\bar{N}^*)$ is not contained in the domain of $(\bar{N}^*)^* = \bar{N}$.

It is of interest to verify that the condition $\dim(\mathfrak{M} \cap \bar{\mathfrak{M}}) = 0$ also implies that N is maximal formally normal, for it is this condition that is used to show that N has no normal extensions in any larger Hilbert space. (Thus any formally normal N such that

https://doi.org/10.4153/CJM-1965-098-8 Published online by Cambridge University Press

$$(2') \qquad \dim \mathfrak{M} = \dim \bar{\mathfrak{M}} > 0, \qquad \dim(\mathfrak{M} \cap \bar{\mathfrak{M}}) = 0,$$

is maximal formally normal and has no normal extensions.) If N, N_1 are formally normal in $\mathfrak{H}$, and $N \subset N_1$, then it will be shown that $\mathfrak{M}_1 \subset \mathfrak{M} \cap \bar{\mathfrak{M}}$. Therefore, if $\dim(\mathfrak{M} \cap \bar{\mathfrak{M}}) = 0$, $N = N_1$; see (3). We have indicated that $\mathfrak{M}_1 \subset \mathfrak{M}$. To show that $\mathfrak{M}_1 \subset \bar{\mathfrak{M}}$ we note that, since $\mathfrak{D}(N_1) \subset \mathfrak{D}(N_1^*)$, $\mathfrak{M}_1 \subset \mathfrak{D}(N_1^*) = \mathfrak{D}(N) + \bar{\mathfrak{M}}_1$. If $\phi \in \mathfrak{M}_1$ we may write $\phi = f + \psi$, where $f \in \mathfrak{D}(N)$, $\psi \in \bar{\mathfrak{M}}_1$. Then

$$\begin{aligned}(N_1 \phi, N_1 f) &= \frac{1}{4} \sum_{k=1}^{4} i^k \, ||N_1(\phi + i^k f)||^2 \\ &= \frac{1}{4} \sum_{k=1}^{4} i^k \, ||N_1^*(\phi + i^k f)||^2 \\ &= (N_1^* \phi, N_1^* f),\end{aligned}$$

or

$$(N_1 \phi, Nf) = (\bar{N}f, \bar{N}f) + (N_1^* \psi, \bar{N}f).$$

Thus

$$(N^* N_1 \phi, f) = ||\bar{N}f||^2 + (\bar{N}^* N_1^* \psi, f),$$

and using the definitions of $\mathfrak{M}_1$ and $\bar{\mathfrak{M}}_1$ we see that

$$-(\phi, f) = ||\bar{N}f||^2 - (\psi, f),$$

or

$$||\bar{N}f||^2 + (\phi - \psi, f) = ||\bar{N}f||^2 + ||f||^2 = 0.$$

This implies that $f = 0$ and consequently that $\phi = \psi$, showing that $\mathfrak{M}_1 \subset \bar{\mathfrak{M}}$.

The fact that $\dim(\mathfrak{M} \cap \bar{\mathfrak{M}}) = 0$ implies that N has no normal extension in any Hilbert space $\mathfrak{H} \oplus \mathfrak{K}$ was pointed out in (**1**, Theorem 4, Corollary). We sketch the reasoning briefly. If $\mathcal{N}_1$ is a normal extension of a formally normal N in $\mathfrak{H} \oplus \mathfrak{K}$, then $\bar{N} \subset \mathcal{N}_1^*$, and a consideration of graphs shows that

$$\begin{aligned}\mathfrak{D}(\mathcal{N}_1) &= \mathfrak{D}(N) + \mathfrak{L}, \qquad & \mathfrak{L} &= \mathfrak{N}(P + N^* P \mathcal{N}_1), \\ \mathfrak{D}(\mathcal{N}_1^*) &= \mathfrak{D}(N) + \bar{\mathfrak{L}}, \qquad & \bar{\mathfrak{L}} &= \mathfrak{N}(P + \bar{N}^* P \mathcal{N}_1^*),\end{aligned}$$

where P is the orthogonal projection of $\mathfrak{H} \oplus \mathfrak{K}$ onto $\mathfrak{H}$. An argument similar to the one above, where we showed that $\mathfrak{M}_1 \subset \mathfrak{M} \cap \bar{\mathfrak{M}}$, now can be used to show that $P\mathfrak{L} \subset \mathfrak{M} \cap \bar{\mathfrak{M}}$. But $\mathfrak{M} \cap \bar{\mathfrak{M}} = \{0\}$, and $\mathcal{N}_1$ normal implies that N must be normal, a contradiction.

3. The example. Let L denote the formal ordinary differential operator on $0 < x < \infty$ given by

$$Lu = u''' + u'' - 3x^{-2}u' + (3x^{-3} - 2x^{-2})u.$$

https://doi.org/10.4153/CJM-1965-098-8 Published online by Cambridge University Press

Let N_0 be the operator in the Hilbert space $\mathfrak{H} = \mathfrak{L}_2(0, \infty)$ with domain $\mathfrak{D}(N_0) = C_0^\infty(0, \infty)$, the set of complex-valued functions on $0 < x < \infty$ of class C^∞ which vanish outside compact subsets of $0 < x < \infty$, and defined by $N_0 u = Lu$ for $u \in \mathfrak{D}(N_0)$. Let N be the closure of N_0 in $\mathfrak{H}$. This N is formally normal and $\mathfrak{M} = \mathfrak{N}(I + N^*\bar{N}^*)$ satisfies (2).

We observe that L may be written as $L = L_3 + L_2$, where

$$L_3 u = u''' - 3x^{-2}u' + 3x^{-3}u,$$
$$L_2 u = u'' \quad - 2x^{-2}u,$$

and these operators have formal adjoints L_3^+, L_2^+, satisfying $L_3^+ = -L_3$, $L_2^+ = L_2$, which implies that $L^+ = -L_3 + L_2$. Moreover, L_2 and L_3 formally commute, that is

$$L_2 L_3 u = L_3 L_2 u, \qquad u \in C^\infty(0, \infty);$$

a fact which was pointed out by J. L. Burchnall and T. W. Chaundy in **(2)**. This and Green's formula now imply that

$$\|Lf\| = \|L^+f\|, \qquad f \in \mathfrak{D}(N_0). \tag{4}$$

Indeed, for such f we have

$$(L_2 f, L_3 f) = -(L_3 L_2 f, f) = -(L_2 L_3 f, f) = -(L_3 f, L_2 f),$$

and therefore

$$\|Lf\|^2 = \|(L_3 + L_2)f\|^2 = \|L_3 f\|^2 + \|L_2 f\|^2 + (L_2 f, L_3 f) + (L_3 f, L_2 f)$$
$$= \|L_3 f\|^2 + \|L_2 f\|^2 = \|(-L_3 + L_2)f\|^2 = \|L^+ f\|^2.$$

From the equality (4) we see that if $f \in \mathfrak{D}(N)$, and $f_n \in \mathfrak{D}(N_0)$, $f_n \to f$, $Lf_n \to g$, then $L^+ f_n$ tends to some limit g^+. Thus f is in the domain of the closure of $\bar{N}_0$, the operator L^+ defined on $\mathfrak{D}(N_0)$, and this closure is contained in $N_0^* = N^*$. For $\mathfrak{D}(N^*)$ is the set of all $u \in \mathfrak{H}$ such that $u \in C^2(0, \infty)$, u'' is absolutely continuous, and $L^+ u \in \mathfrak{H}$; moreover, $N^* u = L^+ u$ for $u \in \mathfrak{D}(N^*)$. Thus $\mathfrak{D}(N) \subset \mathfrak{D}(N^*)$. The operator $\bar{N}$ is just L^+ defined on $\mathfrak{D}(N)$, and $\bar{N}_0^* = \bar{N}^*$ is L defined on $\mathfrak{D}(\bar{N}^*)$, which is the set of all $u \in \mathfrak{H}$ such that $u \in C^2(0, \infty)$, u'' is absolutely continuous, and $Lu \in \mathfrak{H}$. From (4) it now follows that $\|Nf\| = \|Lf\| = \|L^+ f\| = \|N^* f\|$, for all $f \in \mathfrak{D}(N)$. We have thus verified that N is formally normal.

The space $\mathfrak{M} = \mathfrak{N}(I + N^*\bar{N}^*)$ consists of all solutions u of the differential equation

$$(I + L^+ L)u = 0 \tag{5}$$

satisfying $u \in \mathfrak{H}$, $Lu \in \mathfrak{H}$; whereas $\bar{\mathfrak{M}} = \mathfrak{N}(I + \bar{N}^* N^*)$ consists of all solutions u of the same differential equation satisfying $u \in \mathfrak{H}$, $L^+ u \in \mathfrak{H}$. Note that all solutions of this question are analytic on $0 < x < \infty$ since L and L^+

https://doi.org/10.4153/CJM-1965-098-8 Published online by Cambridge University Press

have analytic coefficients. To compute the dimensions of the spaces $\mathfrak{M}$, $\bar{\mathfrak{M}}$, and $\mathfrak{M} \cap \bar{\mathfrak{M}}$, we introduce the function ϕ defined by

$$\phi(x, \lambda) = (x^{-1} - \lambda)e^{\lambda x},$$

for $0 < x < \infty$, and all complex λ. It is readily verified that this function satisfies

$$L_3 \phi = \lambda^3 \phi, \qquad L_2 \phi = \lambda^2 \phi.$$

Thus

$$L\phi = p(\lambda)\phi = (\lambda^3 + \lambda^2)\phi, \qquad L^+\phi = p^+(\lambda)\phi = (-\lambda^3 + \lambda^2)\phi,$$

and

$$(I + L^+L)\phi = q(\lambda)\phi = (p^+(\lambda)p(\lambda) + 1)\phi.$$

The polynomial $q(\lambda) = -\lambda^6 + \lambda^4 + 1$ has no pure imaginary roots; for, if $\lambda = -\bar{\lambda}$,

$$q(\lambda) = p^+(-\bar{\lambda})p(\lambda) + 1 = \overline{p(\lambda)}p(\lambda) + 1 \geqslant 1.$$

If λ is a root of q so are $\bar{\lambda}$, $-\lambda$, and $-\bar{\lambda}$. The roots of q are distinct, and there is one negative real root λ_1 such that $1 < |\lambda_1| < \sqrt{2}$. Two other roots λ_2, $\lambda_3 = \bar{\lambda}_2$ have negative real parts, and $|\lambda_2| = |\lambda_3| < 1$. The other roots are $\lambda_4 = -\lambda_1$, $\lambda_5 = -\lambda_2$, $\lambda_6 = -\bar{\lambda}_2$, and have positive real parts.

Let $\phi_k(x) = \phi(x, \lambda_k)$, $k = 1, \ldots, 6$, where the λ_k are the roots of q. The functions $\phi_1, \ldots, \phi_6$ form a basis for the solutions of the equation (5) on $0 < x < \infty$. Indeed, if we have constants $c_1, \ldots, c_6$ such that, on $0 < x < \infty$,

$$0 = \sum_{k=1}^{6} c_k \phi_k(x) = \sum_{k=1}^{6} c_k(x^{-1} - \lambda_k)e^{\lambda_k x},$$

then

$$\sum_{k=1}^{6} c_k(1 - \lambda_k x)e^{\lambda_k x} = 0,$$

and a differentiation gives

$$\sum_{k=1}^{6} c_k \lambda_k^2 e^{\lambda_k x} = 0.$$

Since the λ_k are distinct, and none are equal to zero, this implies that $c_1 = c_2 = \ldots = c_6 = 0$. The functions ϕ_1, ϕ_2, ϕ_3 are in $\mathfrak{L}_2(1, \infty)$, whereas ϕ_4, ϕ_5, ϕ_6 are not in this space. It is easy to see that the solutions of equation (5) that are in $\mathfrak{L}_2(1, \infty)$ are spanned by ϕ_1, ϕ_2, ϕ_3. Thus, if ϕ satisfies (5) and $\phi \in \mathfrak{H} = \mathfrak{L}_2(0, \infty)$, we must have

$$\phi(x) = \sum_{k=1}^{3} c_k \phi_k(x) = \sum_{k=1}^{3} c_k(x^{-1} - \lambda_k)e^{\lambda_k x},$$

for some constants c_1, c_2, c_3. This function has the form

$$\phi(x) = (c_1 + c_2 + c_3)x^{-1} + \tilde{\phi}(x),$$

where $\tilde{\phi}$ is analytic at the origin. Thus $\phi \in \mathfrak{H}$ if and only if

$$c_1 + c_2 + c_3 = 0. \tag{6}$$

Since

$$L\phi = \sum_{k=1}^{3} c_k L\phi_k = \sum_{k=1}^{3} c_k\, p(\lambda_k)\phi_k,$$

we see that $L\phi \in \mathfrak{H}$ if and only if

$$c_1 p(\lambda_1) + c_2 p(\lambda_2) + c_3 p(\lambda_3) = 0. \tag{7}$$

Similarly, $L^+\phi \in \mathfrak{H}$ if and only if

$$c_1 p^+(\lambda_1) + c_2 p^+(\lambda_2) + c_3 p^+(\lambda_3) = 0. \tag{8}$$

Thus $\phi \in \mathfrak{M}$ if and only if (6) and (7) are valid, and $\phi \in \bar{\mathfrak{M}}$ if and only if (6) and (8) hold.

The conditions (6) and (7) are independent. An easy way to see this is to note that if $\lambda_j \neq \lambda_k$, then $p(\lambda_j) \neq p(\lambda_k)$. Suppose, if possible that $\lambda_j \neq \lambda_k$ and $p(\lambda_j) = p(\lambda_k)$. Then, since $p^+(\lambda_j) = -[p(\lambda_j)]^{-1}$, we have $p^+(\lambda_j) = p^+(\lambda_k)$, and this implies that $\lambda_j^2 = \lambda_k^2$ and $\lambda_j^3 = \lambda_k^3$. Thus $(\lambda_j/\lambda_k)^2 = (\lambda_j/\lambda_k)^3 = 1$. Since $(\lambda_j/\lambda_k) \neq 1$, we must have $(\lambda_j/\lambda_k) = -1$; but this contradicts $(\lambda_j/\lambda_k)^3 = 1$. Similarly, $\lambda_j \neq \lambda_k$ implies that $p^+(\lambda_j) \neq p^+(\lambda_k)$, which in turn yields the independence of the conditions (6) and (8). We have now proved that $\dim \mathfrak{M} = \dim \bar{\mathfrak{M}} = 1$. The function $\phi \in \mathfrak{M} \cap \bar{\mathfrak{M}}$ if and only if (6), (7), and (8) are fulfilled. These constitute three independent conditions, for the determinant of the coefficients c_1, c_2, c_3 is just

$$\begin{vmatrix} 1 & 1 & 1 \\ p(\lambda_1) & p(\lambda_2) & p(\lambda_3) \\ \dfrac{-1}{p(\lambda_1)} & \dfrac{-1}{p(\lambda_2)} & \dfrac{-1}{p(\lambda_3)} \end{vmatrix} = [p(\lambda_1)p(\lambda_2)p(\lambda_3)]^{-1} \begin{vmatrix} 1 & 1 & 1 \\ p(\lambda_1) & p(\lambda_2) & p(\lambda_3) \\ p^2(\lambda_1) & p^2(\lambda_2) & p^2(\lambda_3) \end{vmatrix},$$

which is not zero, since $p(\lambda_j) \neq p(\lambda_k)$, $j \neq k$. Therefore $\dim(\mathfrak{M} \cap \bar{\mathfrak{M}}) = 0$, and we have verified that $\mathfrak{M}$ and $\bar{\mathfrak{M}}$ for N satisfy (2).

4. Remarks on the example.

(i) Using the example N, which was exhibited in § 3, we can construct other examples of maximal formally normal operators having no normal extensions. Let S denote the maximal symmetric operator defined as the closure in $\mathfrak{L}_2(0, \infty)$ of the operator $i\, d/dx$ on $C_0^\infty(0, \infty)$. Its $\mathfrak{M}$-space, which is identical with its $\bar{\mathfrak{M}}$-space, is $\mathfrak{N}(I + S^{*2})$, which has dimension one. Consider the operator

$$N_1 = N \oplus \ldots \oplus N \oplus S \oplus \ldots \oplus S,$$

where there are $p \geqslant 1$ N's and $q \geqslant 0$ S's in the sum. The operator N_1 acts in the Hilbert space $\mathfrak{H}_1$, which is the direct sum of $p + q$ copies of $\mathfrak{L}_2(0, \infty)$. Clearly,

$$\begin{aligned} N_1^* &= N^* \oplus \ldots \oplus N^* \oplus S^* \oplus \ldots \oplus S^*, \\ \bar{N}_1 &= \bar{N} \oplus \ldots \oplus \bar{N} \oplus S \oplus \ldots \oplus S, \\ \bar{N}_1^* &= \bar{N}^* \oplus \ldots \oplus \bar{N}^* \oplus S^* \oplus \ldots \oplus S^*. \end{aligned}$$

Any formally normal extension N_2 of N_1 in $\mathfrak{H}_1$ must satisfy $N_1 \subset N_2 \subset \bar{N}_1^*$, and thus must be of the form

$$N_2 = N' \oplus \ldots \oplus N' \oplus S' \oplus \ldots \oplus S',$$

where N', S' are formally normal extensions of N, S, respectively. Since N, S are maximal formally normal, $N' = N$, $S' = S$, and thus N_1 is maximal formally normal. The $\mathfrak{M}$-space for N_1 is the direct sum of those for the N's and the S's, and this implies that

$$\dim \mathfrak{M} = \dim \mathfrak{N}(I + N_1^* \bar{N}_1^*) = p + q.$$

Thus N_1 is not normal. Moreover, we have

$$\dim(\mathfrak{M} \cap \bar{\mathfrak{M}}) = q.$$

Now N_1 can have no normal extension in any larger Hilbert space, since it was shown in (**1**, Theorem 9), that a necessary condition for such an extension is that $\mathfrak{M} = \bar{\mathfrak{M}}$ in case $\dim \mathfrak{M} < \infty$. Therefore, we have exhibited formally normal operators N_1, having no normal extensions, for which $\dim \mathfrak{M}$ may be any finite integer, and for which $\dim(\mathfrak{M} \cap \bar{\mathfrak{M}})$ may be any integer between zero and $\dim \mathfrak{M} - 1$. inclusive. We do not know of any such example for which $\mathfrak{M} = \bar{\mathfrak{M}}$.

(ii) Let S_1, S_2 denote the real and imaginary parts of the operator N of § 3; thus,

$$S_1 = (N + \bar{N})/2, \qquad S_2 = (N - \bar{N})/2i,$$

and hence $S_1 = L_2$ on $\mathfrak{D}(N)$, whereas $S_2 = -iL_3$ on $\mathfrak{D}(N)$. These operators are symmetric, but not necessarily closed. Their deficiency spaces (and those for their closures) are the spaces

$$\begin{aligned} \mathfrak{E}_1(\pm i) &= \{u \in \mathfrak{D}(S_1^*) \mid S_1^* u = \pm iu\}, && \text{for } S_1, \\ \mathfrak{E}_2(\pm i) &= \{u \in \mathfrak{D}(S_2^*) \mid S_2^* u = \pm iu\}, && \text{for } S_2. \end{aligned}$$

The dimensions of these spaces may be readily computed with the aid of the function ϕ introduced in § 3. Indeed, $S_1^* = L_2$ and $S_2^* = -iL_3$ on their respective domains, and so

$$\begin{aligned} \mathfrak{E}_1(\pm i) &= \{u \in \mathfrak{L}_2(0, \infty) \mid L_2 u = \pm iu\}, \\ \mathfrak{E}_2(\pm i) &= \{u \in \mathfrak{L}_2(0, \infty) \mid -iL_3 u = \pm iu\}. \end{aligned}$$

https://doi.org/10.4153/CJM-1965-098-8 Published online by Cambridge University Press

Now $L_2\phi = \lambda^2\phi = i\phi$ if $\lambda^2 = i$. Let λ_1, λ_2 be the two roots of $\lambda^2 - i$, with $\operatorname{Re}\lambda_1 < 0$, $\lambda_2 = -\lambda_1$, and let $\phi_1(x) = \phi(x, \lambda_1)$, $\phi_2(x) = \phi(x, \lambda_2)$. The solutions of $L_2 u = iu$ are spanned by ϕ_1, ϕ_2. Since $\phi_1 \in \mathfrak{L}_2(1, \infty)$, $\phi_2 \notin \mathfrak{L}_2(1, \infty)$, the solutions that are in $\mathfrak{L}_2(0, \infty)$ must be of the form $c\phi_1$, for some constant c. But this function behaves like c/x near the origin, and therefore cannot be in $\mathfrak{L}_2(0, \infty)$ unless $c = 0$. Thus $\dim \mathfrak{E}_1(+i) = 0$, and similarly $\dim \mathfrak{E}_1(-i) = 0$, which implies that the closure of S_1 is self-adjoint. An analogous argument for S_2 leads to the result that $\dim \mathfrak{E}_2(+i) = 0$, but $\dim \mathfrak{E}_2(-i) = 1$, so that the closure of S_2 is maximal symmetric but not self-adjoint.

(iii) We mentioned in Remark (i) above that a necessary condition for a maximal formally normal N (which is not normal) to have a normal extension in a larger space is that $\mathfrak{M} = \bar{\mathfrak{M}}$ in case $\dim \mathfrak{M} < \infty$, and consequently $\mathfrak{D}(\bar{N}^*) = \mathfrak{D}(N^*)$ must be valid. It is interesting to note that for the N of § 3 ($N = S_1 + iS_2$, in the notation of Remark (ii)) none of the domains $\mathfrak{D}(\bar{N}^*)$, $\mathfrak{D}(N^*)$, $\mathfrak{D}(S_1^*)$, $\mathfrak{D}(S_2^*)$ are comparable—none is included in any of the others. The function α, given by

$$\alpha(x) = (x^{-1} + 1)e^{-x} - x^{-1},$$

is in $\mathfrak{D}(\bar{N}^*)$ but in none of the other domains. Let β be a function defined by

$$\beta(x) = \begin{cases} (x^{-1} - 1)e^{x} - x^{-1}, & 0 < x \leqslant 1, \\ 0, & 2 \leqslant x < \infty, \end{cases}$$

and β smoothed to be of class $C^\infty(0, \infty)$. This β is in $\mathfrak{D}(N^*)$, but in none of the other domains. If γ is given by

$$\gamma(x) = \begin{cases} x^2, & 0 < x \leqslant 1, \\ 0, & 2 \leqslant x < \infty, \end{cases}$$

and of class $C^\infty(0, \infty)$, then $\gamma \in \mathfrak{D}(S_1^*)$, but is in none of the remaining domains. Finally, if δ is defined as

$$\delta(x) = \begin{cases} x, & 0 < x \leqslant 1, \\ 0, & 2 \leqslant x < \infty, \end{cases}$$

and $\delta \in C^\infty(0, \infty)$, then $\delta \in \mathfrak{D}(S_2^*)$, but is in none of the other domains.

5. Further remarks. Let $N = S_1 + iS_2$, where $S_1 = \operatorname{Re} N$, $S_2 = \operatorname{Im} N$, be formally normal in $\mathfrak{H}$. Here we consider some situations where the domains of $\bar{N}^*$, N^*, S_1^*, S_2^* are comparable, and show that in these cases N has a normal extension in $\mathfrak{H}$. The closure of an operator T in $\mathfrak{H}$ will be denoted by $\tilde{T}$.

First, we note that *if $\tilde{S}_1$ is self-adjoint, and $\mathfrak{D}(S_1^*) = \mathfrak{D}(\bar{N}^*)$, then N must be normal.* This can be seen by observing that the mapping $\{u, S_1^* u\} \to \{u, \bar{N}^*u\}$ is a closed mapping of the Banach space $\mathfrak{G}(S_1^*)$ into the Banach space $\mathfrak{G}(\bar{N}^*)$. The closed graph theorem then implies that this mapping is continuous, and therefore there is a constant c such that

https://doi.org/10.4153/CJM-1965-098-8 Published online by Cambridge University Press

$$(9) \qquad ||\bar{N}^* u||^2 \leqslant c(||S_1^* u||^2 + ||u||^2), \qquad u \in \mathfrak{D}(S_1^*).$$

Thus

$$||Nu||^2 \leqslant c(||S_1 u||^2 + ||u||^2), \qquad u \in \mathfrak{D}(S).$$

From this it follows that if $u \in \mathfrak{D}(\tilde{S}_1) = \mathfrak{D}(S_1^*)$, then $u \in \mathfrak{D}(\tilde{N}) = \mathfrak{D}(N)$. Consequently, we have $\mathfrak{D}(N) = \mathfrak{D}(S_1) \subset \mathfrak{D}(\tilde{S}_1) = \mathfrak{D}(\bar{N}^*) \subset \mathfrak{D}(N)$, and hence $\mathfrak{D}(\bar{N}^*) = \mathfrak{D}(N)$, which implies that N is normal.

The same result is valid if $\tilde{S}_1$ is self-adjoint and $\mathfrak{D}(S_1^*) = \mathfrak{D}(N^*)$. Thus, in the Fuglede, or Nelson, examples mentioned in § 1, it must be true that the domains of $\tilde{S}_1$ or $\tilde{S}_2$ are not equal to the domains of $\bar{N}^*$ or N^*.

The above argument can be carried a bit further in case $\dim \mathfrak{M} < \infty$. Indeed, *suppose N, N_1 are operators in $\mathfrak{H}$ having all the properties of formally normal operators, except that they are not necessarily closed, and let*

$$N \subset N_1, \qquad \dim[\mathfrak{D}(\bar{N}^*)/\mathfrak{D}(\tilde{N})] < \infty,$$

$$S_1 = \operatorname{Re} N, \qquad S_2 = \operatorname{Im} N, \qquad T_1 = \operatorname{Re} N_1, \qquad T_2 = \operatorname{Im} N_1.$$

If $\tilde{T}_1$ is self-adjoint, and $D(S_1^*) = D(\bar{N}^*)$ (or $D(S_1^*) = D(N^*)$), then $\tilde{N}_1$ is normal.

Both $\tilde{N}$ and $\tilde{N}_1$ are formally normal; it remains to check that

$$\mathfrak{D}(\tilde{N}_1) = \mathfrak{D}(\tilde{N}_1^*) = \mathfrak{D}(N_1^*).$$

The equality of the domains of S_1^* and $\bar{N}^*$ implies, as before, an inequality (9). Since $N \subset N_1$ we have

$$N \subset N_1 \subset \bar{N}_1^* \subset \bar{N}^*,$$

$$\bar{N} \subset \bar{N}_1 \subset N_1^* \subset N^*,$$

and thus

$$S_i \subset T_i \subset T_i^* \subset S_i^* \qquad (i = 1, 2).$$

An application of (9) to $u \in \mathfrak{D}(T_1) = \mathfrak{D}(N_1)$ shows that $\mathfrak{D}(\tilde{T}_1) \subset \mathfrak{D}(\tilde{N}_1)$, and using this inequality for $u \in \mathfrak{D}(S_1) = \mathfrak{D}(N)$, we obtain $\mathfrak{D}(\tilde{S}_1) \subset \mathfrak{D}(\tilde{N})$. But, for $u \in \mathfrak{D}(T_1)$, we have

$$||T_1 u|| = \tfrac{1}{2}||(N_1 + \bar{N}_1)u|| \leqslant \tfrac{1}{2}(||N_1 u|| + ||\bar{N}_1 u||) = ||N_1 u||,$$

and this yields $\mathfrak{D}(\tilde{N}_1) \subset \mathfrak{D}(\tilde{T}_1)$; similarly $\mathfrak{D}(\tilde{N}) \subset \mathfrak{D}(\tilde{S}_1)$. Therefore

$$(10) \qquad \mathfrak{D}(\tilde{S}_1) = \mathfrak{D}(\tilde{N}), \qquad \mathfrak{D}(\tilde{T}_1) = \mathfrak{D}(\tilde{N}_1).$$

The symmetric operator $\tilde{S}_1$ has a self-adjoint extension $\tilde{T}_1$. Consequently,

$$(11) \qquad \mathfrak{D}(S_1^*) = \mathfrak{D}(\tilde{S}_1) + \mathfrak{N}(S_1^* + iI) + \mathfrak{N}(S_1^* - iI),$$

with

$$(12) \qquad \dim \mathfrak{N}(S_1^* + iI) = \dim \mathfrak{N}(S_1^* - iI) = k,$$

say. Then we know that

$$\mathfrak{D}(\tilde{T}_1) = \mathfrak{D}(\tilde{S}_1) + \mathfrak{K}_1, \qquad \dim \mathfrak{K}_1 = k. \tag{13}$$

Also, since $\tilde{N}$ has the formally normal extension $\tilde{N}_1$, we have

$$\mathfrak{D}(\tilde{N}^*) = \mathfrak{D}(\tilde{N}) + \mathfrak{M}_1 + \mathfrak{M}_2, \tag{14}$$

a direct sum, with

$$\mathfrak{D}(\tilde{N}_1) = \mathfrak{D}(\tilde{N}) + \mathfrak{M}_1, \qquad \mathfrak{D}(N_1^*) = \mathfrak{D}(\tilde{N}) + \bar{\mathfrak{M}}_1, \tag{15}$$

where

$$\mathfrak{M}_1 \subset \bar{\mathfrak{M}}_1 = \tilde{N}^*\mathfrak{M}_2, \qquad \dim \bar{\mathfrak{M}}_1 = \dim \mathfrak{M}_2; \tag{16}$$

see Theorem 2, and the remark following the proof of this result, in **(1)**. Thus (10)–(16) yield

$$\dim \mathfrak{M}_1 + \dim \bar{\mathfrak{M}}_1 = \dim[\mathfrak{D}(\tilde{N}^*)/\mathfrak{D}(\tilde{N})] = \dim[\mathfrak{D}(S_1^*)/\mathfrak{D}(\tilde{S}_1)] = 2k,$$
$$\dim \mathfrak{M}_1 = \dim[\mathfrak{D}(\tilde{N}_1)/\mathfrak{D}(\tilde{N})] = \dim[\mathfrak{D}(\tilde{T}_1)/\mathfrak{D}(\tilde{S}_1)] = k,$$

which implies $\dim \bar{\mathfrak{M}}_1 = k = \dim \mathfrak{M}_1$. Since $\mathfrak{M}_1$, $\bar{\mathfrak{M}}_1$ are finite-dimensional, and $\mathfrak{M}_1 \subset \bar{\mathfrak{M}}_1$, we have $\mathfrak{M}_1 = \bar{\mathfrak{M}}_1$. Then (15) shows that $\mathfrak{D}(\tilde{N}_1) = \mathfrak{D}(N_1^*)$, and we have proved that $\tilde{N}_1$ is normal.

The argument is entirely similar if $\mathfrak{D}(S_1^*) = \mathfrak{D}(N^*)$. Instead of (9) we have an inequality

$$||N^* u||^2 \leqslant c' (||S_1^* u||^2 + ||u||^2), \qquad u \in \mathfrak{D}(S_1^*), \tag{9'}$$

and use is made of the fact that

$$\dim[\mathfrak{D}(\tilde{N}^*)/\mathfrak{D}(\tilde{N})] = \dim[\mathfrak{D}(N^*)/\mathfrak{D}(\tilde{N})].$$

The above result may be applied to the case of regular ordinary differential operators. Let L_1, L_2 be formally self-adjoint ordinary differential operators

$$L_1 = a_n D^n + \ldots + a_0, \qquad D = d/dx,$$
$$L_2 = b_m D^m + \ldots + b_0, \qquad m \leqslant n,$$

with coefficients a_k, b_k of class C^∞ on some finite, closed interval $a \leqslant x \leqslant b$, and $a_n(x) \neq 0$, $b_m(x) \neq 0$ there. Suppose $L_1 L_2 u = L_2 L_1 u$ for all $u \in C^\infty(a, b)$. Let S_i be L_i defined on $C_0^\infty(a, b)$, $i = 1, 2$. Then, in the Hilbert space $\mathfrak{L}_2(a, b)$, the operator $N = S_1 + iS_2$ has all the properties of a formal normal operator, except that it is not closed. Moreover, it is easy to see that $\mathfrak{D}(S_1^*) = \mathfrak{D}(\tilde{N}^*) = \mathfrak{D}(N^*)$, and $\dim[\mathfrak{D}(\tilde{N}^*)/\mathfrak{D}(\tilde{N})] = 2n$. The symmetric operator $\tilde{S}_1$ has self-adjoint extensions in $\mathfrak{L}_2(a, b)$. If T_1 is a symmetric extension of S_1 such that $\tilde{T}_1$ is self-adjoint, and $N_1 = T_1 + iT_2$ is formally normal, but not necessarily closed, then $\tilde{N}_1$ is normal. Thus an example of the Fuglede, or Nelson, type cannot be found among regular ordinary differential operators.

References

1. G. Biriuk and E. A. Coddington, *Normal extensions of unbounded formally normal operators*, J. Math. Mech., *12* (1964), 617–638.
2. J. L. Burchnall and T. W. Chaundy, *Commutative ordinary differential operators*, Proc. London Math. Soc. (Ser. 2), *21* (1923), 420–440.
3. E. Nelson, *Analytic vectors*, Ann. Math., *70* (1959), 572–615.

University of Copenhagen,
University of California, Los Angeles