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EMBEDDING PARTIAL GRAPH DESIGNS, BLOCK DESIGNS, AND TRIPLE SYSTEMS WITH $\lambda > 1$

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ABSTRACT. A general embedding technique for graph designs and block designs is developed, which transforms the embedding problem for partial designs with $\lambda > 1$ into the embedding problem for partial designs with $\lambda = 1$. Given an embedding technique for n-element partial block designs with $\lambda = 1$ into block designs with $f(n)$ elements, the transformation produces a technique which embeds an n-element partial design with $\lambda > 1$ and block size k into a design with at most $f(3^{k-1}\lambda n^2)$ elements. For graph designs and block designs with $k > 3$, a finite embedding method results. For triple systems, a quadratic embedding technique is obtained immediately; the best previous result here was exponential. Finally, for partial triple systems, Mendelsohn triple systems, and directed triple systems, these quadratic embeddings are improved to linear using a colouring technique.

1. **Introduction**. A *(directed) graph design* $B[G, \lambda; v]$ for a (directed) graph G is a pair (V, B); V is a v-set of elements and B is a collection of graphs isomorphic to G with vertices chosen from V. Each (directed) pair from V appears in precisely λ of the graphs in B. A *balanced incomplete block design* $B[k, \lambda; v]$ is a graph design whose graph is the complete k-vertex graph. A *triple system* is a block design with $k = 3$. A *directed triple system* is a directed graph design with G being the transitive tournament of order 3. A *Mendelsohn triple system* is a directed graph design with G being the cyclic tournament of order 3. A *partial* design relaxes the condition on pairs, so that (directed) pairs appear in *at most* λ graphs, or blocks.

Embedding techniques for partial block designs have been the subject of a large amount of research in combinatorial design theory (see, for example, [5,10]). In the case of *Steiner systems* ($\lambda = 1$), significant progress has been made. For Steiner triple systems ($B[3, 1; v]$ designs), one can embed a partial Steiner triple system of order n in a Steiner triple system of order v for every $v \geq 4n + 1$, $v \equiv 1, 3 \pmod 6$ [1]. Partial Steiner systems can always be finitely embedded [6, 12], but in this case the containing Steiner system is exponentially larger than the partial system. More generally, Wilson's theorem [12] gives a finite embedding for partial (directed) graph designs with $\lambda = 1$.

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The state of affairs for designs with $\lambda > 1$ is much less satisfying. For triple systems $B[3, \lambda; v]$, finite embeddings are known for every λ [11]. In the case of even λ, embeddings exist for which the size of the containing system is quadratic in the size of the partial system [3]; for $\lambda = 2$, this can be improved to linear [8]. Both results rely on the fact that every triple system with even λ can be transformed into a directed triple system [4]; hence, they provide no improvement in the case of odd λ. Finally, for block designs $B[k, \lambda; v]$ in general, no embedding result seems to be known.

In this paper, we describe a general technique for solving the embedding problem for $B[G, \lambda; v]$ designs by reducing it to λ instances of the embedding problem for $B[G, 1; v]$ designs. This provides a small (quadratic) embedding technique for partial triple systems with any λ. It further gives a finite embedding technique for partial graph designs with any k and any λ. Finally, we generalize the method of [8] to produce linear embeddings for triple systems, Mendelsohn triple systems, and directed triple systems.

2. **The Embedding Theorem for Graph Designs**. The central idea of the proof is to decompose the partial design with $\lambda > 1$ into λ partial designs with $\lambda = 1$, to embed each of these into a design with $\lambda = 1$, and to recombine to form the containing design. There is a major stumbling block here: there are partial designs with arbitrarily large λ which cannot be decomposed into λ partial designs with $\lambda = 1$ [2]. This problem is overcome here by first replacing each block by a collection of blocks in such a way that the resulting system is certainly decomposable. Having completed the embedding, this collection of blocks is then replaced by a set containing the original block. We formalize this approach in the remainder of this section.

THEOREM 2.1. *For any (directed) graph G, there is a (directed) graph H having an induced (directed) subgraph isomorphic to G; moreover, H has two distinct edge-decompositions into copies of G.*

PROOF. Let $G = (V, E)$ be a directed graph on n vertices. If G has no arcs, the statement is trivial; otherwise, label the n vertices with labels $\{1, 2, \ldots, n\}$ so that $(1, 2)$ is an arc and $(2, 1)$ is *not* an arc. Whenever such an arc cannot be chosen as $(1, 2)$, G is essentially an undirected graph; in this case, we select any *edge* $\{1, 2\}$ and replace "arc" by "edge" in what follows.

We next produce a family of graphs $F(G)$ containing the original graph G and a graph for each non-arc of G, constructed as follows. For every pair (x, y) of vertices which is not an arc of G, we include a graph isomorphic to G on $n - 2$ new vertices, together with $\{x, y\}$. The mapping of G onto this vertex set is arbitrary, except that $(1, 2)$ maps to (x, y). If G has t non-arcs, this produces a family $F(G)$ of graphs, each isomorphic to G, on $n + t(n - 2)$ vertices, which has the property that every pair from $\{1, 2, \ldots, n\}$ appears in exactly one graph. This family is a partial graph design with $\lambda = 1$, and so can be finitely embedded in a graph design $F'(G)$ [12].

We can similarly construct a different family of graphs $K(G)$, each isomorphic to G, by constructing graphs as above for non-arcs *and arcs* of G; unlike $F(G)$, this family does not contain G, but like $F(G)$ covers all pairs from $\{1, 2, \ldots, n\}$. Once again, we embed this family by Wilson's theorem into a graph design $H'(G)$, ensuring that $F'(G)$ and $H'(G)$ are on the same number of vertices, z. Thus the z-vertex complete digraph has two distinct edge-decompositions into copies of G; these are distinct since $F'(G)$ contains the original graph G whereas $H'(G)$ does not.

In general, however, the z-vertex complete digraph contains no induced subgraph isomorphic to G. To obtain this latter condition as well, we produce $F''(G)$ from $F'(G)$ by omitting all graphs (isomorphic copies of G) which contain an ordered pair which is a *non*-arc of the original graph G. $H''(G)$ is produced similarly from $H'(G)$. Then $F''(G)$ and $H''(G)$ decompose the same z-vertex directed graph, and this digraph has an induced copy of G on vertices $\{1, 2, \ldots, n\}$. □

We now employ theorem 2.1 to obtain finite embeddings for (directed) graph designs with $\lambda > 1$.

THEOREM 2.2. *A partial (directed) graph design can be finitely embedded in a (directed) graph design with the same* λ.

PROOF. Suppose we are given a partial directed graph design PD with index λ. If $\lambda = 1$, Wilson's theorem [12] produces the required embedding. If $\lambda > 1$, we proceed as follows. Let $PD = \{G_1, \ldots, G_S\}$. Each G_i is isomorphic to a given digraph G. Using theorem 2.1, we produce a graph H with two different edge-decompositions into copies of G; in addition, H has an induced subgraph isomorphic to G. Each G_i is replaced by a copy of H, H_i; the induced subgraph of H isomorphic to G is identified with G_i and all other vertices in H_i are new.

Each H_i is then decomposed into copies of G in such a way that none of the original G_i are chosen. The resulting partial graph design D can be easily decomposed into λ partial designs with index 1, since the only pairs appearing more than once are pairs of original vertices, and each graph in the partial graph design contains at most two of the original vertices. To carry out this decomposition, form a graph whose vertices represent the graphs in the partial graph design D. An edge connects two vertices whenever the corresponding graphs share an arc. Now observe that this graph is the disjoint union of some isolated vertices and some cliques of size at most λ. This graph is therefore vertex-colourable with λ colours; moreover, such a colouring provides a decomposition, as each colour corresponds to blocks of a partial graph design with $\lambda = 1$.

Having decomposed D into λ systems with index 1, each is embedded separately using Wilson's theorem, into complete digraphs of the same order. The union of these forms a completed graph design with index λ. Finally, each of the edge-decompositions of H_i is replaced with the alternate edge-decomposition, i.e. the one which contains the graph G_i. The result is a directed graph design which contains the partial graph design supplied. □

Theorem 2.2 establishes the existence of finite embeddings for a very wide variety of embedding problems. These embeddings are, however, astronomical in size, since the two applications of Wilson's theorem each dramatically increases the magnitude. In subsequent sections, we improve on the sizes of the embeddings obtained in special cases.

3. **Embedding Block Designs**. Wilson's theorem produces directed graphs with two distinct edge-decompositions, which we required in the proof of theorem 2.2; however, the graphs obtained are very large. In the case of block designs, we can employ an interesting family of graphs to avoid this.

LEMMA 3.1. There is a graph G_k with 3^{k-1} vertices having exactly two different edge-decompositions into k-cliques. Moreover, any two k-cliques of G_k have at most two vertices in common.

PROOF: Define a graph whose vertex set $V = \{(x_1, \ldots, x_k) \mid x_i \in \{-1, 0, 1\}, \Sigma_{i=1}^{k} x_i \equiv 0 \pmod 3\}$. Two vertices are adjacent if the corresponding vectors agree everywhere except in two positions. This graph has 3^{k-1} vertices and has exactly two different edge-decompositions into k-cliques; moreover, two distinct k-cliques have at most a pair in common [9]. □

This lemma gives us the necessary ingredient to prove a bound on the embedding which improves on that obtained from theorem 2.2.

THEOREM 3.2. *Suppose a partial Steiner system $B[k, 1; n]$ with n elements can be embedded in a Steiner system with $f(n)$ elements. Then a partial block design $B[k, \lambda; n]$ with n elements can be embedded in a block design with at most $f(3^{k-1}\lambda n^2)$ elements.*

PROOF. Let $B = \{b_1, \ldots, b_s\}$ be the collection of blocks in a partial block design with n elements, block size k, and $\lambda > 1$. We produce a new set of blocks B' as follows. Each block b_i involves k elements, say $x_{i1}, \ldots, x_{ik}$. We replace the block b_i by a copy of the graph G_k from lemma 3.1, in such a way that k of the vertices in the G_k corresponding to a k-clique in one of the two edge-decompositions are identified with the elements $x_{i1}, \ldots, x_{ik}$; the other vertices correspond to new elements which are added to the partial design. This is done for every block in B; for each, different new elements are added. Then the collection B' of blocks is obtained by edge-decomposing each G_k into k-cliques so that none of the original blocks from B is included.

The collection B contains at most λn^2 blocks, and hence the collection B' has at most $3^{k-1}\lambda n^2$ elements. Moreover, B' can be decomposed into λ disjoint sets of blocks $L_1, \ldots, L_\lambda$, each of which is a partial Steiner system, as follows. Consider the graph obtained by treating each block of B' as a point, with adjacency between points when the corresponding blocks share a pair. Each block in B' contains at most two of the original n elements, and these are the only elements in which blocks can intersect;

hence, the graph obtained consists of cliques of size at most λ. A proper colouring of this graph in λ colours produces the required decomposition, since each colour class induces a partial Steiner system.

We next embed each partial Steiner system in $f(r)$ elements, where $r \leq 3^{k-1}\lambda n^2$ is the number of elements in the blocks of B'. Combining the Steiner systems which result gives a $B[k, \lambda; f(r)]$ design. In this design, we replace the blocks arising from each G_k by the blocks produced in the other edge-decomposition of the G_k. Note first that this replacement yields a block design, since exactly the same pairs are used. Finally, note that the original blocks from B all appear in this design, and hence we have the required embedding. □

COROLLARY 3.3. *Partial block designs* $B[k, \lambda; v]$ can be finitely embedded.

4. **Embedding Partial Triple Systems**. Each of the embeddings produced so far has been obtained by describing a graph which has two distinct edge-decompositions. In the case of triple systems, one can employ a graph with only six vertices: the 6-vertex cocktail party graph CP (i.e., K_6 omitting a 1-factor) has two edge-decompositions into triangles. This graph can be used in the construction of theorem 3.2 to improve the constant for triple systems; however, the embedding would still be quadratic.

A different observation is required in order to obtain a linear embedding. In the embedding, each triple is replaced by a copy of CP using the method previously developed in [3, 8]; in the process, three new vertices are added. It is important to observe that these three vertices need not be different for every copy of CP to be added; this idea forms the basis for the following result.

THEOREM 4.1. *A partial triple system of order* v *and index* λ *can be embedded in a triple system of order at most* $4(3\lambda/2 + 1)v + 1$.

PROOF. Consider the partial triple system PT. PT is the edge-decomposition of some multigraph M into triangles. We produce an edge-colouring of M as follows. For a triangle (a, b, c) of PT, we colour (a, b) with c, (a, c) with b, and (b, c) with a. This assigns v colours to the edges of M; each colour i induces a subgraph $C(i)$ with maximum degree λ. Each $C(i)$ can be properly edge-coloured in $3\lambda/2$ colours, hence giving a proper $v(3\lambda/2)$-edge-colouring of M.

These colours on M are now interpreted as names of new elements. That is, $v(3\lambda/2)$ new elements are added to the v elements of PT. Each triangle of PT is replaced by a copy of the cocktail party graph — the six vertices employed are the three original vertices, and the three vertices corresponding to the colours of the edges in the original triangle.

The resulting collection of triples has $(3\lambda/2 + 1)v$ elements; moreover, triples intersect in pairs only in pairs of original elements. Each triple involves at most two original elements, and hence the collection can easily be decomposed into λ partial Steiner triple systems. We embed each partial Steiner triple system on $(3\lambda/2 + 1)v$

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elements into a Steiner triple system on $4(3\lambda/2 + 1)v + 1$ elements using the Andersen-Hilton-Mendelsohn technique [1], and recombine to form a triple system with index λ. As before, we now replace each edge-decomposition of CP by the alternate edge-decomposition in order to reintroduce the original triples. □

These techniques can be easily adapted to partial Mendelsohn triple systems and partial directed triple systems, producing linear embeddings for every index λ in each case. In particular, using the "$18v + 3$" embeddings of Hamm, Lindner and Rodger [8], we obtain the following two results.

LEMMA 4.2. A partial Mendelsohn triple system of order v and index λ can be embedded in a Mendelsohn triple system of index λ on $6(3\lambda/2 + 1)v + 3$ elements.

LEMMA 4.3: A partial directed triple system of order v and index λ can be embedded in a directed triple system of index λ on $18(3\lambda + 1)v + 3$ elements.

In lemma 4.3, the constant is twice that of lemma 4.2 due to a technicality: the graph induced by a single colour may have degree 2λ in the directed case, unlike the Steiner and Mendelsohn cases. Otherwise, the details parallel theorem 4.1 quite closely and so we have omitted them here.

5. **Concluding Remarks**. Theorem 2.2 demonstrates the existence of finite embeddings of partial graph designs with arbitrary index; these results extend those for $\lambda = 1$ which follow from Wilson's theorem. Theorem 3.2 demonstrates that embedding block designs for $\lambda > 1$ does not differ substantially from the embedding of Steiner systems. In fact, whenever an embedding result is available for Steiner systems, it can be exploited with only a polynomial increase in size to handle block designs in general. It is important to note that the technique employed here will work equally well for t-designs in general; in this case, the role of G_k can be served by any t-uniform hypergraph with two different edge-decompositions into k-cliques. Any t-design could serve as such a t-uniform hypergraph.

In view of the results here, the fundamental question in embedding is now to produce a small embedding technique for Steiner systems in general. This would, along with theorem 3.2, provide small embeddings in all cases for block designs.

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