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MATHEMATICS, STATISTICS, AND PROBABILITY
NOVEL-RESULT

Hodge Representations

Xiayimei Han and Colleen Robles

Mathematics Department, Duke University, Box 90320, Durham, NC 27708-0320, USA
E-mail address: xiayimei.han@duke.edu; robles@math.duke.edu

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Abstract
Green–Griffiths–Kerr introduced Hodge representations to classify the Hodge groups of polarized Hodge structures, and the corresponding Mumford–Tate subdomains. We summarize how, given a fixed period domain $\mathcal{D}$, to enumerate the Hodge representations and corresponding Mumford–Tate subdomains $D \subset \mathcal{D}$. The procedure is illustrated in two examples: (i) weight two Hodge structures with $p_g = h^{2,0} = 2$; and (ii) weight three CY-type Hodge structures.

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1. Introduction

1.1. *Hodge groups*

Fix a period domain $\mathcal{D} = \mathcal{D}_{\mathrm{h}} = \mathcal{G}_{\mathbb{R}}/\mathcal{G}_{\mathbb{R}}^0$ parameterizing Q–polarized Hodge structures on a rational vector space V with Hodge numbers $\mathrm{h} = \left(h^{n,0}, \ldots, h^{0,n}\right)$. Here

$$\mathcal{G}_{\mathbb{R}} = \mathrm{Aut}(V_{\mathbb{R}}, Q)$$

is either an orthogonal group $\mathrm{O}(a,\ 2b)$ (if n is even) or a symplectic group $\mathrm{Sp}\,(2r,\ \mathbb{R})$ (if n is odd), and $\mathcal{G}_{\mathbb{R}}^0$ is the compact stabilizer of a fixed $\varphi \in \mathcal{D}$. To each Hodge structure $\varphi \in \mathcal{D}$ is associated a ($\mathbb{Q}$–algebraic) Hodge group $\mathbf{G}_\varphi \subset \mathrm{Aut}(V,\ Q)$, and a Mumford–Tate domain $D = D_\varphi = G_\varphi \cdot \varphi \subset \mathcal{D}$, where $G_\varphi = \mathbf{G}_\varphi(\mathbb{R})$. Briefly, the Hodge structure $\varphi \in \mathcal{D}$ determines a homomorphism of $\mathbb{R}$–algebraic groups $\varphi : S^1 \to \mathrm{Aut}(V_{\mathbb{R}},\ Q)$, and the Hodge group $\mathbf{G}_\varphi$ is the $\mathbb{Q}$–algebraic closure of $\varphi(S^1)$. The Hodge group $\mathbf{G}_\varphi$ may be equivalently defined as the stabilizer of the Hodge tensors of φ.

1.2. *Motivations*

The geometric considerations motivating a classification of the Hodge groups for a given period domain $\mathcal{D}$ include the following. For generic choice of $\varphi \in D$, the Hodge group $\mathbf{G}_\varphi$ is the full automorphism group $\mathrm{Aut}(V,\ Q)$. So when containment $\mathbf{G}_\varphi \subsetneq \mathrm{Aut}(V,\ Q)$ is strict, the Hodge structure has nongeneric Hodge tensors. (And, because $\mathbf{G}_{\varphi'} \subset \mathbf{G}_\varphi$ for all $\varphi' \in D_\varphi$, the Mumford–Tate domain D_φ will parameterize Hodge structures with nongeneric Hodge tensors.) An extreme example here is the case that D_φ is a point $\{\varphi\}$; this is the case if and only if $\mathbf{G}_\varphi$ is a torus; equivalently, End $(V,\ \varphi)$ is a CM field. When containment

$\mathbf{G}_\varphi \subsetneq \mathrm{Aut}(V, Q)$ is strict and the Hodge structure is realized by the cohomology of an algebraic variety, the variety and its self-products "should" admit nongeneric arithmetic properties (such as extra Hodge classes, or automorphisms, *et cetera*). In general, the Hodge group can have significant geometric consequences; for example, it plays a key role in Ribet's study (Ribet, 1983) of the Hodge conjecture for principally polarized abelian varieties (expanding upon earlier work of Tanke'ev's, 1981; 1982).

Likewise much geometric motivation for the classification of the Mumford–Tate domains comes from the moduli of algebraic varieties. In general, the period domain is not Hermitian. Two significant exceptions are the period domains arising when considering moduli spaces of principally polarized abelian varieties and K3 surfaces. The Hermitian symmetric structure of $\mathcal{D}$ in these two cases, along with global Torelli theorems, is the underlying structure that has made Hodge theory such a powerful tool in the study of these moduli spaces and their compactifications (Laza, 2016). Even when the period domain $\mathcal{D}$ is not Hermitian, it may contain Hermitian symmetric Mumford–Tate subdomains D. (For example, every horizontal subdomain is Hermitian symmetric.) If a moduli space $\mathcal{M}$ has an injective period map $\Phi : \mathcal{M} \to \Gamma\backslash\mathcal{D}$ (a Torelli-type theorem) and the image is a Zariski open subset of the locally Hermitian symmetric $\pi(D) \subset \Gamma\backslash\mathcal{D}$, with $\pi : \mathcal{D} \to \Gamma\backslash\mathcal{D}$ the quotient, then Hodge theory is again a significant tool in the study of $\mathcal{M}$ and its compactifications, *cf.* Allcock et al. (2002, 2011), Borcea (1997), Garbagnati and van Geeme, (2010), Kondo (2000), Laza et al. (2018), Pearlstein and Zhang (2019), Rohde (2009), and Voisin (1983). Reciprocally, given a Hermitian symmetric Mumford–Tate domain $D \subset \mathcal{D}$ it is a very interesting problem to find geometric (or motivic) realizations of the domain; work in this direction includes Kerr and Pearlstein (2016), Pearlstein and Zhang (2019), and Zhang (2014, 2016).

1.3. Objective and approach

The principal goal of this paper is to present an expository discussion of the Green–Griffiths–Kerr (Green et al., 2012) prescription to identify the real algebraic groups $G_\varphi = \mathbf{G}_\varphi(\mathbb{R})$ that may arise. More precisely, Green–Griffiths–Kerr identify the underlying real Lie algebra $\mathfrak{g}_\mathbb{R}$. This determines G_φ to finite data, and suffices to identify the domains D_φ as intrinsic G_φ^{ad}–homogeneous spaces. (See Patrikis, 2016 for the classification of general $\mathbf{G}_\varphi$.)

Example 1.1. The case of weight one Hodge representations is classical (Deligne, 1979; Milne, 2005): The real form $\mathfrak{g}_\mathbb{R}$ is one of $\mathfrak{sp}_{2r}\mathbb{R}$, $\mathfrak{u}(a, b)$, $\mathfrak{su}(a, a)$, $\mathfrak{so}(2, m)$ and $\mathfrak{so}^*(2r)$. See Example 3.3 for the corresponding Hodge representations.

Remark 1.2. We are aware of only a few cases in which the classification of the $\mathbf{G}_\varphi$ as $\mathbb{Q}$–algebraic groups has been completely worked out. These include Zarhin's classification (Zarhin, 1983) of the Hodge groups of K3 surfaces (see Example 5.3 for the corresponding Hodge representations), and Green–Griffiths–Kerr classification (Green et al., 2012, §7) for period domains $\mathcal{D}$ with Hodge numbers $\mathbf{h} = (2,2)$ and $\mathbf{h} = (1,1,1,1)$.

Green–Griffiths–Kerr (Green et al., 2012) showed that the Hodge groups $\mathbf{G} = \mathbf{G}_\varphi$ and Mumford–Tate domains $D = D_\varphi \subset \mathcal{D}_\mathrm{h}$ are in bijection with Hodge representations.

$S^1 \xrightarrow{\phi} \boldsymbol{G}(\mathbb{R}),\ \mathbf{G} \hookrightarrow \mathrm{Aut}(V, Q),.$

with Hodge numbers $\mathbf{h}_\phi \le \mathbf{h}$. (Given $\mathbf{h}_1 = \left(h_1^{n,0},, h_1^{0,n}\right)$ and $\mathbf{h}_2 = \left(h_2^{n,0},, \ h_2^{0,n}\right)$, we write $\mathbf{h}_1 \le \mathbf{h}_2$ if $h_1^{p,q} \le h_2^{p,q}$ for all $p,\ q$.) Effectively one may say that *the Hodge groups and Mumford–Tate domains are classified by the Hodge representations*: given a fixed $\mathcal{D} = \mathcal{D}_\mathrm{h}$ (with specified Hodge numbers $\mathbf{h}$), one identifies all possible Hodge domains $D \subset \mathcal{D}$ by enumerating the Hodge representations with $\mathbf{h}_\phi \le \mathbf{h}$.

Remark 1.3. There are some obvious subdomains that can be identified without Hodge representations: (products of) period subdomains. If $\mathcal{D}_i$ is the period domain for Hodge numbers $\mathbf{h}_i$ and $\mathbf{h}_1 + \cdots + \mathbf{h}_\ell \le \mathbf{h}$, then $\mathcal{D}_1 \times \cdots \times \mathcal{D}_\ell$ is a Mumford–Tate subdomain of $\mathcal{D}$.

Green–Griffiths–Kerr's characterization of the Hodge representations is formulated as Theorem 3.1, which asserts that the induced (real Lie algebra) Hodge representations

$$\mathbb{R} \to \mathfrak{g}_\mathbb{R} \to \mathrm{End}\,(V_\mathbb{R},\ Q) \tag{1.4}$$

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are enumerated by tuples $(\mathfrak{g}^{ss}_{\mathbb{C}}, E^{ss}, \mu, c)$ consisting of:

(i) a semisimple complex Lie algebra $\mathfrak{g}^{ss}_{\mathbb{C}} = [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$,
(ii) an element $E^{ss} \in \mathfrak{g}^{ss}_{\mathbb{C}}$ with the property that ad E^{ss} acts on $\mathfrak{g}^{ss}_{\mathbb{C}}$ diagonalizably with integer eigenvalues,
(iii) a highest weight μ of $\mathfrak{g}^{ss}_{\mathbb{C}}$, and
(iv) a constant $c \in \mathbb{Q}$ satisfying $\mu(E^{SS}) + c \in \frac{1}{2}\mathbb{Z}$.

The real form $\mathfrak{g}^{ss}_{\mathbb{R}}$ is the Lie algebra of the image G^{ad}_{φ} of $\mathrm{Ad} : G_{\varphi} \to \mathrm{Aut}(\mathfrak{g}_{\mathbb{R}})$. We have $D_{\varphi} = G^{ad}_{\varphi} \cdot \varphi$, and E^{ss} is essentially equivalent to the isotropy group $\mathrm{Stab}_{G^{ad}_{\varphi}}(\varphi)$.

1.4. Examples and special cases

1.4.1. Horizontal Hodge domains

As discussed above (§1.2) the identification of the horizontal subdomains is of particular interest. These are the domains that satisfy the infinitesimal period relation (IPR, a.k.a. Griffiths' transversality). It is well-known that horizontal subdomains are necessarily Hermitian, and as such their structure as intrinsic homogeneous complex manifolds is classical and well-understood. These results are reviewed in §3.2. The Hodge representations with horizontal D_{ϕ} are characterized in Proposition 3.7.

1.4.2. Weight two Hodge representations

In §4 we apply the prescription of Theorem 3.1 to identify all Hodge representations and Mumford–Tate subdomains D of the period domain $\mathcal{D}$ parameterizing Q–polarized, (effective) weight $n = 2$ Hodge structures with $p_g = h^{2,0} = 2$ (Theorems 4.1, 4.3, 4.4 and Theorem 4.6). This period domain is chosen as our primary example for two reasons. First, it is in a certain sense the simplest example of a period domain that is not Hermitian symmetric. (The infinitesimal period relation is a contact subbundle of $T\mathcal{D}$.) Second, it is the period domain arising when considering families of Horikawa surfaces (Horikawa, 1978; 1978/79), in which there has been much interest recently (Franciosi et al., 2015; 2017; Pearlstein & Zhang, 2019).

1.4.3. Hodge representations of Calabi–Yau type

Hodge representations of CY-type (those with first Hodge number $h^{n,0} = 1$) are of considerable interest and have been studied by several authors, including Friedman and Laza (2013, 2014), Gross (1994), and Sheng and Zuo (2010). Much of this work is over $\mathbb{R}$, but Friedman and Laza (2014) have identified some rational forms $\mathbf{G}_{\varphi}(\mathbb{Q})$ admitting Hodge representations of CY 3-fold type. In §5 we classify the (Lie algebra) Hodge representations of CY-type (Theorem 5.2). The CY-Hodge representations with D Hermitian are well- known, and those with $\mathfrak{g}_{\mathbb{R}}$ semisimple have been classified (Robles, 2014, Proposition 6.1); so the content of Theorem 5.2 is to drop the hypothesis that $\mathfrak{g}_{\mathbb{R}}$ be semisimple from the classification. This result is used in Han (2021) to enumerate the set of all Hodge representations of CY 3-fold type. Those with horizontal (and therefore Hermitian) domain $D \subset \mathcal{D}$ are listed in Example 5.4, completing the classification begun in Friedman and Laza (2013, §2.3).

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2. Hodge representations

What follows is a laconic review of the necessary background material on Hodge representations. References for more detailed discussion include Green et al. (2012), Robles (2014, §§2–3) and Robles (2016, §§2–3).

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2.1. Basics

Let

$$\phi : S^1 \to G_{\mathbb{R}} \text{ and } G_{\mathbb{R}} \to \text{ Aut } (V_{\mathbb{R}}, Q) \tag{2.1}$$

be the data of a (real) Hodge representation (Green et al., 2012). Without loss of generality, we may suppose that the induced Lie algebra representation

$$\mathfrak{g}_{\mathbb{R}} \hookrightarrow \text{End } (V_{\mathbb{R}}, Q) \tag{2.2}$$

is faithful. The associated Hodge decomposition

$$V_{\mathbb{C}} = \oplus V_{\phi}^{p,q}$$

is the ϕ–eigenspace decomposition; that is,

$$V_{\phi}^{p,q} = \left\{ v \in V_{\mathbb{C}} | \phi(z)(v) = z^{p-q} v, \ \forall z \in S^1 \right\}$$

The associated grading element $\mathrm{E}_{\phi} \in i\mathfrak{g}_{\mathbb{R}}$ (or infinitesimal Hodge structure) (Robles, 2014) is defined by $\mathrm{E}_{\phi}(v) = \frac{1}{2}(p-q)v$ for all $v \in V_{\phi}^{p,q}$; that is, $\mathrm{E}_{\phi} \in \mathrm{End}(V_{\mathbb{C}})$ is defined so that $V_{\phi}^{p,q}$ is the E_{ϕ}–eigenspace with eigenvalue $\frac{1}{2}(p-q)$; for this reason it is sometimes convenient to write

$$V^{p,q} = V_{(p-q)/2}$$

Remark 2.3. Together the grading element E and Lie algebra representation (2.2) determine the group representation (2.1) up to finite data.

Definition 2.4. We call the pair $(\mathfrak{g}_{\mathbb{R}} \hookrightarrow \mathrm{Aut}(V_{\mathbb{R}}, Q), \mathrm{E})$ the data of a *real, Lie algebra Hodge representation* ($\mathbb{R}$-LAHR).

Remark 2.5. A key point here is that a Hodge representation (2.1) determines a grading element $\mathrm{E}_{\phi} \in i\mathfrak{g}_{\mathbb{R}}$. Conversely a complex reductive Lie algebra $\mathfrak{g}_{\mathbb{C}}$, a grading element $\mathrm{E} \in \mathfrak{g}_{\mathbb{C}}$ determines both a real form $\mathfrak{g}_{\mathbb{R}}$ (§ 2.3.2) and a Hodge representation (§ 2.3.3).

Notice that ϕ is a level n Hodge structure on $V_{\mathbb{R}}$ if and only if the E_{ϕ}–eigenspace decomposition is

$$V_{\mathbb{C}} = V_{n/2} \oplus V_{n/2-1} \oplus \cdots \oplus V_{1-n/2} \oplus V_{-n/2}. \tag{2.6}$$

Remark 2.7. The Hodge structure is of level zero (equivalently, $V_{\mathbb{C}} = V_{\phi}^{0,0}$) if and only if ϕ is trivial. We assume this is not the case.

Remark 2.8 (Period domains). The Hodge domain D determined by (2.1) is a period domain if and only if $G_{\mathbb{R}} = \mathrm{Aut}(V_{\mathbb{R}}, Q)$.

2.2. Induced Hodge representation

There is an induced Hodge representation on the Lie algebra $\mathfrak{g}_{\mathbb{R}}$. Define

$$\mathfrak{g}_{\phi}^{\ell,-\ell} ; = \left\{ \xi \in \mathfrak{g}_{\mathbb{C}} | \xi\left(V_{\phi}^{p,q} \right) \subset V_{\phi}^{p+\ell, q-\ell}, \ \forall p, \ q \right\}.$$

Then

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{\ell \in \mathbb{Z}} \mathfrak{g}_{\phi}^{\ell,-\ell} \tag{2.9}$$

is a weight zero Hodge structure on $\mathfrak{g}_{\mathbb{R}}$ that is polarized by $-\kappa$, with κ the Killing form.

The Jacobi identity implies

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$$\left[\mathfrak{g}_\phi^{k,-k},\ \mathfrak{g}_\phi^{\ell,-\ell}\right] \subset \mathfrak{g}_\phi^{k+\ell,-k-l}.$$

The subalgebra

$$\mathfrak{g}_{\phi,\mathbb{C}}^{\text{even}}; = \underset{\ell\in Z}{\oplus}\ \mathfrak{g}_\phi^{2\ell,-2\ell}$$

is the complexification $\mathfrak{k}_\phi \otimes_{\mathbb{R}} \mathbb{C}$ of the (unique) maximal compact subalgebra $\mathfrak{k}_\phi \subset \mathfrak{g}_{\mathbb{R}}$ containing the Lie algebra $\mathfrak{g}_{\mathbb{R}}^0 = \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{g}_\phi^{0,0}$ of the stabilizer/centralizer $G_{\mathbb{R}}^0$ of ϕ.

2.3. Grading elements

Hodge structures are closely related to grading elements. This relationship is briefly reviewed here; see Robles (2014, §§2–3) and Robles (2016, §§2–3) for details.

Remark 2.10. Here grading elements are essentially linearizations of the circle $\phi : S^1 \hookrightarrow G_{\mathbb{R}}$ in the Hodge representation (2.1). The essential observation of this section is that the data $(\mathfrak{g}_{\mathbb{C}},\ \mathrm{E})$ determines the real form $\mathfrak{g}_{\mathbb{R}}$, and the Hodge domain and compact dual $D \subset \check{D}$ (as intrinsic homogeneous spaces. They are not represented as subdomains of a period domain $\mathcal{D}$ until we select the second half $G_{\mathbb{R}} \hookrightarrow \mathrm{Aut}\,(V_{\mathbb{R}},\ Q)$ of the Hodge representation (2.1)).

2.3.1. Definition

Fix a complex reductive Lie algebra $\mathfrak{g}_{\mathbb{C}}$. A *grading element* is any element $\mathrm{E} \in \mathfrak{g}_{\mathbb{C}}$ with the property that $\mathrm{ad(E)} \in \mathrm{End}\,(\mathfrak{g}_{\mathbb{C}})$ acts diagonalizably on $\mathfrak{g}_{\mathbb{C}}$ with integer eigenvalues; that is,

$$\mathfrak{g}_{\mathbb{C}} = \underset{\ell\in Z}{\oplus}\ \mathfrak{g}^{\ell,-\ell}, \text{ with } \mathfrak{g}^{\ell,-\ell} = \{\xi \in \mathfrak{g}_{\mathbb{C}} | [\mathrm{E},\ \xi] = \ell\xi\}. \tag{2.11}$$

Remark 2.12. The notation $\mathfrak{g}^{\ell,-\ell}$ is meant to be suggestive. The grading element E determines a weight zero (real) Hodge decomposition that is polarized by $-\kappa$, with κ the Killing form (§2.3.3).

Remark 2.13. The data $(\mathfrak{g}_{\mathbb{C}},\ \mathrm{E})$ determines a parabolic subgroup $P_{\mathrm{E}} \subset G_{\mathbb{C}}$ with Lie algebra $\mathfrak{p}_{\mathrm{E}} = \oplus_{\ell\geq 0}\mathfrak{g}^{\ell,-\ell}$. The resulting generalized grassmannian $\check{D} = G_{\mathbb{C}}/P_{\mathrm{E}}$ (or rational homogeneous variety) is the compact dual of the Hodge domain (as an intrinsic homogeneous space).

2.3.2. Grading elements versus real forms

Fix a complex reductive Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Given $\mathfrak{g}_{\mathbb{C}}$ and E, there is a unique real form $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{C}}$ such that (2.11) is a weight zero Hodge structure on $\mathfrak{g}_{\mathbb{R}}$ that is polarized by $-\kappa$ (Robles, 2016, §3.1.2). The real form $\mathfrak{g}_{\mathbb{R}}$ is determined by the condition that $\oplus_\ell \mathfrak{g}^{2\ell,-2\ell}$ is the complexification $\mathfrak{k}_{\mathbb{C}}$ of a maximal compact subalgebra $\mathfrak{k} \subset \mathfrak{g}_{\mathbb{R}}$.

See §3.2 for a discussion of the examples that are of the most interest here.

2.3.3. Grading elements versus Hodge representations

Given the data of §2.3.2, the grading element E acts on any representation $G_{\mathbb{R}} \to \mathrm{Aut}\,(V_{\mathbb{R}})$ by rational eigenvalues. The E–eigenspace decomposition $V_{\mathbb{C}} = \oplus_{k\in\mathbb{Q}} V_k$ is a Hodge decomposition (polarized by some Q), with $V^{p,q} = V_{(p-q)/2}$ as in §2.1, if and only if those eigenvalues lie in $\frac{1}{2}\mathbb{Z}$ (Green et al., 2012). The corresponding Hodge representation is given by the circle $\phi : S^1 \to G_{\mathbb{R}}$ defined by

$$\phi(z)v := z^{p-q}v,\ \ z \in S^1,\ \ v \in V^{p,q} = V_{(p-q)/2}.$$

Note that $\mathrm{E} = \mathrm{E}_\phi$.

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2.3.4. Normalization of grading element

The Lie algebra $\mathfrak{g}$ of G is reductive. Let

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}^{\mathrm{SS}} \tag{2.14}$$

denote the decomposition of $\mathfrak{g}$ into its center $\mathfrak{z}$ and semisimple factor $\mathfrak{g}^{\mathrm{SS}} = [\mathfrak{g}, \ \mathfrak{g}]$. Let $\mathrm{E} = \mathrm{E}' + \mathrm{E}^{\mathrm{SS}}$ be the decomposition given by (2.14). The Hodge domain D is determined by $\mathfrak{g}^{\mathrm{SS}}$ and E^{SS}.

Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$ that contains E_{ϕ}, is contained in $\mathfrak{g}^{0,0}_{\phi}$ and that is defined over $\mathbb{R}$. Then

$$\mathfrak{h} = \mathfrak{z} \oplus \mathfrak{h}^{\mathrm{SS}},$$

where $\mathfrak{h}^{\mathrm{SS}} = \mathfrak{h} \cap \mathfrak{g}^{\mathrm{SS}}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathrm{SS}}_{\mathbb{C}}$. Choose simple roots $\{\alpha_1,, \ \alpha_r\} \in \left(\mathfrak{h}^{\mathrm{SS}}\right)^*$ of $\mathfrak{g}^{\mathrm{SS}}_{\mathbb{C}}$ so that $\alpha_j(\mathrm{E}^{\mathrm{ss}}) \geq 0$, for all j. Without loss of generality, we may assume that $\alpha_j(\mathrm{E}) \in \{0,1\}$ (Robles, 2014, §3.3).[1] This is equivalent to the condition that the infinitesimal period relation $T^hD \subset TD$ is bracket–generating; equivalently, $\mathfrak{g}^{1,-1}$ generates $\mathfrak{g}^{+,-} = \oplus_{\ell>0}\mathfrak{g}^{\ell,-\ell}$ as a Lie algebra.

2.4. Reduction to irreducible

V. If $V_{\mathbb{R}} = V_1 \oplus V_2$ is reducible as a real representation, then the associated domain D factors $D = D_1 \times D_2$ into the product of the domains D_i for the V_i. So without loss of generality we may assume that $V_{\mathbb{R}}$ is irreducible.[2] The Schur lemma (and our hypothesis that (2.2) is faithful) implies

$$\dim \mathfrak{z} \ \in \{0,1\}. \tag{2.15}$$

Remark 2.16. Note that $\mathfrak{z} = \text{span}\ \{\mathrm{E}'\}$, so that $\mathfrak{g} = \mathfrak{g}^{\mathrm{SS}}$ if and only if $\mathrm{E}' = 0$.

Given an irreducible real representation $V_{\mathbb{R}}$ there exists a (unique) irreducible representation U of $G_{\mathbb{C}}$ such that one of the following holds:

$$V_{\mathbb{R}} \otimes \mathbb{C} = \begin{cases} U \quad \text{and } U = U^* & (U \text{ is } \textit{real} \text{ w.r. t. } \mathfrak{g}_{\mathbb{R}}), \\ U \oplus U^* \text{ and } U \neq U^* & (U \text{ is } \textit{complex} \text{ w.r. t. } \mathfrak{g}_{\mathbb{R}}), \\ U \oplus U^* \text{ and } U = U^* & (U \text{ is } \textit{quaternionic} \text{ w.r. t. } \mathfrak{g}_{\mathbb{R}}). \end{cases} \tag{2.17}$$

Let μ, $\mu^* \in \mathfrak{h}^*$ denote the highest weights of U and U^* respectively. When we wish to emphasize the highest weight of U, we will write $U = U_{\mu}$.

Remark 2.18 (Period domains). In this case of Remark 2.8, we have $V_{\mathbb{C}} = U_{\omega_1}$, with $\mu = \omega_1$ the first fundamental weight.

Remark 2.19. It follows from Remark 2.16 that action of the center $\mathfrak{z} \subset \mathfrak{g}_{\mathbb{R}}$ on $V_{\mathbb{R}}$ is determined by the action of E' The latter acts on U by scalar multiplication by $c = \mu(\mathrm{E}') \in \mathbb{Q}$. In particular, $\mathfrak{z} \neq 0$ if and only if $c \neq 0$. Moreover, E' necessarily acts on the dual by $-c = \mu^*(\mathrm{E}')$. So $\mu \neq \mu^*$, and U is complex with respect to $\mathfrak{g}_{\mathbb{R}}$ whenever $\mathfrak{g}_{\mathbb{R}}$ has a nontrivial center $(\mathfrak{z} \neq 0)$.

2.5. Real, complex and quaternionic representations

Note that U is complex if and only if $\mu \neq \mu^*$ By Remark 2.19, this is always the case when $\mathfrak{z} \neq 0$; equivalently, $\mathfrak{g} \neq \mathfrak{g}^{\mathrm{SS}}$. When $\mathfrak{z} = 0$ $\left(\text{equivalently, } \mathfrak{g} \ = \mathfrak{g}^{\mathrm{SS}} \text{ is semisimple}\right)$ the real and quaternionic

[1]There is a typo in Robles (2014, Proposition 3.4): in general one may assert only that the group F is $\mathbb{R}$–algebraic (not $\mathbb{Q}$–algebraic).

[2]Note that a rational representation may be irreducible over $\mathbb{Q}$, but not over $\mathbb{R}$. So this assumption would be too strong if one were looking for a classification over $\mathbb{Q}$.

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representations may be distinguished as follows. Recall the conventions of §2.3.4, and let $\{A^1,, A^r\} \subset \mathfrak{h}^{ss}$ be the basis dual to the simple roots $\{\alpha_1,, \alpha_r\}$. Then

$$\mathrm{E}_\phi^{\mathrm{SS}} = \sum \alpha_i(\mathrm{E}_\phi)\mathrm{A}^i, \text{ with } \alpha_i(\mathrm{E}_\phi) \in \{0,1\}.$$

Define

$$\mathrm{T}_\phi := 2 \sum_{\alpha_i\left(\mathrm{E}_\phi^{\mathrm{ss}}\right)=0} \mathrm{A}^i.$$

If $\mu = \mu^*$, then U is real if and only if $\mu(\mathrm{T}_\phi)$ is even, and is quaternionic if and only if $\mu(\mathrm{T}_\phi)$ is odd (Green et al., 2012).

2.6. *Eigenvalues and level of the Hodge structure*

Set

$$m := \mu(\mathrm{E}_\phi) \text{ and } m^* := \mu^*(\mathrm{E}_\phi).$$

Then the nontrivial eigenvalues of E_ϕ on U are

$$\{m,\ m-1,\ m-2, \ldots,\ 2-m^*,\ 1-m^*,\ -m^*\}.$$

Equation (2.6) implies

$$2m,\ 2m^* \in \mathbb{Z}.$$

The Hodge structure ϕ on $V_{\mathbb{R}}$ is of level

$$n = 2\max\{m,\ m^*\}.$$

2.7. *Reductive* versus *semisimple*

Let $(\mathfrak{g}_{\mathbb{C}},\ \mathrm{E}_\phi,\ \mu)$ be a triple underlying a Hodge representation; $\mathfrak{g}_{\mathbb{C}}$ is a complex reductive Lie algebra, $\mathrm{E}_\phi \in \mathfrak{g}_{\mathbb{C}}$ is a grading element (determining a real form $\mathfrak{g}_{\mathbb{R}}$, §2.3.2), and μ is the highest weight of an irreducible $\mathfrak{g}_{\mathbb{C}}$–module $U = U_\mu$. The purpose of this section is to observe that such triples are equivalent to tuples $\left(\mathfrak{g}_{\mathbb{C}}^{\mathrm{ss}},\ \mathrm{E}_\phi^{\mathrm{ss}},\ \mu^{\mathrm{ss}},\ c\right)$ with $\mathfrak{g}_{\mathbb{C}}^{\mathrm{ss}}$ a complex semisimple Lie algebra, $\mathrm{E}_\phi^{\mathrm{ss}} \in \mathfrak{g}_{\mathbb{C}}^{\mathrm{ss}}$ a grading element, μ^{ss} the highest weight of an irreducible $\mathfrak{g}_{\mathbb{C}}^{\mathrm{ss}}$–module, and $c \in \mathbb{Q}$.

Recall the notations of §2.3.4. As discussed in Remark 2.19, the central factor E'_ϕ acts on the irreducible U by a scalar

$$c = \mu\left(\mathrm{E}'_\phi\right) \in \mathbb{Q},$$

and on U^* by $-c$. It follows that $(\mathfrak{g}_{\mathbb{C}},\ \mathrm{E}_\phi,\ \mu)$ and $\left(\mathfrak{g}_{\mathbb{C}}^{\mathrm{ss}},\ \mathrm{E}_\phi^{\mathrm{ss}},\ \mu^{\mathrm{ss}},\ c\right)$ carry the same data. (Here $\mu^{\mathrm{ss}} = \mu|_{\mathfrak{h}^{\mathrm{ss}}}$ is the highest weight of U as a $\mathfrak{g}_{\mathbb{C}}^{\mathrm{ss}}$–module.)

As noted in Remark 2.19, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^{\mathrm{ss}}$ is semisimple if and only if $c = 0$.

The remainder of this section is devoted to discussing the relationship between the $\mathrm{E}_\phi^{\mathrm{ss}}$–eigenspace decomposition of U and the Hodge decomposition (§2.1) of $V_{\mathbb{R}}$.

Let

$$U = U_{\mu\left(\mathrm{E}_\phi^{\mathrm{ss}}\right)} \oplus \cdots \oplus U_{-\mu^*\left(\mathrm{E}_\phi^{\mathrm{ss}}\right)} \tag{2.20}$$

be the $\mathrm{E}_\phi^{\mathrm{ss}}$–eigenspace decomposition of U. We have

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$$m = \mu(\mathrm{E}_\phi) = \mu\left(\mathrm{E}_\phi^{\mathrm{ss}}\right) + c.$$

Likewise, the $\mathrm{E}_\phi^{\mathrm{ss}}$–eigenspace decomposition of U^* is

$$U^* = U^*_{\mu^*\left(\mathrm{E}_\phi^{\mathrm{ss}}\right)} \oplus \cdots \oplus U^*_{-\mu\left(\mathrm{E}_\phi^{\mathrm{ss}}\right)}.$$

It is a general fact from representation theory that $\mu\left(\mathrm{E}_\phi^{\mathrm{SS}}\right)$ and $-\mu^*\left(\mathrm{E}_\phi^{\mathrm{SS}}\right)$ are both elements of $\mathbb{Q}$, and any two nontrivial $\mathrm{E}_\phi^{\mathrm{ss}}$–eigenvalues of U differ by an integer.

(a) If U_μ is real, then $\mu = \mu^*$ and $V_{\mathbb{C}} = U_\mu$ imply that $c = 0$ and

$$V^{p,q} = U_{(p-q)/2}.$$

(In this case, we have $\mathfrak{z} = 0$.)

(b) If U is complex or quaternionic, then

$$V^{p,q} = U_{(p-q)/2-c} \oplus U^*_{(p-q)/2+c}.$$

Remark 2.21. From (2.20), we see that the number of nontrivial E–eigenvalues for U_μ is precisely $e(\mu, \mathrm{E}) = (\mu + \mu^*)(\mathrm{E}) + 1$. By (2.6) and (2.17), we have $e(\mu, \mathrm{E}) \leq n+1$. And by Remark 2.7, $e(\mu, \mathrm{E}) \geq 2$. Thus

$$2 \leq e(\mu, \mathrm{E}) = (\mu + \mu^*)(\mathrm{E}) + 1 \leq n+1.$$

3. Identification of Hodge domains: general strategy

3.1. Main result

Given a complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ with Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$, and irreducible $\mathfrak{g}_{\mathbb{C}}$–representation U and a rational number $c \in \mathbb{Q}$, let $\mathrm{E}' = c\,\mathrm{Id} \in \mathrm{End}(U)$ be the operator acting on U by scalar multiplication. We specify that $\mathrm{E}' = -c\,\mathrm{Id} \in \mathrm{End}(U^*)$ act by $-c$ on the dual representation. Then

$$\widetilde{\mathfrak{g}}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}} \oplus \mathrm{span}_{\mathbb{C}}\{\mathrm{E}'\}$$

is a reductive Lie algebra (semisimple if $c = 0$), with semisimple factor $\mathfrak{g}_{\mathbb{C}}$ and center $\mathfrak{z}$ spanned by E' (We are essentially making a change of notation here, replacing the reductive/semisimple pair $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{g}_{\mathbb{C}}^{\mathrm{ss}}$ of the previous sections with (possibly) reductive/semisimple pair $\widetilde{\mathfrak{g}}_{\mathbb{C}}$, $\mathfrak{g}_{\mathbb{C}}$. This is done for notational simplicity: it is cleaner to drop the $^{\mathrm{ss}}$ superscript.)

The upshot of the discussions in §§2.3–2.7 is.

Theorem 3.1 (Green–Griffiths–Kerr [Green et al., 2012]). *In order to identify the Hodge representations* (2.1) *with specified Hodge numbers* $\mathbf{h} = \left(h^{n,0}, \ldots, h^{0,n}\right)$, *it suffices to identify tuples* $(\mathfrak{g}_{\mathbb{C}}, \mathrm{E}, \mu, c)$ *consisting of a complex semisimple Lie algebra* $\mathfrak{g}_{\mathbb{C}}$, *a grading element* $\mathrm{E} \in \mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$ (as in §2.3.2 and §2.3.4), *the highest weight* $\mu \in \mathfrak{h}^*$ *of an a irreducible* $\mathfrak{g}_{\mathbb{C}}$– *module* U, *and* $c \in \mathbb{Q}$ *that satisfy the following conditions*: $m; = \mu(\mathrm{E}) + c \in \frac{1}{2}\mathbb{Z}$, *and the irreducible representation* $V_{\mathbb{R}}$ (§2.4) *of the real form* $\mathfrak{g}_{\mathbb{R}}$ *determined by* E (§2.3.2) *has* $(\mathrm{E} + \mathrm{E}')$*–eigenspace decomposition of the form* (2.6) *with* $\dim V_{(p-q)/2} = h^{p,q}$.

Example 3.2 (Period domains). The domain D_ϕ is a period domain if and only if the tuple $(\mathfrak{g}_{\mathbb{C}}, \mathrm{E}, \mu, c)$ is of one of the following two forms:

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(i) $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sp}_{2r}\mathbb{C}; \mu = \omega_1$, so that $U = \mathbb{C}^{2r}$ is the standard representation; $\alpha_r(\mathrm{E}) = 1$ and $c = 0$.
(ii) $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}_m\mathbb{C}; \mu = \omega_1$, so that $U = \mathbb{C}^m$ is the standard representation; and $c = 0$. If $m = 2r$ is even, then we also have $(\alpha_{r-1} + \alpha_r)(\mathrm{E}) \in \{0, 2\}$.

Example 3.3 (Weight $n = 1$). The weight $n = 1$ Hodge representations are well understood (Deligne, 1979; Milne, 2005). The corresponding tuples $(\mathfrak{g}_{\mathbb{C}}, \mathrm{E}, \mu, c)$ are.

(i) $(\mathfrak{sp}_{2r}\mathbb{C}, \mathrm{A}^r, \omega_1, 0)$, with $\widetilde{\mathfrak{g}}_{\mathbb{R}} = \mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}_{2r}\mathbb{R}$. The corresponding Hodge domain D is the period domain $\mathcal{D}$ parameterizing polarized Hodge structures with $\mathbf{h} = (r, r)$.
(ii) $\left(\mathfrak{sl}_{r+1}\mathbb{C}, \mathrm{A}^1, \omega_i, \frac{i}{r+1} - \frac{1}{2}\right)$, with $\widetilde{\mathfrak{g}}_{\mathbb{R}} = \mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(1, r)$ if $2i = r + 1$, and $\widetilde{\mathfrak{g}}_{\mathbb{R}} = \mathfrak{u}(1, r)$ otherwise.
(iii) $\left(\mathfrak{sl}_{a+b}\mathbb{C}, \mathrm{A}^a, \omega_1, \frac{1}{2} - \frac{b}{a+b}\right)$, with $\widetilde{\mathfrak{g}}_{\mathbb{R}} = \mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(a, a)$ if $a = b$, and $\widetilde{\mathfrak{g}}_{\mathbb{R}} = \mathfrak{u}(a, b)$ otherwise.
(iv) $\left(\mathfrak{so}_{m+2}\mathbb{C}, \mathrm{A}^1, \omega_r, 0\right)$, with $m + 2 \in \{2r, 2r + 1\}$ and $\widetilde{\mathfrak{g}}_{\mathbb{R}} = \mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(2, m)$.
(v) $(\mathfrak{so}_{2r}\mathbb{C}, \mathrm{A}^r, \omega_1, 0)$, with $\widetilde{\mathfrak{g}}_{\mathbb{R}} = \widetilde{\mathfrak{g}}_{\mathbb{R}} = \mathfrak{so}^*(2r)$.

Remark 3.4. Note that the two tuples $(\mathfrak{g}_{\mathbb{C}}, \mathrm{E}, \mu, c)$ and $(\mathfrak{g}_{\mathbb{C}}, \mathrm{E}, \mu^*, -c)$ determine the same Hodge representation (§§2.4 & 2.7).

Remark 3.5. One consequence of Remark 1.3 is that in any particular example – that is, the case of a fixed period domain $\mathcal{D}$ with specified Hodge numbers $\mathbf{h}$ – it suffices to identify the *irreducible* Hodge domains D with Hodge numbers $\mathbf{h}' \le \mathbf{h}$. For example, in §4, where we consider the case that $\mathbf{h} = \left(2, h^{1,1}, 2\right)$, it will suffice to consider the two cases that $h' = (1, h, 1)$ and $h' = (2, h, 2)$ with $h \le h^{1,1}$.

3.2. Horizontal Hodge domains

Theorem 3.1 identifies all the Hodge subdomains D of the period domain $\mathcal{D}_{\mathrm{h}}$. We are especially interested in the horizontal subdomains, which are necessarily Hermitian. These are the domains that satisfy the infinitesimal period relation (IPR, a.k.a. Griffiths' transversality). These distinguished subdomains may be identified as follows.

It is a consequence of the normalization in §2.3.4 that the Hodge subdomain $D \subset \mathcal{D}$ is horizontal if and only if the induced Hodge decomposition (2.9) is of the form

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\phi}^{1,-1} \oplus \mathfrak{g}_{\phi}^{0,0} \oplus \mathfrak{g}_{\phi}^{-1,1} \ ; \tag{3.6}$$

that is, $\mathfrak{g}_{\phi}^{\ell,-\ell} = 0$ for all $|\ell| \ge 2$, *cf.* Čap and Slovák (2009) and Robles (2014, §§2–3). This is a condition on the grading element:

$$\widetilde{\alpha}(\mathrm{E}_{\phi}) = 1,$$

where $\widetilde{\alpha}$ is the highest root. All such domains are necessarily Hermitian symmetric.

For the simple, complex Lie groups $\mathfrak{g}_{\mathbb{C}}$ the set of all such grading elements (see §2.5 for notation), the corresponding compact duals $\check{D}$, the real forms $\mathfrak{g}_{\mathbb{R}}$, and the maximal compact subalgebra $\mathfrak{k} \subset \mathfrak{g}_{\mathbb{R}}$ are listed in Table 3.1. Here $\mathrm{Gr}\left(a, \mathbb{C}^{a+b}\right)$ is the grassmannian of a–plane in $\mathbb{C}^{a+b}$, $\mathcal{Q}^d \subset \mathbb{P}^{d+1}$ is the quadric hypersurface, and $\mathrm{Gr}^Q\left(r, \mathbb{C}^{2r}\right)$ is the Lagrangian grassmannian of Q–isotropic r plane in $\mathbb{C}^{2r}$. The following proposition is immediate and well-known.

Proposition 3.7. *If $(\mathfrak{g}_{\mathbb{C}}, \mathrm{E}, \mu, c)$ is a tuple indexing a Hodge representation* (2.1) (*cf.* Theorem 3.1), *then the resulting Hodge domain D_{ϕ} is horizontal if and only if $(\mathfrak{g}_{\mathbb{C}}, \mathrm{E})$ is a sum of those pairs listed in Table 3.1.*

In general the Hodge domains $D \subset \check{D}$ are cut out by positivity conditions defined by a Hermitian form $\mathcal{H}$. For example, in the case of period domains, the compact dual essentially encodes the first Hodge–

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Table 3.1 Data underlying irreducible Hermitian symmetric Hodge domains

$\mathfrak{g}_{\mathbb{C}}$	E	$\check{D}=G_{\mathbb{C}}/P_{\mathrm{E}}$	$\mathfrak{g}_{\mathbb{R}}$	$\mathfrak{k}$
$\mathfrak{sl}(a+b,\ \mathbb{C})$	A^a	$\mathrm{Gr}(a,\ \mathbb{C}^{a+b})$	$\mathfrak{su}(a,\ b)$	$\mathfrak{s}(\mathfrak{u}(a)\oplus\mathfrak{u}(b))$
$\mathfrak{sl}(d+2,\ \mathbb{C})$	A^1	$\mathcal{Q}^d$	$\mathfrak{so}(2,\ d)$	$\mathfrak{s}(\mathfrak{o}(2)\oplus\mathfrak{o}(d))$
$\mathfrak{sp}(2r,\ \mathbb{C})$	A^r	$\mathrm{Gr}^Q(r,\mathbb{C}^{2r})$	$\mathfrak{sp}(2r,\ \mathbb{R})$	$\mathfrak{u}(r)$
$\mathfrak{so}(2r,\ \mathbb{C})$	A^r	Spinor variety	$\mathfrak{so}*(2r)$	$\mathfrak{u}(r)$
$\mathfrak{e}_6$	A^6	Cayley plane	EIII	$\mathfrak{so}(10)\oplus\mathbb{R}$
$\mathfrak{e}_7$	A^7	Freudenthal variety	EVII	$\mathfrak{e}_6\oplus\mathbb{R}$.

Riemann bilinear relation, and the second Hodge–Riemann bilinear relation is the positivity condition cutting out D. To illustrate this, we describe the Hodge domains for the first three rows of Table 3.1.

(1) In the case of $\check{D}=\mathrm{Gr}\left(a,\ \mathbb{C}^{a+b}\right)$, we note that $\mathbb{C}^{a+b}$ has an underlying real structure, and we fix a nondegenerate Hermitian form $\mathcal{H}$ on $\mathbb{C}^{a+b}$ of signature $(a,\ b)$. Then

$$D=\left\{E\in\mathrm{Gr}\left(a,\ \mathbb{C}^{a+b}\right)|\mathcal{H}|_E\text{is pos def}\right\}.$$

(2) In the case that $\check{D}=\mathcal{Q}^d=\mathrm{Gr}^Q\left(1,\ \mathbb{C}^{d+2}\right)$ we define a Hermitian form $\mathcal{H}$ on $\mathbb{C}^{d+2}$ by $\mathcal{H}(u,\ v)=-Q(u,\overline{v})$. Then

$$D=\left\{E\in\mathrm{Gr}^Q\left(1,\ \mathbb{C}^{d+2}\right)|\mathcal{H}|_E\text{ is pos def}\right\}.$$

(3) In the case that $\check{D}=\mathrm{Gr}^Q\left(r,\ \mathbb{C}^{2r}\right)$ we define a Hermitian form $\mathcal{H}$ on $\mathbb{C}^{2r}$ by $\mathcal{H}(u,\ v)=\mathbf{i}Q(u,\overline{v})$. Then

$$D=\left\{E\in\mathrm{Gr}^Q\left(r,\ \mathbb{C}^{2r}\right)|\mathcal{H}|_E\text{is pos def}\right\}.$$

4. Example: Hodge domains for level 2 Hodge structures

The purpose of this section is to illustrate the application of the strategy outlined in §3 in the case that $\mathcal{D}=\mathcal{D}_{\mathrm{h}}$ is the period domain parameterizing Q–polarized, (effective) weight two Hodge structures on $V_{\mathbb{R}}$ with Hodge numbers

$$\mathbf{h}=\left(h^{2,0},\ h^{1,1},\ h^{0,2}\right)=\left(2,\ h^{1,1},2\right).$$

Equivalently, $\varphi\in\mathcal{D}$ parameterizes Hodge decompositions

$$V_{\mathbb{C}}=V^{2,0}\oplus V^{1,1}\oplus V^{0,2},$$

with

$$\dim_{\mathbb{C}}\ V^{2,0}=2=\dim_{\mathbb{C}}\ V^{0,2}.$$

(We assume throughout that $h^{1,1}=\dim_{\mathbb{C}}\ V^{1,1}\neq 0$.) Geometrically such Hodge structures arise when studying smooth projective surfaces with $p_g=2$.

We have

$$\mathcal{G}_{\mathbb{R}}=\mathrm{Aut}(V_{\mathbb{R}},Q)=\mathrm{O}\left(h^{1,1},4\right).$$

As discussed in §1.3 it suffices to identify the irreducible Hodge representations (2.1) with either $\mathbf{h}_\phi=(1,\ h,\ 1)$ or $\mathbf{h}_\phi=(2,\ h,\ 2)$, and $h\leq h^{1,1}$. (Each such Hodge representation corresponds to a Hodge

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subdomain $D=G_{\mathbb{R}}\cdot\phi$ of the period domain $\mathcal{D}=\mathcal{D}_{\mathrm{h}_\phi}$ parameterizing Q–polarized Hodge structures on $V_{\mathbb{R}}$ with Hodge numbers $\mathbf{h}_\phi$.) The analysis decomposes into three parts:

(A) We begin with the simplifying assumptions that $\mathfrak{g}_{\mathbb{C}}$ is simple and that D is horizontal. This has the strong computational advantage that we may take the grading element E to be as listed in Table 3.1. The resulting domains are enumerated in Theorems 4.1 and 4.3.

(B) Continuing to assume that $\mathfrak{g}_{\mathbb{C}}$ is simple, we turn to the case that horizontality fails; the domains are enumerated in Theorem 4.4.

(C) Finally we consider in Theorem 4.6 the case that $\mathfrak{g}_{\mathbb{C}}$ is semisimple (but not simple).

Together Theorems 4.1, 4.3, 4.4 and 4.6 give a complete list of the irreducible Hodge representations (2.1) with $\mathbf{h}_\phi\le\mathbf{h}=(2,\ h^{1,1},2)$.

Theorem 4.1. *The irreducible Hodge representations* (2.1) *with* $\mathfrak{g}_{\mathbb{C}}$ *simple,* $\mathbf{h}_\phi=(1,\ h,\ 1)$ *and horizontal Hodge domain* $D\ \subset\ \mathcal{D}_{(1,h,1)}$ *are given by the following tuples* $(\mathfrak{g}_{\mathbb{C}},\ \mathrm{E},\ \mu,\ c)$:

(i) *Period domains*: $\left(\mathfrak{so}\ (h+2,\ \mathbb{C}),\ \mathbf{A}^1,\ \omega_1,0\right)$, *with* $\mathbf{h}_\phi=(1,\ h,\ 1)$.

(ii) *Complex balls*: *both tuples*

$$\left(\mathfrak{sl}(1+r,\ \mathbb{C}),\ \mathbf{A}^1,\ \omega_r,\ -1/(r+1)\right)\ \textit{and}\ (\mathfrak{sl}(1+r,\ \mathbb{C}),\ \mathbf{A}^r,\ \omega_1,\ -1/(r+1))$$

yield Hodge representations with $\mathbf{h}_\phi=(1,2r,\ 1)$.

Remark 4.2 (Geometric realizations). Pearlstein and Zhang (2019) have exhibited geometric realizations of $G_\varphi=G_1\times G_2$ with G_i one of $\mathrm{SO}(2,\ h_i)$ or $\mathrm{U}(1,\ r_i)$, corresponding to the two cases/factors of Theorem 4.1.

Theorem 4.3. *The irreducible Hodge representations* (2.1) *with*$\mathfrak{g}_{\mathbb{C}}$*simple,* $\mathbf{h}_\phi=(2,\ h,\ 2)$*and horizontal Hodge domain* $D\ \subset\ \mathcal{D}_{(2,h,2)}$ *are all grassmannian Hodge domains* (corresponding to the first row of Table 3.1), *and are given by the following tuples* $(\mathfrak{sl}(a+b,\ \mathbb{C}),\ \mathrm{A}^a,\ \mu,c)$*:*

(i) *The tuples*

$$\left(\mathfrak{sl}(3,\ \mathbb{C}),\ \mathbf{A}^1,\ \omega_2,2/3\right)\ \textit{and}\ \left(\mathfrak{sl}(3,\ \mathbb{C}),\ \mathbf{A}^1,\ \omega_1,\ -2/3\right)$$

yield Hodge representations with $\mathbf{h}_\phi=(2,2,2)$.

(ii) *The tuple* $\left(\mathfrak{sl}(r+1,\ \mathbb{C}),\ \mathbf{A}^2,\ \omega_1,2/(r+1)\right)$ *yields a Hodge representation with* $\mathbf{h}_\phi=(2,2r-2,2)$.

Theorem 4.4. *The irreducible Hodge representations* (2.1) *with* $\mathfrak{g}_{\mathbb{C}}$ *simple and* $\mathbf{h}_\phi\le\mathbf{h}=\left(2,\ h^{1,1},2\right)$, *for which the Hodge domain*$D\ \subset\ \mathcal{D}_{\mathbf{h}}$ *is* not *horizontal are given by the following tuples* $(\mathfrak{g}_{\mathbb{C}},\ \mathrm{E},\ \mu,\ c)$*:*

(i) *Period domains*: $\left(\mathfrak{so}\ (h+4,\ \mathbb{C}),\ \mathrm{A}^2,\ \omega_1,0\right)$, *with* $\mathbf{h}_\phi=(2,\ h,\ 2)$.

(ii) *Special Linear contact domains*: $\left(\mathfrak{sl}(r+1,\ \mathbb{C}),\ \mathrm{A}^1+\mathrm{A}^r,\ \omega_1,0\right)$, *with* $\mathbf{h}_\phi=(2,2r-2,2)$.

(iii) *Special Linear contact domains*: $\left(\mathfrak{sl}(4,\ \mathbb{C}),\ \mathrm{A}^1+\mathrm{A}^3,\ \omega_2,0\right)$, *with* $\mathbf{h}_\phi=(2,2,2)$.

(iv) *Spinor contact domains*: $\left(\mathfrak{so}(5,\ \mathbb{C}),\ \mathrm{A}^2,\ \omega_2,0\right)$*and*$\left(\mathfrak{so}\ (7,\ \mathbb{C}),\ \mathrm{A}^2,\ \omega_3,0\right)$, *both with* $\mathbf{h}_\phi=(2,4,2)$. (The first is quaternionic, the second is real.)

(v) *Symplectic contact domains*: $\left(\mathfrak{sp}(2\mathrm{r},\ \mathbb{C}),\ \mathrm{A}^1,\ \omega_1,0\right)$, *with* $\mathbf{h}_\phi=(2,4(r-1),\ 2)$.

(vi) *Exceptional contact domains*: $\left(\mathfrak{g}_2,\ \mathrm{A}^2,\ \omega_1,0\right)$*with* $\mathbf{h}_\phi=(2,3,2)$.

See Han and Robles (2020, §A.8) for further discussion of the domains D appearing in Theorem 4.4 as homogeneous spaces.

Remark 4.5. The Spinor contact domain given by the tuple $\left(\mathfrak{so}\ (5,\ \mathbb{C}),\ \mathrm{A}^2,\ \omega_2,0\right)$ in Theorem 4.3 (iv) is a special case of Theorem 4.3(v) under the isomorphism $\mathfrak{so}(5,\ \mathbb{C})\simeq\mathfrak{sp}(4,\ \mathbb{C})$.

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Theorem 4.6. *The irreducible Hodge representations* (2.1) *with* $\mathbf{h}_\phi \le \mathbf{h} = \left(2,\ h^{1,1},2\right)$ *and* $\mathfrak{g}_\mathbb{C}$ *semi-simple (but not simple) are given by:*

(i) $\mathfrak{g}_\mathbb{C} = \mathfrak{sl}_2\mathbb{C} \oplus \mathfrak{sl}_2\mathbb{C}$ *acting on* $U = \mathbb{C}^2 \otimes \mathbb{C}^2$ *with* $\mathrm{E} = \mathrm{A}^1 + \mathrm{A}^2$; *and*
(ii) $\mathfrak{g}_\mathbb{C} = \mathfrak{sl}_2\mathbb{C} \oplus \mathfrak{sl}_4\mathbb{C}$ *acting on* $U = \mathbb{C}^2 \otimes \mathbb{C}^4$ *with* $\mathrm{E} = \mathrm{A}^1 + \mathrm{A}^3$.

Each of these Hodge representations is real (implying). The Hodge numbers are and $\mathbf{h} = (1,2,1)$ *and* $\mathbf{h} = (2,4,2)$ $c = 0$ *respectively.*

The theorems are proved in the Appendix (Han & Robles, 2020, §A).

5. Hodge representations of Calabi–Yau type

We say that a Hodge representation (2.1) is of *Calabi–Yau type* (or *CY-type* if the first Hodge number $h_\phi^{n,0} = 1$. The irreducible CY-Hodge representations with $\mathfrak{g}_\mathbb{R}$ semisimple are classified in (Robles, 2014, Proposition 6.1). They are precisely the tuples $(\mathfrak{g}_\mathbb{C},\ \mathrm{E},\ \mu,\ c)$ of Theorem 3.1 with $c = 0$ (§2.7), and such that:

(a) $\mu^i = 0$ whenever $\alpha_i(\mathrm{E}) = 0$, where α_i are the simple roots of (the semisimple) $\mathfrak{g}_\mathbb{C}$ and the $0 \le \mu^i \in \mathbb{Z}$ are the coefficients of $\mu = \mu^i \omega_i$ as a linear combination of the fundamental weights ω_i;
(b) either the representation is real (equivalently, $U = U^*$ and $\mu(\mathrm{T}_\phi)$ is an even integer), or
(c) $\mu(\mathrm{E}_\phi) \ne \mu^*(\mathrm{E}_\phi)$, and U is necessarily complex.

Remark 5.1. The condition (a) above is equivalent to the statement that $\dim U_{\mu(\mathrm{E})} = 1$; equivalently, $U_{\mu(\mathrm{E})}$ is a highest weight line.

Theorem 5.2. *An irreducible Hodge representation* (2.1) *is of CY-type if and only if the corresponding tuple* $(\mathfrak{g}_\mathbb{C},\ \mathrm{E},\ \mu,\ c)$ *of Theorem 3.1 has the properties*:

(i) *The condition* (a) *above holds.*
(ii) If U_μ *is not real (with respect to the semisimple* $\mathfrak{g}_\mathbb{R}$*), then* $\mu(\mathrm{E}) + c > \mu^*(\mathrm{E}) - c$.

Proof. It is straightforward to deduce the theorem from the proof of Robles (2014, Proposition 6.1) and the discussion of §2.7. Details are left to the reader. □

Example 5.3. The (rational) Hodge groups of K3 type (CY 2-fold type) were determined by Zarhin (1983). The corresponding (real) Hodge representations (2.1) are those with Hodge numbers $\mathbf{h}_\phi = (1,\ h,\ 1)$. The list of all associated tuples (Theorem 3.1) is.

(i) $\left(\mathfrak{so}_{h+2}\mathbb{C},\ \mathrm{A}^1,\ \omega_1, 0\right)$, with $\mathrm{h}_\phi = (1,\ h,\ 1)$ and $h \ge 3$.[3]
(ii) $\left(\mathfrak{sl}_2\mathbb{C} \oplus \mathfrak{sl}_2\mathbb{C},\ \mathrm{A}^1 + \mathrm{A}^2,\ \omega_1 + \omega_2, 0\right)$, with $\mathbf{h}_\phi = (1,2,1)$.[4]
(iii) $\left(\mathfrak{sl}_{r+1}\mathbb{C},\ \mathrm{A}^1,\ \omega_1,\ \frac{1}{r+1}\right)$ with $\mathbf{h}_\phi = (1, 2r,\ 1)$ and $r \ge 2$.
(iv) $\left(\mathfrak{sl}_2\mathbb{C},\ \mathrm{A}^1, 2\omega_1, 0\right)$ with $\mathbf{h}_\phi = (1,1,1)$.
(v) $\left(\mathfrak{sl}_4\mathbb{C},\ \mathrm{A}^2,\ \omega_2, 0\right)$, with $\mathbf{h}_\phi = (1,4,1)$.

Example 5.4. The set of all Hodge representations (2.1) of CY 3-fold type (Hodge numbers $\mathbf{h}_\phi = (1,\ h,\ h,\ 1)$) is enumerated in Han (2021). In the case that D_ϕ is *horizontal* (and therefore Hermitian) these are of particular interest (Friedman & Laza, 2013; 2014; Gross, 1994). In fact, this completes the classification begun in Friedman and Laza (2013, §2.3); our (v), (vii) and (x) are omitted from

[3] The associated domain D_ϕ is the period $\mathcal{D}$ parameterizing polarized Hodge structures of K3-type with $\mathrm{h}_\phi = (1,\ h,\ 1)$, *cf.* Example 5.2.

[4] Recall that $\mathfrak{go}_4\mathbb{C} = \mathfrak{gl}_2\mathbb{C} \oplus \mathfrak{gl}_2\mathbb{C}$ is semisimple. Here D_ϕ is again the period domain.

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Friedman and Laza (2013, Corollary 2.29). The corresponding (horizontal) tuples $(\mathfrak{g}_{\mathbb{C}},\ \mathrm{E},\ \mu,\ c)$ of Theorem 3.1 are.

(i) $(\mathfrak{sl}_2\mathbb{C},\ \mathrm{A}^1, 3\omega_1, 0)$ with $\mathbf{h}_\phi = (1,1,1,1)$.
(ii) $(\mathfrak{sl}_2\mathbb{C},\ \mathrm{A}^1,\ \omega_1, 0)^{\oplus 3}$ with $\mathbf{h}_\phi = (1,3,3,1)$.
(iii) $(\mathfrak{sl}_6\mathbb{C},\ \mathrm{A}^3,\ \omega_3, 0)$ with $\mathbf{h}_\phi = (1,9,9,1)$.
(iv) $\left(\mathfrak{sl}_{r+1}\mathbb{C},\ \mathrm{A}^1,\ \omega_1,\ \frac{3}{2} - \frac{r}{r+1}\right)$ with $\mathbf{h}_\phi = (1,\ r,\ r,\ 1)$.
(v) $\left(\mathfrak{sl}_{r+1}\mathbb{C},\ \mathrm{A}^1, 2\omega_1,\ \frac{3}{2} - \frac{2r}{r+1}\right)$ with $\mathbf{h}_\phi = (1,\ h,\ h,\ 1)$ and $h+1 = \frac{1}{2}(r+1)(r+2)$.
(vi) $\left(\mathfrak{sl}_{r+1}\mathbb{C},\ \mathrm{A}^2,\ \omega_2,\ \frac{3}{2} - \frac{2(r-1)}{r+1}\right)$ with $\mathbf{h}_\phi = (1,\ h,\ h,\ 1)$ and $h+1 = \frac{1}{2}r(r+1)$.
(vii) $(\mathfrak{sl}_{r+1}\mathbb{C},\ \mathrm{A}^1,\ \omega_1) \oplus (\mathfrak{sl}_{r'+1}\mathbb{C},\ \mathrm{A}^1,\ \omega_1)$ and $c = \frac{3}{2} - \frac{r}{r+1} - \frac{r'}{r+1}$, with $\mathbf{h}_\phi = (1,\ h,\ h,\ 1)$ and $h = r + r' + rr'$.
(viii) $(\mathfrak{sl}_2\mathbb{C},\ \mathrm{A}^1,\ \omega_1, 0) \oplus (\mathfrak{g}'_{\mathbb{C}},\ \mathrm{E}',\ \mu',\ c')$ with $\mathbf{h}_\phi = (1,\ h'+1,\ h'+1, 1)$, where $(\mathfrak{g}'_{\mathbb{C}},\ \mathrm{E}',\ \mu',\ c')$ is any tuple of Example 5.3 with Hodge numbers $\mathbf{h}' = (1,\ h',\ 1)$.
(ix) $(\mathfrak{sp}_6\mathbb{C},\ \mathrm{A}^3,\ \omega_3, 0)$ with $\mathbf{h}_\phi = (1,6,6,1)$.
(x) $(\mathfrak{so}_m\mathbb{C},\ \mathrm{A}^1,\ \omega_1, 1/2)$ with $\mathbf{h}_\phi = (1,\ m-1,\ m-1, 1)$.
(xi) $(\mathfrak{so}_{10}\mathbb{C},\ \mathrm{A}^5,\ \omega_5, 1/4)$ with $\mathbf{h}_\phi = (1,15,15,1)$.
(xii) $(\mathfrak{so}_{12}\mathbb{C},\ \mathrm{A}^6,\ \omega_6, 0)$ with $\mathbf{h}_\phi = (1,15,15,1)$.
(xiii) $(\mathfrak{e}_6,\ \mathrm{A}^6,\ \omega_6, 1/6)$ with $\mathbf{h}_\phi = (1,26,26,1)$.
(xiv) $(\mathfrak{e}_7,\ \mathrm{A}^7,\ \omega_7, 0)$ with $\mathbf{h}_\phi = (1,27,27,1)$.

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References

Allcock, D., Carlson, J. A., & Toledo, D. (2002). The complex hyperbolic geometry of the moduli space of cubic surfaces. *Journal of Algebraic Geometry*, **11**, 659–724.

Allcock, D., Carlson, J. A., & Toledo, D. (2011). The moduli space of cubic threefolds as a ball quotient. *Memoirs of the American Mathematical Society*, **209**, xii+70.

Borcea, C. (1997). K3 surfaces with involution and mirror pairs of Calabi-Yau manifolds. In *Mirror symmetry, II*, AMS/IP Studies Advanced Mathematics, **1** (pp. 717–743). American Mathematical Society.

Čap, A., & Slovák, J. (2009). *Parabolic geometries. I. Background and general theory*, Mathematical Surveys and Monographs, **154**. American Mathematical Society.

Deligne, P. (1979). Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques. In *Automorphic forms, representations and L-functions (Proceedings Symposia in Pure Mathematics, Oregon State University, Corvallis, Ore., 1977), Part 2*, Proceedings of Symposia in Pure Mathematics, **XXXIII** (pp. 247–289). American Mathematical Society.

Franciosi, M., Pardini, R., & Rollenske, S. (2015). Log-canonical pairs and Gorenstein stable surfaces with $K_X^2 = 1$. *Compositio Mathematica*, **151**, 1529–1542.

Franciosi, M., Pardini, R., & Rollenske, S. (2017). Gorenstein stable surfaces with $K_X^2 = 1$ and $p_g > 0$. *Mathematische Nachrichten*, **290**, 794–814. arXiv:1511.03238.

Friedman, R., & Laza, R. (2013). Semialgebraic horizontal subvarieties of Calabi-Yau type. *Duke Mathematical Journal*, **162**, 2077–2148.

Friedman, R., & Laza, R. (2014). On some Hermitian variations of Hodge structure of Calabi-Yau type with real multiplication. *The Michigan Mathematical Journal*, **63**, 83–99.

Garbagnati, A., & vanGeemen, B. (2010). Examples of Calabi-Yau threefolds parametrised by Shimura varieties. *Rendiconti del Seminario Matematico Università e Politecnico di Torino*, **68**, 271–287.

Green, M., Griffiths, P., & Kerr, M. (2012). *Mumford-Tate groups and domains: their geometry and arithmetic*, Annals of Mathematics Studies, **183**. Princeton University Press.

Gross, B. H. (1994). A remark on tube domains. *Mathematical Research Letters*, **1**, 1–9.

Han, X. (2021). *Hodge representations of CY 3-fold type.* (Bachelor's thesis). Duke University, Durham.

Han, X., & Robles, C. (2020). Hodge representations: Appendix. Manuscript submitted for publication.

Horikawa, E. (1978). Algebraic surfaces of general type with small c_1^2. III. *Inventiones Mathematicae*, **47**, 209–248.

Horikawa, E. (1978/79). Algebraic surfaces of general type with small c_1^2. IV. *Inventiones Mathematicae*, **50**, 103–128.

Kerr, M., & Pearlstein, G. (2016). Boundary components of Mumford-Tate domains. *Duke Mathematical Journal*, **165**, 661–721.

Kondo, S. (2000). A complex hyperbolic structure for the moduli space of curves of genus three. *Journal für die Reine und Angewandte Mathematik*, **525**, 219–232.

Laza, R. (2016). Perspectives on the construction and compactification of moduli spaces. In *Compactifying moduli spaces*, Advanced Courses in Mathematics—CRM Barcelona (pp. 1–39). Birkhäuser/Springer.

Laza, R., Pearlstein, G., & Zhang, Z. (2018). On the moduli space of pairs consisting of a cubic threefold and a hyperplane. *Advances in Mathematics*, **340**, 684–722.

Milne, J. S. (2005). Introduction to Shimura varieties. In *Harmonic analysis, the trace formula, and Shimura varieties*, Clay Mathematics Proceedings, **4** (pp. 265–378). American Mathematical Society.

Patrikis, S. (2016). Mumford-Tate groups of polarizable Hodge structures. *Proceedings of the American Mathematical Society*, **144**, 3717–3729.

Pearlstein, G., & Zhang, Z. (2019). A generic global Torelli theorem for certain Horikawa surfaces. *Algebraic Geometry*, **6**, 132–147.

Ribet, K. A. (1983). Hodge classes on certain types of abelian varieties. *American Journal of Mathematics*, **105**, 523–538.

Robles, C. (2014). Schubert varieties as variations of Hodge structure. *Selecta Mathematica (NS)*, **20**, 719–768.

Robles, C. (2016). Classification of horizontal SL(2)s. *Compositio Mathematica*, **152**, 918–954.

Rohde, J. C. (2009). *Cyclic coverings, Calabi-Yau manifolds and complex multiplication*, Lecture Notes in Mathematics, **1975**. Springer-Verlag.

Sheng, M., & Zuo, K. (2010). Polarized variation of Hodge structures of Calabi-Yau type and char- acteristic subvarieties over bounded symmetric domains. *Mathematische Annalen*, **348**, 211–236.

Tankeev, S. G. (1981). Algebraic cycles of simple 5-dimensional abelian varieties. *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, **45**, 793–823, 928.

Tankeev, S. G. (1982). Cycles on simple abelian varieties of prime dimension. *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, **46**, 155–170, 192.

Voisin, C. (1983). Miroirs et involutions sur les surfaces K3. *Astérisque*, **218**, 273–323.

Zarhin, Y. G. (1983). Hodge groups of K3 surfaces. *Journal für die Reine und Angewandte Mathematik*, **341**, 193–220.

Zhang, Z. (2014). *On geometric and motivic realizations of variations of Hodge structure over Hermitian symmetric domains.* (PhD thesis). State University of New York at Stony Brook, New York.

Zhang, Z. (2016). A realization for a $\mathbb{Q}$-Hermitian variation of Hodge structure of Calabi-Yau type with real multiplication. *Mathematical Research Letters*, **22**, 967–982.

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Peer Reviews

Reviewing editor: Dr. Adrian Clingher[1,2]

[1]University of Missouri at Saint Louis, Mathematics and Computer Science, One University Blvd, St. Louis, Missouri, United States, 63121
[2]UMSL

This article has been accepted because it is deemed to be scientifically sound, has the correct controls, has appropriate methodology and is statistically valid, and met required revisions.

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Review 1: Hodge Representations

Reviewer: Prof. Matt Kerr

Washington University in Saint Louis, Mathematics and Statistics, 1 Brookings Drive, Campus Box 1146, Saint Louis, Missouri, United States, 63130-4899
Reviewer declares none.

Date of review: 10 August 2020

Conflict of interest statement. Reviewer declares none.

Comments to the Author: Given a classifying space for polarized Hodge structures with given Hodge numbers **h**, its Mumford-Tate subdomains are the loci on which the Hodge (or Mumford-Tate) group drops. From the standpoint of algebraic geometry, having a classification in hand is valuable given the control exerted by the Hodge group on monodromy, algebraic cycles, degenerations, and arithmetic. In particular, the horizontal subdomains and the compactifications of their quotients have played a central role in recent studies of completions of period maps, and of families of algebraic varieties parametrized by automorphic data. The classification of all real M-T subdomain types provided herein for **h**=(2,n,2), and the authors' completion of the Friedman-Laza classification of (real types of) horizontal subdomains for Hodge numbers **h**=(1,n,n,1), are thus an important addition to the literature: we may think of each of the Hodge representations they enumerate (more precisely, each of its **Q**-forms) as an algebro-geometric realization problem.

Score Card

Presentation

5.0 /5

Is the article written in clear and proper English? (30%)	5/5
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Does the title suitably represent the article? (25%)	5/5
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Is the conclusion consistent with the results and discussion? (40%)	5/5
Are the limitations of the experiment as well as the contributions of the experiment clearly outlined? (20%)	5/5

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Review 2: Hodge Representations

Reviewer: Zheng Zhang

Colorado Mesa University John U Tomlinson Library, Mathematics

Date of review: 22 August 2020

Comments to the Author: Mumford-Tate/Hodge groups are the fundamental symmetry groups in Hodge theory. Mumford-Tate subdomains are defined via Mumford-Tate/Hodge groups, and are parameter spaces for Hodge structures with extra symmetry. Green, Griffiths and Kerr developed a general algorithm for classifying Mumford-Tate subdomains contained in a given period domain using Hodge representations. The paper under review gives a nice summary of Green-Griffiths-Kerr's algorithm, and carries it out in full detail (see the appendix) in two cases: (1) weight two Hodge structures with h^{2,0}=2 and (2) weight three Calabi-Yau type Hodge structures. Classically, Hodge theory has been successfully applied to study the moduli spaces of principally polarized abelian varieties and K3 surfaces. One important reason why Hodge theory is a powerful tool in the study of these moduli spaces is that the corresponding period domains are Hermitian symmetric. Weight two Hodge structures with h^{2,0}=2 and weight three Hodge structures of Calabi-Yau type are the simplest cases where the corresponding period domains are not Hermitian symmetric. The paper under review (and some upcoming paper) lists all the horizontal Mumford-Tate subdomains (which are Hermitian symmetric), and can be thought of as the first step for finding non-classical applications of Hodge theory in the study of the moduli spaces of algebraic surfaces with geometric genus 2 and Calabi-Yau threefolds. As far as I can see, the paper is correct and quite nice. I recommend it for publication in Experimental Results.

Score Card

Presentation

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Is the article written in clear and proper English? (30%)	5/5
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Context

4.8 /5

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Is the objective of the experiment clearly defined? (25%)	5/5

Analysis

4.8 /5

Does the discussion adequately interpret the results presented? (40%)	5/5
Is the conclusion consistent with the results and discussion? (40%)	5/5
Are the limitations of the experiment as well as the contributions of the experiment clearly outlined? (20%)	4/5

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