A Catalogue of Periodic Orbits in the Elliptic Restricted 3-Body Problem

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Abstract. Results of families of periodic orbits in the elliptic restricted problem are shown for some specific resonances. They are calculated for all mass ratios $0 \leq \mu \leq 1.0$ of the primary bodies and for all values of the eccentricity of the orbit of the primaries $e \leq 1.0$. The grid size is of 0.01 for both parameters. The classification of the stability is undertaken according to the usual one and the results are compared with the extensive studies by Contopoulos (1986) in different galactical models.

1. Introduction

Since the fundamental work by Poincaré (1891) the study of periodic orbits (PO) is one of the most important topics of dynamical systems. The determination of their stability is fundamental to understand the structure of the phase space. We now know that this space is separated into 2 more or less well defined regions: the zone of regular (quasiperiodic) motion and the zone of stochasticity, where two orbits which have almost the same initial conditions for an instant t_0 depart from each other hyperbolically. It is evident that the determination of POs (and their stability) plays a dominant role especially for the knowledge of the long term behaviour of any dynamical system, because stable POs (in contrary to unstable ones) are always imbedded in regular zones (which may be in fact very small).

For problems in Celestial Mechanics the restricted 3-body problem is commonly used: a massless body ($m_3 = 0$) moves in the gravitational field of two primaries having circular orbits. The motion of asteroids, comets, satellites and even planetesimals in the early Solar system can be treated within this framework (since a circular orbit is a good approximation for Jupiter's motion). But the elliptic restricted 3-body problem is more realistic, and sometimes the only simple physical model to understand the dynamics of celestial bodies (e.g. the origin of the Kirkwood gaps, Dvorak,1991 this volume). Therefore we have taken this model and have calculated stability diagramms of POs for some resonances.

2. The Model and the Stability of Periodic Orbits

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \frac{r}{p}\left(x - m_1\frac{x - x_1}{r_1^3} - m_2\frac{x - x_2}{r_2^3}\right) \\ \ddot{y} + 2\dot{x} &= \frac{r}{p}\left(y - m_1\frac{y}{r_1^3} - m_2\frac{y}{r_2^3}\right) \end{aligned} \tag{1}$$

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are the equations of motion in rotating/pulstating coordinates. Here m_1 and m_2 are the primaries' masses, p the orbital parameter, r_1 and r_2 the distances between the third massless body m_3 from m_1 and m_2.

Due to the ellipticity of the primaries' orbit 5 initial conditions determine a specific orbit (x, $\dot{x}$, y, $\dot{y}$ and v, the true anomaly of the primaries). Consequently the dynamical system can be completely described in a 5 dimensional phase space. It is important to keep in mind, that in general the condition for an orbit to be a periodic one is simple, that it has to pass through a point in phase space 2 times (and consequently an infinite number of times). A more restrictive definition is given by Broucke (1969): *An orbit is periodic if it has two perpendicular crossings with the syzygy-axis, and if the crossing are at moments when the primaries are at an apse.*

The stability of such a PO is calculated according to Déprit and Price (1965) and Hénon and Guyot (1970). The linear stability is described by the eigenvalues of the fundamental matrix which is the solution of the variational equations. The eigenvalues are the roots of the characteristic equation of this matrix

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_1\lambda + 1 = 0 \tag{2}$$

and when one uses the stability parameter b from Contopoulos (1986)

$$b_{1/2} = \frac{1}{2}(a_1 \pm \sqrt{a_1^2 - 4(a_2 - 2)}) \tag{3}$$

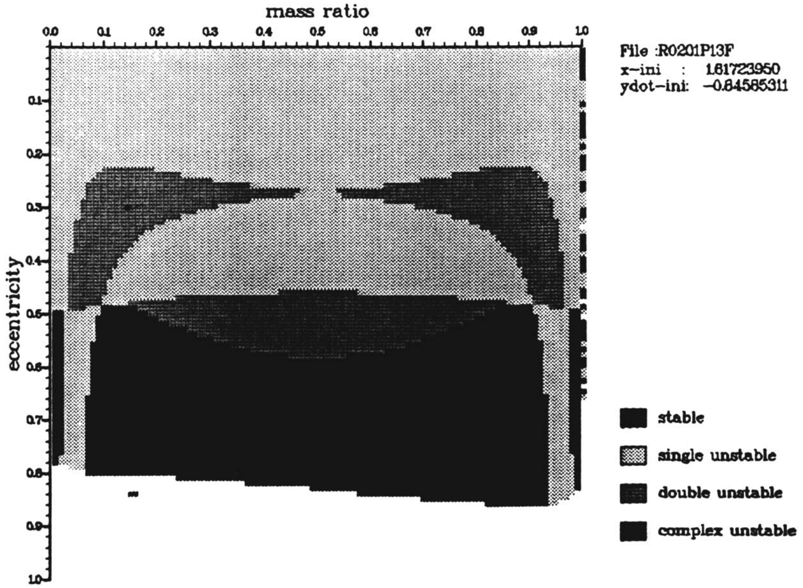
As a consequence there exist 4 different stability characteristics of POs:
stable (S) if $|b_1| < 2$ and $|b_2| < 2$
simple unstable (U) if $|b_1| > 2$ or $|b_2| > 2$
double unstable (DU) if $|b_1| > 2$ and $|b_2| > 2$
complex unstable (CU) if b_1 and b_2 are complex

3. The Results

We used a Bulirsch-Stoer intergration method to solve the equations of motions and started always perpendicular to the syzygy axe when the primaries are at an periapse or an apoapse. We present only 2 (out of 20) stability diagramms: the 2/1 Periapse and the 3/1 Apoapse. (Fig. 1), the others will be published elsewhere. The upper graph (Fig.1) shows the symmetry of the 2/1 resonant POs, which can be explained by the fact of mirror symmetry of the initial conditions around the vertical line $\mu = 0.5$. We recognize that stable orbits are very rare, they exist only for very small (or big) mass ratios with relatively high eccentricities. Most of the orbits are single unstable up to the eccentricity 0.5; whereas most of them are complex unstable for $e \geq 0.5$. White points mean that it was impossible to find any PO, although a sophisticated technique of finding them starting from the neighbouring POs was developed.

Characteristic is the big hole of DU in the region in then middle, which separates the CU and SU area. Vice versa for moderate eccentricities the big SU sea is more or less separated by two big smoothly shaped islands of DU. This phenomena is visible in different diagramms which we calculated. In the lower graph (Fig.1) we

Stability Diagram for the 2: 1 Resonance (Periapsis)



Stability Diagram for the 3: 1 Resonance (Apoapsis)

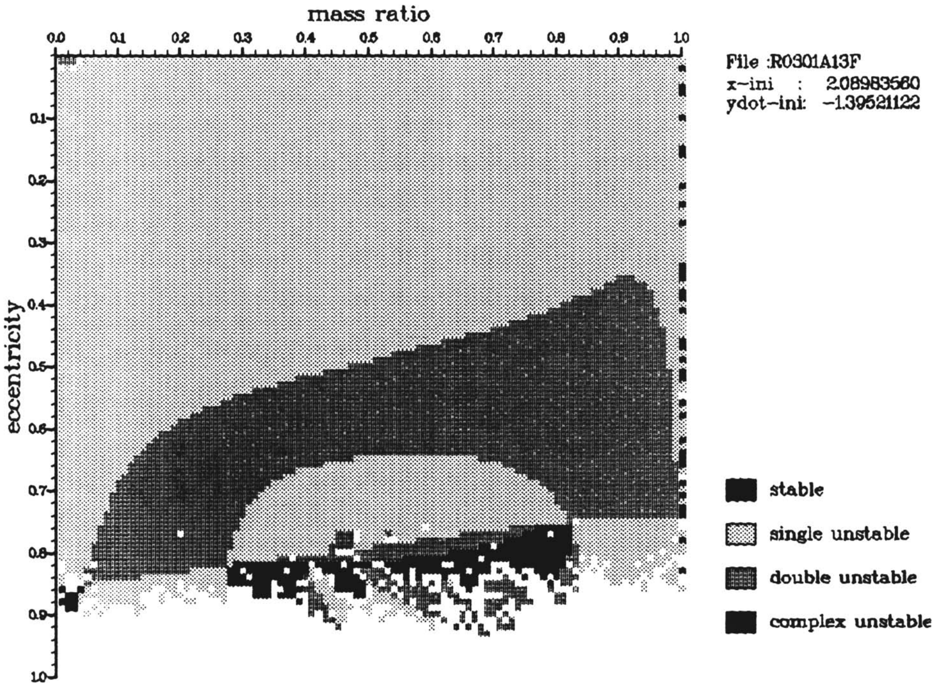


Fig. 1. Stability diagramm for the 2:1 resonant Periodic Orbits (upper graph). . Starting point is always when the primaries are in the periapse position. Stability diagramm for the 3:1 resonant Periodic Orbits (lower graph). Starting point is always when the primaries are in the apoapse position.

see the diagram of the stability of the 3/1 resonant POs starting in the Apoapse. According to this resonance the picture is asymmetric. We see only few stable POs, but that time most of them for rather high eccentricities of the primaries. Again a large sea of SU is divided by the island of DU, which form the characteristic hole. Many of the diagramms have a very similiar appearance (like the 5/3 and the 5/2, but some of them are completely different.

Table 1. shows the relation between the different stability modes for the 2 stability diagramms of fig.1. Note that stable orbits are very seldom and that the U area is by far the largest one! Checking the change of the stability mode on the border lines one can see, that there occur no direct change from DU to S and U to CU and vice versa. This picture corresponds to work of Contopoulos (1986) where he studied also a dynamical system of $2\frac{1}{2}$ degrees of freedom for the better understanding of galactic dynamics.

Table 1: Stability of Orbits for some special Resonances

Resonance	initial x	No. of POs	S	U	DU	CU
2/1	1.617	8248	1.6%	50.2%	17.2%	30.9%
3/1	2.090	8591	0.1%	69.7%	27.2%	3.0%

4. Conclusions

This catalogue of POs in the elliptic restricted problem is only a first step in understanding the whole complexity of POs in this model. Some features seem to arise in many of the stability diagrams (like the strange hole in the sea of SU orbits) but unfortunately we are not able to give any theoretical explanation or physical meaning of them up to now. Maybe some general rules can be derived when we look up the diagrams order by order concerning their resonant character, but this has to be left for future investigations.

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