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CONSISTENCY OF CONSTRUCTIONS FOR CELL DIVISION PROCESSES

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Abstract

For a class of cell division processes in the Euclidean space $\mathbb{R}^d$, spatial consistency is investigated. This addresses the problem whether the distribution of the generated structures, restricted to a bounded set V, depends on the choice of a larger region $W \supset V$ where the construction of the cell division process is performed. This can also be understood as the problem of boundary effects in the cell division procedure. It is known that the STIT tessellations are spatially consistent. In the present paper it is shown that, within a reasonable wide class of cell division processes, the STIT tessellations are the only ones that are consistent.

Keywords: Stochastic geometry; random tessellation; iteration/nesting of tessellations; STIT tessellation; spatial consistency

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1. Introduction

In stochastic geometry, the well established reference models for random tessellations of the Euclidean space $\mathbb{R}^d$, $d \geq 2$, are the Poisson–Voronoi tessellations and the Poisson hyperplane tessellations; see [1], [8]. Aside from these models there are many suggestions in the literature to construct tessellations by sequential division of the cells, i.e. of the polytopes which constitute a tessellation. A systematization including many of such constructions can be found in [2]. Usually, these tessellations are constructed in a bounded window $W \subset \mathbb{R}^d$. This yields a key problem for this kind of cell division processes: are the *constructions* consistent in space, i.e. does the distribution of the resulting tessellation depend on the window in which the construction is performed? More precisely, if $(Y \wedge W)_t$ and $(Y \wedge V)_t$ are the random tessellations generated by cell division until time $t > 0$ in bounded windows $V \subset W$, are then $(Y \wedge W)_t \wedge V$ and $(Y \wedge V)_t$ identically distributed? The consistency of a model implies the existence of a tessellation Y_t of the whole space $\mathbb{R}^d$ such that the restrictions $Y_t \wedge W$ have the same law as $(Y \wedge W)_t$. In this paper we consider a certain class of cell division processes and provide sufficient conditions for consistency. With a further restriction to homogeneity (spatial stationarity) and continuous time, we show that in this class the STIT (stable with respect to iteration) tessellations (introduced in [7]) are the only ones which are consistent. This confirms a conjecture in [9], where a rather general approach to cell division processes is treated.

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2. Random tessellations and consistency

For the d-dimensional Euclidean space $\mathbb{R}^d$, $d \geq 2$, denote by ∂B, $\mathrm{int}(B)$, and $\mathrm{cl}(B)$ the boundary, the interior, and the topological closure, respectively, of a set $B \subset \mathbb{R}^d$. Let $[\mathcal{H}, \mathfrak{H}]$ denote the measurable space of hyperplanes in $\mathbb{R}^d$ with its Borel σ-algebra $\mathfrak{H}$ with respect to (w.r.t.) the topology of closed convergence for closed subsets of $\mathbb{R}^d$; see [8]. For a set $B \subset \mathbb{R}^d$, we write $[B] = \{h \in \mathcal{H} : B \cap h \neq \varnothing\}$. Furthermore, let $\mathfrak{P}$ denote the set of all polytopes (i.e. convex hulls of finite point sets) with *nonempty interior* in $\mathbb{R}^d$. A set $\{C_1, C_2, \ldots\}$ with $C_i \in \mathfrak{P}$ is a tessellation if $\bigcup_{i=1}^{\infty} C_i = \mathbb{R}^d$ and $\mathrm{int}(C_i) \cap \mathrm{int}(C_j) = \varnothing$ for $i \neq j$. Moreover, a local finiteness condition must be satisfied, $\#\{i : C_i \cap B \neq \varnothing\} < \infty$ for all bounded $B \subset \mathbb{R}^d$, i.e. the number of polytopes intersecting a bounded set is finite.

A tessellation can be considered as a set $\{C_1, C_2, \ldots\}$ of polytopes—referred to as the cells—as well as a closed set $\bigcup_{i=1}^{\infty} \partial C_i \subset \mathbb{R}^d$, the union of the cell boundaries. There is an obvious one-to-one relation between both descriptions of a tessellation, and also the σ-algebras which are used for them can be related appropriately. Let $\mathbb{T}$ denote the set of all tessellations of $\mathbb{R}^d$. By $y \in \mathbb{T}$ we mean the closed set of cell boundaries of the tessellation y. Then $\mathbb{T}$ can be endowed with the Borel σ-algebra $\mathcal{B}(\mathbb{T})$ of the topology of closed convergence. A random tessellation Y is a random variable with values in $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$. For $W \in \mathfrak{P}$, the set of tessellations restricted to W is denoted by $\mathbb{T} \wedge W$, where we assume that ∂W belongs to these tessellations. On the other hand, for $y \in \mathbb{T}$ the boundary ∂W does not necessarily belong to $y \cap W$, but $(y \cap W) \cup \partial W \in \mathbb{T} \wedge W$. Here, W is referred to as a window.

Let $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. By $\stackrel{\mathrm{D}}{=}$ we denote the identity of the distributions of random variables. Our investigation into the consistency of random tessellations is based on the following proposition. In [8, Theorem 2.3.1], a more general form is given; we specify it here for $\mathbb{R}^d$.

Theorem 1. (Schneider and Weil.) *Let $(Z_i : i \in \mathbb{N})$ be a sequence of random closed sets in $\mathbb{R}^d$, and $(G_i : i \in \mathbb{N})$ a sequence of open, bounded sets with $\mathrm{cl}(G_i) \subset G_{i+1}$ for $i \in \mathbb{N}$ and $\bigcup_{i=1}^{\infty} G_i = \mathbb{R}^d$. If $Z_m \cap \mathrm{cl}(G_i) \stackrel{\mathrm{D}}{=} Z_i$ for all $m > i$, then there exists a random closed set Z in $\mathbb{R}^d$ with*

$$Z \cap \mathrm{cl}(G_i) \stackrel{\mathrm{D}}{=} Z_i \quad \text{for all } i \in \mathbb{N}.$$

The assertion of Theorem 1 motivates the following definition.

Definition 1. A family $(Y(W), W \in \mathfrak{P})$ of random tessellations with $Y(W) \in \mathbb{T} \wedge W$ is called consistent if and only if for any two windows $V, W \in \mathfrak{P}$ with nonempty interiors and $V \subset W$ $Y(V) \stackrel{\mathrm{D}}{=} Y(W) \wedge V$ holds

Obviously, if Y is a random tessellation of the whole space $\mathbb{R}^d$, then $(Y(W) := Y \wedge W, W \in \mathfrak{P})$ is a consistent family of tessellations. This applies, e.g. to Poisson line (or hyperplane) tessellations, Poisson–Voronoi tessellations, and tessellations introduced by Georgii *et al.* [3]. On the other hand, from Theorem 1 for any consistent family $(Y(W), W \in \mathfrak{P})$ there exists a random tessellation Y such that $Y \wedge W \stackrel{\mathrm{D}}{=} Y(W)$. Given the law of a random infinite tessellation, one can ask whether there is a construction in any bounded window which generates the correct distribution within the window, e.g. for the Poisson line tessellation there is an obvious construction in any window; but for the Poisson–Voronoi tessellation there is no construction known which is based only on the information inside a bounded window.

In this paper we will study the particular setting that the *construction* of $Y \wedge W$ is performed strictly inside W, no information outside W is available, and the tessellation is generated by

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sequential cell division. (Note that Poisson line (or hyperplane) tessellations and Poisson–Voronoi tessellations are not generated by division of individual cells, i.e. they do not belong to the types of tessellations considered here.) The motivation for the restriction to this class of tessellations is that such constructions (described inside bounded windows) have already attracted interest over recent decades, but there are only a few theoretical results available (mostly only simulation studies). On the other hand, the 'branching random tessellations' by Georgii *et al.* [3] are also generated by cell division, but they are based on (the STIT) tessellations of the *whole space* $\mathbb{R}^d$, and, therefore, they are consistent, but they cannot be constructed inside bounded windows (unless their law is identical to STIT).

Since cell divisions are assumed to appear sequentially, it is appropriate to study random processes (in time) of tessellations. Therefore, the consistency of finite-dimensional distributions of these processes has also to be considered.

Definition 2. For any $W \in \mathfrak{P}$ let $((Y \wedge W)_t, t > 0)$ be a random process of tessellations with values in $\mathbb{T} \wedge W$. The family of tessellation processes $((Y \wedge W)_t, t > 0),\ W \in \mathfrak{P})$ is called consistent in space if and only if for any two windows $V, W \in \mathfrak{P}$ with $V \subset W$ and for all $0 < t_1 < \cdots < t_n,\ n \in \mathbb{N}$, the following holds:

$$((Y \wedge V)_{t_1}, \ldots, (Y \wedge V)_{t_n}) \stackrel{\mathrm{D}}{=} ((Y \wedge W)_{t_1} \wedge V, \ldots, (Y \wedge W)_{t_n} \wedge V).$$

Again, if $(Y_t, t > 0)$ is a random process of tessellations of $\mathbb{R}^d$, then $(((Y \wedge W)_t, t > 0),\ W \in \mathfrak{P})$ is a consistent family of tessellation processes. Vice versa, if $(((Y \wedge W)_t, t > 0),\ W \in \mathfrak{P})$ is a consistent family of tessellation processes, then for all $0 < t_1 < \cdots < t_n,\ n \in \mathbb{N}$ exist tessellations $Y_{t_1}, \ldots, Y_{t_n}$ with

$$(Y \wedge W)_{t_1}, \ldots, (Y \wedge W)_{t_n}) \stackrel{\mathrm{D}}{=} (Y_{t_1} \wedge W, \ldots, Y_{t_n} \wedge W) \quad \text{for all } W \in \mathfrak{P}.$$

Because the laws of $(Y_{t_1}, \ldots, Y_{t_n})$ with $0 < t_1 < \cdots < t_n,\ n \in \mathbb{N}$, form a projective family of distributions, and the measurable space $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ of tessellations is a Polish space (see [5]), Kolmogorov's extension theorem (see, e.g. [4]) yields the existence of a process $(Y_t, t > 0)$ with the respective finite-dimensional distributions.

3. A class of cell division processes

Inspired by Cowan's paper [2] we study a certain class of random tessellation processes which are generated by sequential cell division. But we will not restrict ourselves to discrete time. A cell division process is defined by the distributions of lifetimes (that in bounded windows somehow correspond to Cowan's selection rule) and by a division rule for the extant cells.

We will make the following assumptions which we formalize later. Let $W, C \in \mathfrak{P}$ with $\operatorname{int}(W \cap C) \neq \varnothing$. The construction is performed within the window W.

Any $C \cap W$ has a random lifetime $\tau(C \cap W)$ with a distribution that depends on $C \cap W$ only. At the end of its lifetime, $C \cap W$ is divided by a random hyperplane $h(C \cap W)$ with a distribution $\Lambda_{C \cap W}$ that depends on $C \cap W$ only and is also independent from the lifetime. Given a current tessellation, the lifetimes for different cells are conditionally independent, and the distributions of the dividing hyperplanes are conditionally independent as well.

This seems to be rather restrictive for a cell division process, and it would be desirable to consider procedures where the cell division may depend also on the environment of a given cell. But since the assumption of spatial consistency refers to arbitrary windows W, we have also to deal with the case that for a given cell C we have $W \subset C$, i.e. the window is completely

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contained in a cell, and we can use only this information about the tessellation at a given time. Therefore, for an investigation of consistent constructions, our assumptions seem to be reasonable.

In [3] and [9] the approach to cell division processes is considerably more general, in particular allowing for dependencies of the lifetimes and distributions of the dividing hyperplanes on the whole tessellation. But there, the problem of consistency is not addressed, except in the conjecture in [9] saying that only STIT tessellations and some related constructions are consistent. Moreover, several constructions in those papers use the STIT tessellations as initial tessellations, i.e. they rely on the existence of the STIT tessellation. This indicates that the STIT tessellations form a sound basis for the development of a much wider class of tessellation models which can be adapted to real structures. Recently, a fracture model using modifications of STIT was presented in [6].

3.1. A single division step

In order to have consistency for the distributions of the lifetimes and of the dividing hyperplanes we begin by considering a single division.

Assumption 1. *Let* $(\Lambda_D,\ D \in \mathfrak{P})$ *be a family of probability measures on* $(\mathcal{H}, \mathfrak{H})$ *satisfying, for all* $W, C \in \mathfrak{P}$:

$$\operatorname{int}(C \cap W) \neq \varnothing \quad \Longrightarrow \quad \Lambda_{C\cap W}([C \cap W]) = 1, \qquad \Lambda_{C\cap W} \ll \Lambda_W$$

and, for all $A \in \mathfrak{H} \cap [C \cap W]$,

$$\Lambda_{C\cap W}(A) = \frac{\Lambda_C(A)}{\Lambda_C([C \cap W])}.$$

Assumption 2. *Let* $(\tau(D), D \in \mathfrak{P})$ *be a family of random variables satisfying, for all* $D \in \mathfrak{P}$, *and for all* $s \geq 0$,

$$\mathbb{P}(\tau(D) \geq s) > 0, \tag{1}$$

and the following equation for the failure rates. For continuous time, for all $W, C \in \mathfrak{P}$: $\operatorname{int}(C \cap W) \neq \varnothing$, *and for all* $s \geq 0$,

$$\begin{aligned} &\mathbb{P}(\tau(C \cap W) \leq s + \mathrm{d}s \mid \tau(C \cap W) \geq s) \\ &\qquad = \mathbb{P}(\tau(C) \leq s + \mathrm{d}s \mid \tau(C) \geq s)\Lambda_C([C \cap W]) + o(\mathrm{d}s), \end{aligned} \tag{2}$$

and for discrete time for all $W, C \in \mathfrak{P}$ *such that* $\operatorname{int}(C \cap W) \neq \varnothing$, *for all* $s \in \mathbb{N}$, $s \geq 1$,

$$\mathbb{P}(\tau(C \cap W) = s \mid \tau(C \cap W) \geq s) = \mathbb{P}(\tau(C) = s \mid \tau(C) \geq s)\Lambda_C([C \cap W]). \tag{3}$$

Assumption 1 means that the distribution $\Lambda_{C\cap W}$ coincides with the conditional distribution Λ_C, given that $C \cap W$ is intersected. Moreover, it implies that the Radon–Nikodym density is

$$\frac{d\Lambda_{C\cap W}}{d\Lambda_C}(h) = \frac{\mathbf{1}_{\{C\cap W\}}(h)}{\Lambda_C([C \cap W])}, \qquad h \in \mathcal{H},$$

where $\mathbf{1}$ is the indicator function.

Concerning the lifetimes, (2) means that the conditional probability for the division of $C \cap W$ in an interval $(s, s + \mathrm{d}s)$ equals the probability that C is divided at this time and the dividing hyperplane for C also intersects $C \cap W$, assuming that the random dividing hyperplane and the random life time are independent. For continuous time, this is formulated for time intervals of a small length $\mathrm{d}s$ only, because this condition refers to a single division of C.

Firstly, we consider a consequence of Assumption 1.

Corollary 1. *Let be $V, W, Z \in \mathfrak{P}$, $V \subseteq W \subseteq Z$, and $A \in \mathfrak{H} \cap [V]$ with $\Lambda_W(A) > 0$. Then Assumption1 implies that $\Lambda_Z([V]) > 0$ and*

$$\frac{\Lambda_Z([W])}{\Lambda_Z([V])} = \frac{\Lambda_V(A)}{\Lambda_W(A)}.$$

Let $V_0 = [0, 1]^d$ and $W_1 \subset W_2 \subset \cdots$ be a sequence of windows with $\bigcup_{n=1}^{\infty} W_n = \mathbb{R}^d$. Now we consider a function Λ with

$$\Lambda(A) = \lim_{n\to\infty} \frac{\Lambda_{W_n}(A)}{\Lambda_{W_n}([V_0])}, \qquad A \in \mathfrak{H}. \tag{4}$$

Lemma 1. *Assumption 1 implies the following.*

(a) *For all $A \in \mathfrak{H}$, the sequence $\Lambda_{W_n}(A)/\Lambda_{W_n}([V_0])$, $n = 1, 2, \ldots$, is monotone and the limit (4) exists.*

(b) *The limit does not depend on the particular choice of the sequence $W_1 \subset W_2 \subset \cdots$ with $\bigcup_{n=1}^{\infty} W_n = \mathbb{R}^d$.*

(c) *The function Λ is a measure on $(\mathcal{H}, \mathfrak{H})$.*

(d) *For all $D \in \mathfrak{P}$ it holds that $0 < \Lambda([D]) < \infty$ and*

$$\Lambda_D(A) = \frac{\Lambda(A \cap [D])}{\Lambda([D])}, \qquad A \in \mathfrak{H}. \tag{5}$$

Up to now, we have shown that Assumption 1 implies that all the distributions Λ_D, for dividing hyperplanes in a polytope D are necessarily normalized restrictions of a 'universal' measure Λ on $(\mathcal{H}, \mathfrak{H})$. This measure must not be translation invariant, and in particular it can have atoms. With our assumptions, we have $\Lambda([D]) > 0$ for all polytopes D with nonempty interiors. This condition could be relaxed, but then the assumptions and proofs would require a more sophisticated structure of case-by-case analysis.

Let us consider a counterexample. For $D \in \mathfrak{P}$ define Λ_D in the following way: throw a random point uniformly into D, and then choose a random hyperplane through this point with a certain directional distribution (see [2]). Then Assumption 1 is not satisfied. This can be illustrated as follows. Choose $d = 2$ and $W = [0, 1]^2$. For $C = [0.8, 2.8] \times [0.4, 0.6]$ we have $C \cap W = [0.8, 1] \times [0.4, 0.6]$, a square. Let the directional distribution of the dividing lines be concentrated on the vertical and the horizontal directions, with probability 0.5 each. Then $\Lambda_{C\cap W}$ yields random lines with the same directional distribution. On the other hand, applying Λ_C, and observing which distribution of lines appears in $C \cap W$, the probability for a horizontal line is $\frac{0.50}{0.55}$ and for a vertical line is $\frac{0.05}{0.55}$, which contradicts Assumption 1. Moreover, it can easily be seen that for this family $(\Lambda_D, D \in \mathfrak{P})$ of measures there is no measure Λ on $(\mathcal{H}, \mathfrak{H})$ such that (5) is satisfied.

Let us now consider the lifetime distributions.

Lemma 2. *For a continuous-time process, Assumption 1 and (1) and (2) from Assumption 2 imply that there exists a function $\gamma\colon [0, \infty) \to [0, \infty)$ such that for all $D \in \mathfrak{P}$ and all $s \geq 0$,*

$$\mathbb{P}(\tau(D) \leq s + \mathrm{d}s \mid \tau(D) \geq s) = \gamma(s)\Lambda([D])\,\mathrm{d}s + o(\mathrm{d}s), \tag{6}$$

where the measure Λ is given by (4).

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For a discrete-time process, Assumption 1 and (1) and (3) from Assumption 2 yield that Λ *must be a finite measure on* $(\mathcal{H}, \mathfrak{H})$ *and*

$$\mathbb{P}(\tau(D) = s \mid \tau(D) \geq s) = \gamma(s)\Lambda([D]). \tag{7}$$

Note that the function γ does not depend on the polytope D, i.e. there is a 'universal' failure-rate function γ for all polytopes which is weighted by the factor $\Lambda([D])$.

We will assume, for continuous time,

$$\gamma(s) > 0 \quad \text{for all } s > 0,$$

and for discrete time,

$$\gamma(s) > 0 \quad \text{for all } s \in \mathbb{N},\ s \geq 1.$$

Let us consider a counterexample. For $d = 2$ assume that $\tau(D)$ is exponentially distributed with parameter 1 for all $D \in \mathfrak{P}$. Choose Λ as a translation invariant measure on the space of lines with the directional distribution concentrated on the vertical and the horizontal directions, with probability 0.5 each. Choose $V = [0, 1]^2$ and $W = [-5, 5]^2$. For $C = [0.8, 2.8] \times [0.4, 0.6]$, we have $C \cap V = [0.8, 1] \times [0.4, 0.6]$ and $C \cap W = C$. If the construction is performed in the window V, the time until a dividing line appears in $C \cap V$ is exponentially distributed with parameter 1. In contrast, if the construction is performed in the window W, the lifetime of C is exponentially distributed with parameter 1, but the time until a dividing line appears in $C \cap V$ is exponentially distributed with parameter $\Lambda([C \cap V])/\Lambda([W]) = 0.1$, which contradicts (2) in Assumption 2.

Analogously, we can consider the case that $\tau(D)$ is exponentially distributed with parameter area(D) and see that (2) does not hold. If the parameter of the exponential distribution of $\tau(D)$ is 1 or area(D) respectively, then there is no measure Λ on $(\mathcal{H}, \mathfrak{H})$ and no function γ such that the failure rate of $\tau(D)$ equals $\gamma(s)\,\Lambda([D])$.

3.2. The construction of tessellations in bounded windows

Let us describe a process $((Y \wedge W)_t,\ t \geq 0)$ of random tessellations constructed in a bounded window W. By $\mathfrak{C}(Y \wedge W)_t$ denote the set of its cells. All the cells C of the tessellations are marked by their birth times $\beta(C)$.

For $h \in \mathcal{H}$ we denote by h^+ and h^- the closed half-spaces of $\mathbb{R}^d$ generated by h with the following definition. If $\boldsymbol{u} \in \mathcal{S}_+^{d-1}$ is a vector in the upper half unit-sphere of $\mathbb{R}^d$ and $a \in \mathbb{R}$ such that $h = \{x \in \mathbb{R}^d \colon \langle x, \boldsymbol{u}\rangle = a\}$, then $h^+ := \{x \in \mathbb{R}^d \colon \langle x, \boldsymbol{u}\rangle \geq a\}$ and $h^- := \{x \in \mathbb{R}^d \colon \langle x, \boldsymbol{u}\rangle \leq a\}$.

Definition 3. Fix a window $W \in \mathfrak{P}$. Let $(h(D),\ D \in \mathfrak{P})$ be a family of independent random hyperplanes with the respective distributions $(\Lambda_D,\ D \in \mathfrak{P})$ satisfying Assumption 1. Furthermore, let $(\tau(D),\ D \in \mathfrak{P})$ be a family of independent nonnegative random lifetimes satisfying (2) or (3), respectively, and assume that the families of the hyperplanes and of the lifetimes are independent as well. Now define $\mathfrak{C}(Y \wedge W)_0 = \{W\}$, with birth time $\beta(W) = 0$.

Let σ_n, $n = 1, 2, \ldots$, denote the jump times of $((Y \wedge W)_t,\ t \geq 0)$, given by $\sigma_1 = \tau(W)$, and for $n \geq 2$ let

$$\sigma_{n+1} = \min\{\beta(C) + \tau(C) \colon C \in \mathfrak{C}(Y \wedge W)_{\sigma_n}\}.$$

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At these jump times the new state of the tessellation process is defined as

$$\begin{aligned}\mathfrak{C}(Y \wedge W)_{\sigma_{n+1}} = [&\mathfrak{C}(Y \wedge W)_{\sigma_n} \setminus \{C \in \mathfrak{C}(Y \wedge W)_{\sigma_n} : \beta(C) + \tau(C) = \sigma_{n+1}\}] \\ &\cup \{C \cap h^+(C) \colon C \in \mathfrak{C}(Y \wedge W)_{\sigma_n} : \beta(C) + \tau(C) = \sigma_{n+1}\} \\ &\cup \{C \cap h^-(C) \colon C \in \mathfrak{C}(Y \wedge W)_{\sigma_n} : \beta(C) + \tau(C) = \sigma_{n+1}\},\end{aligned}$$

and all these new cells $C \cap h^{\pm}(C)$ are marked with the birth time σ_{n+1}.

Then define, for all $t > 0$,

$$(Y \wedge W)_t = (Y \wedge W)_{\sigma_n} \quad \text{if } \sigma_n \leq t < \sigma_{n+1}.$$

Note that according to this definition, at least one cell is divided at a jump time. If the life time distributions are continuous, then almost surely exactly one cell is divided at a jump time.

Now we will show that the assumptions of consistency and independence imply a Markov property.

Lemma 3. *Under the assumptions of Definition 3 there exists a constant $\gamma > 0$ such that $\gamma(s) = \gamma$ for all $s \geq 0$ for continuous time, and for all $s \in \mathbb{N},\ s \geq 1$ for discrete time.*

Combined with (6) or (7), this yields that the lifetime $\tau(D)$ is exponentially distributed (for continuous time) or geometrically distributed (for discrete time), respectively, and in both cases the parameter is $\gamma\Lambda([D])$.

This yields the following proposition immediately.

Corollary 2. *For all windows $W \in \mathfrak{P}$ the process $((Y \wedge W)_t,\ t \geq 0)$ (continuous or discrete time) as described in Definition 3 is a Markov pure jump process.*

Recall that $(Y \wedge W)_t$ denotes the random set of cell boundaries of the tessellation in W, including ∂W. Note that $(Y \wedge W)_t \setminus \partial W$ is the random set of cell boundaries in the interior of W, which can even be empty. Even if all consistent families of random sets induce a random set in $\mathbb{R}^d$, this is not necessarily a random tessellation of $\mathbb{R}^d$.

Theorem 2. (Continuous time.) *The family (indexed by W) of continuous-time random tessellation processes $[((Y \wedge W)_t,\ t > 0),\ W \in \mathfrak{P}]$ as given by Definition 3 with $\gamma(t) = \gamma > 0$ for all $t > 0$ provides a consistent family $[(\partial(Y \wedge W)_t \setminus \partial W,\ t > 0),\ W \in \mathfrak{P}]$ of random set processes. Hence, there exists a process $(Y_t,\ t > 0)$ of random sets in $\mathbb{R}^d$ such that $Y_t \wedge W \stackrel{\mathrm{D}}{=} (Y \wedge W)_t$ for all $t > 0$ and all windows W.*

In the following proposition we show that under the additional assumption of translation invariance the STIT processes are the only cell division processes that can be constructed consistently inside bounded windows.

Theorem 3. (Continuous-time, translation invariance.) *Let $[((Y \wedge W)_t,\ t > 0),\ W \in \mathfrak{P}]$ be a family of continuous-time random tessellation processes as given by Definition 3 with $\gamma(t) = \gamma > 0$ for all $t > 0$. Furthermore, suppose that the measure Λ given by (4) is translation invariant and not concentrated on a set of hyperplanes which are all parallel to one line (i.e. the directional distribution is not concentrated on a great subsphere, cf. [8, Section 10.3]). Then there exists a process $(Y_t,\ t > 0)$ of random tessellations of $\mathbb{R}^d$ such that $Y_t \wedge W \stackrel{\mathrm{D}}{=} (Y \wedge W)_t$ for all $t > 0$ and all windows W. This process is a STIT tessellation process governed by $\gamma\Lambda$.*

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Proof. For continuous-time processes, the lifetimes of the cells C are exponentially distributed with parameter $\gamma\Lambda([C])$. Hence, the construction given in Definition 3 is a construction of the STIT tessellation process governed by the measure $\gamma\Lambda$; see [7].

Lemma 4. (Discrete time.) *The family (indexed by W) of discrete-time random tessellation processes* $[((Y \wedge W)_t,\ t \in \mathbb{N}_0,\ W \in \mathfrak{P}]$ *as given by Definition 3 with* $\gamma(t) = \gamma > 0$ *for all* $t \geq 1$ *provides a consistent family* $[(\partial(Y \wedge W)_t \setminus \partial W,\ t > 0),\ W \in \mathfrak{P}]$ *of random set processes if and only if* Λ *is a finite measure on* $(\mathcal{H}, \mathfrak{H})$. *Under these assumptions, there exists a process* $(Y_t,\ t \in \mathbb{N}_0)$ *of random sets in* $\mathbb{R}^d$ *such that* $Y_t \wedge W \overset{\mathrm{D}}{=} (Y \wedge W)_t$ *for all* $t \in \mathbb{N}_0$ *and all windows* W. *None of these random sets is a tessellation of* $\mathbb{R}^d$.

Proof. The proof of consistency is analogous with the proof of Theorem 2. Until any time $t \in \mathbb{N}$ and in any window W there are at most 2^t cells, and, hence, at that time there cannot be infinitely many cells in $\mathbb{R}^d$. Therefore, the random set of boundaries cannot be a tessellation.

For this lemma, we see that in a discrete-time consistent construction the measure Λ cannot be translation invariant, and the corresponding random set in $\mathbb{R}^d$ cannot be spatially stationary (homogeneous).

4. Discussion

Some generalizations of the model considered in the present paper are the following.

In Theorem 3 we assumed translation invariance of the model. It will be interesting to study also the nontranslation invariant case as well and the nonhomogeneous STIT tessellations.

Also, from Assumption 2, we can relax (1) in a sense that for the cells there are upper bounds for their lifetimes, i.e. times at which the cells are almost surely divided (if not divided before). For such models these upper bounds must be very small for large cells and larger for smaller cells.

Furthermore, we can discuss generalizations where the distribution Λ_C or the distribution of $\tau(C)$ may also be depend on the 'past' (e.g. the mother cells) of C *within* the bounded window W of construction. Also, Cowan's [2] selection rules (for discrete time) in general depend on the set of *all* extant cells inside W. But this approach seems not to be appropriate for considering consistency, because there is a strong dependence on the window of construction.

We conjecture that the existence of a hyperplane measure Λ on $(\mathcal{H}, \mathfrak{H})$ for more general cell division models, which governs lifetimes and divisions, remains essential, even if some of the assumptions of this paper are relaxed.

5. Proofs

Proof of Corollary 1. Because $\Lambda_V([V]) = 1$ and $\Lambda_V \ll \Lambda_Z$, we have $\Lambda_Z([V]) > 0$. Furthermore, Assumption 1 implies that

$$\Lambda_V(A) = \frac{\Lambda_Z(A)}{\Lambda_Z([V])}, \qquad \Lambda_W(A) = \frac{\Lambda_Z(A)}{\Lambda_Z([W])}$$

and, hence,

$$\Lambda_V(A)\,\Lambda_Z([V]) = \Lambda_W(A)\,\Lambda_Z([W]).$$

Now, using the assumption that $\Lambda_W(A) > 0$ yields the proposed equation.

Proof of Lemma 1. (a) Let be $A \in \mathfrak{H}$. If $\Lambda_{W_n}(A) = 0$ for all n, then the limit (4) is 0. If there is an n_0 such that $\Lambda_{W_{n_0}}(A) > 0$ then the assumption of absolute continuity in Assumption 1

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implies that $\Lambda_{W_n}(A) > 0$ for all $n \geq n_0$ and we can consider the ratio of two subsequent values (assuming that $W_{n_0} \supset V_0$),

$$\begin{aligned}
\frac{\Lambda_{W_{n+1}}(A)}{\Lambda_{W_{n+1}}([V_0])}\frac{\Lambda_{W_n}([V_0])}{\Lambda_{W_n}(A)} &= \frac{\Lambda_{W_{n+1}}(A\cap[W_{n+1}])}{\Lambda_{W_{n+1}}([V_0])}\frac{\Lambda_{W_n}([V_0])}{\Lambda_{W_n}(A\cap[W_n])} \\
&\quad [\text{apply Corollary 1 for } [V_0],\ V = W_n,\ W = Z = W_{n+1}] \\
&= \frac{1}{\Lambda_{W_{n+1}}([W_n])}\frac{\Lambda_{W_{n+1}}(A\cap[W_{n+1}])}{\Lambda_{W_n}(A\cap[W_n])} \\
&\geq \frac{1}{\Lambda_{W_{n+1}}([W_n])}\frac{\Lambda_{W_{n+1}}(A\cap[W_n])}{\Lambda_{W_n}(A\cap[W_n])} \\
&\quad [\text{apply Corollary 1 for } A\cap[W_n],\ V = W_n,\ W = Z = W_{n+1}] \\
&= \frac{1}{\Lambda_{W_{n+1}}([W_n])}\frac{\Lambda_{W_{n+1}}([W_n])}{\Lambda_{W_{n+1}}([W_{n+1}])} \\
&= 1,
\end{aligned}$$

which proves the proposed monotonicity and hence the existence of the limit (4), which may be infinite.

(b) In order to prove that this limit does not depend on the particular choice of the sequence of windows, consider two such monotone sequences $(W_n, n \in \mathbb{N})$ and $(W'_n, n \in \mathbb{N})$. Then for any $n \in \mathbb{N}$ there is a $m \in \mathbb{N}$ such that $W_n \subset W'_m$ and, hence,

$$\lim_{n\to\infty}\frac{\Lambda_{W_n}(A)}{\Lambda_{W_n}([V_0])} \leq \lim_{n\to\infty}\frac{\Lambda_{W'_n}(A)}{\Lambda_{W'_n}([V_0])}.$$

Exchanging the roles of W_n and W'_n, we see that the opposite inequality holds as well, and, hence, both limits are equal.

(c) Obviously, Λ is nonnegative. Let be $A_1, A_2, \ldots \in \mathfrak{H}$ be a sequence of pairwise disjoint sets. Then

$$\begin{aligned}
\Lambda\left(\bigcup_{i=1}^{\infty} A_i\right) &= \lim_{n\to\infty}\frac{\Lambda_{W_n}(\bigcup_{i=1}^{\infty} A_i)}{\Lambda_{W_n}([V_0])} \\
&= \lim_{n\to\infty}\frac{\Lambda_{W_n}([W_n]\cap\bigcup_{i=1}^{\infty} A_i)}{\Lambda_{W_n}([V_0])} \\
&= \lim_{n\to\infty}\frac{\sum_{i=1}^{\infty}\Lambda_{W_n}([W_n]\cap A_i)}{\Lambda_{W_n}([V_0])} \\
&= \sum_{i=1}^{\infty}\lim_{n\to\infty}\frac{\Lambda_{W_n}([W_n]\cap A_i)}{\Lambda_{W_n}([V_0])} \\
&= \sum_{i=1}^{\infty}\Lambda(A_i),
\end{aligned}$$

where the last but one equation follows from the monotone convergence theorem.

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(d) For $D \in \mathfrak{P}$ let $n_0 \in \mathbb{N}$ be such that $V_0 \subset W_{n_0}$ and $D \subset W_{n_0}$. Then, denoting the convex hull of $D \cup V_0$ by $\text{conv}(D \cup V_0)$, and using Corollary 1, for all $n \geq n_0$,

$$\begin{aligned} 0 &< \Lambda_{\text{conv}(D\cup V_0)}([D]) \\ &= \frac{\Lambda_{\text{conv}(D\cup V_0)}([D])}{\Lambda_D([D])} \\ &= \frac{\Lambda_{W_n}([D])}{\Lambda_{W_n}([\text{conv}(D\cup V_0)])} \\ &\leq \frac{\Lambda_{W_n}([D])}{\Lambda_{W_n}([V_0])} \\ &\leq \frac{\Lambda_{W_n}([\text{conv}(D\cup V_0)])}{\Lambda_{W_n}([V_0])} \\ &= \frac{\Lambda_{V_0}([V_0])}{\Lambda_{\text{conv}(D\cup V_0)}([V_0])} \\ &= \frac{1}{\Lambda_{\text{conv}(D\cup V_0)}([V_0])} \\ &< \infty. \end{aligned}$$

Hence,

$$0 < \Lambda([D]) = \lim_{n\to\infty} \frac{\Lambda_{W_n}([D])}{\Lambda_{W_n}([V_0])} < \infty. \tag{8}$$

Furthermore, for all $A \in \mathfrak{H} \cap [D]$, and $n \geq n_0$, Assumption 1 yields

$$\Lambda_D(A) = \frac{\Lambda_{W_n}(A)}{\Lambda_{W_n}([D])} = \frac{\Lambda_{W_n}(A)}{\Lambda_{W_n}([V_0])}\frac{\Lambda_{W_n}([V_0])}{\Lambda_{W_n}([D])}.$$

Now, due to (8), we obtain

$$\Lambda_D(A) = \lim_{n\to\infty} \frac{\Lambda_{W_n}(A)}{\Lambda_{W_n}([V_0])}\frac{\Lambda_{W_n}([V_0])}{\Lambda_{W_n}([D])} = \lim_{n\to\infty} \frac{\Lambda_{W_n}(A)}{\Lambda_{W_n}([V_0])} \lim_{n\to\infty} \frac{\Lambda_{W_n}([V_0])}{\Lambda_{W_n}([D])} = \frac{\Lambda(A)}{\Lambda([D])}.$$

Proof of Lemma 2. For continuous time, (2) from Assumption 2 and (5) from Lemma 1 imply that, for all $V, W \in \mathfrak{P}$, $V \subseteq W$, and all $s \geq 0$,

$$\frac{\mathbb{P}(\tau(V) \leq s + \mathrm{d}s \mid \tau(V) \geq s)}{\Lambda([V])} = \frac{\mathbb{P}(\tau(W) \leq s + \mathrm{d}s \mid \tau(W) \geq s)}{\Lambda([W])} + o(\mathrm{d}s).$$

Now, for arbitrary $V_1, V_2 \in \mathfrak{P}$, and a window W with $V_i \subset W$, $i = 1, 2$, we have

$$\frac{\mathbb{P}(\tau(V_1) \leq s + \mathrm{d}s \mid \tau(V_1) \geq s)}{\Lambda([V_1])} = \frac{\mathbb{P}(\tau(V_2) \leq s + \mathrm{d}s \mid \tau(V_2) \geq s)}{\Lambda([V_2])} + o(\mathrm{d}s),$$

i.e. up to $o(\mathrm{d}s)$ the ratio $\mathbb{P}(\tau(V) \leq s + \mathrm{d}s \mid \tau(V) \geq s)/\Lambda([V])$ does not depend on $V \in \mathfrak{P}$, but only on s, and we can define, for any such V,

$$\gamma(s) = \lim_{ds\downarrow 0} \frac{\mathbb{P}(\tau(V) \leq s + \mathrm{d}s \mid \tau(V) \geq s)}{\mathrm{d}s\,\Lambda([V])}$$

and this yields (6).

Now consider a discrete-time process. Let $W_1 \subset W_2 \subset \cdots$ be a sequence of windows with $\bigcup_{n=1}^{\infty} W_n = \mathbb{R}^d$. Then for any $s \geq 1$, and $m, n \in \mathbb{N}$,

$$\begin{aligned}\frac{\mathbb{P}(\tau(W_m) = s_0)}{\Lambda([W_m])} &= \frac{\mathbb{P}(\tau(W_m) = s_0 \mid \tau(W_m) \geq s_0)}{\Lambda([W_m])} \\ &= \frac{\mathbb{P}(\tau(W_n) = s_0 \mid \tau(W_n) \geq s_0)}{\Lambda([W_n])} \\ &= \frac{\mathbb{P}(\tau(W_n) = s_0)}{\Lambda([W_n])}.\end{aligned}$$

Because $\lim_{n\to\infty} \Lambda([W_n]) = \Lambda(\mathcal{H}))$ the probability $\mathbb{P}(\tau(W_m) = s_0)$ would be 0 if $\Lambda(\mathcal{H})) = \infty$. Therefore, in a consistent discrete-time cell division process, the measure Λ must be finite.

Proof of Lemma 3. Let be $V, W, C \in \mathfrak{P}$, $V \subset W$, $C \subset W$, $\operatorname{int}(C \cap V) \neq \varnothing$, and $C \not\subseteq V$. Regarding the construction of $((Y \wedge V)_t,\ t \geq 0)$ inside V the birth time of V is $\beta(V) = 0$. This does not change if at time $\tau(W)$ the window W is divided by a hyperplane $h(W)$ with $h(W) \cap V = \varnothing$, an event with positive probability. But with respect to the construction of $((Y \wedge W)_t,\ t \geq 0)$ inside W, the smaller window V is now a subset of a cell, $C_{1,V}^{W}$ say, with birth time $\tau(W)$. Hence, the independence assumptions for the lifetimes and the dividing hyperplanes and (2), (3), and (5) yield

$$\gamma(s)\Lambda([V]) = \gamma(s - \tau(W))\Lambda([C_{1,V}^{W}])\frac{\Lambda([V])}{\Lambda([C_{1,V}^{W}])} = \gamma(s - \tau(W))\Lambda([V]).$$

Because $0 < \Lambda([V]) < \infty$, we obtain $\gamma(s) = \gamma(s - t)$ for all $0 < t < s$ with $\gamma(t) > 0$.

Proof of Theorem 2. Consider two windows $V \subset W$. Since both $((Y \wedge V)_t,\ t > 0)$ and $((Y \wedge W)_t,\ t > 0)$ are Markov processes, starting with $(Y \wedge V)_0 = V = W \cap V = (Y \wedge W)_0 \wedge V$ it suffices to prove the consistency of the transition probabilities. Assume that at a time $t > 0$ the tessellation $(Y \wedge W)_t$ consists of the cells $C_1, \ldots, C_{n_t}$, and, hence, $(Y \wedge W)_t \wedge V$ of those $C_i \cap V$ which have a nonempty interior, $i = 1, \ldots, n_t$. Due to the independence assumptions in Definition 3 it is sufficient to consider the single cells $C_i \cap V$. For the construction inside V the following holds:

$$\begin{aligned}&\mathbb{P}(\tau(C_i \cap V) \leq t + \mathrm{d}t, h(C_i \cap V) \in \mathrm{d}h) + o(\mathrm{d}t) \\ &\quad = \gamma\Lambda([C_i \cap V])\,\mathrm{d}t\mathbf{1}_{\{C_i \cap V\}}(h)\frac{\Lambda(\mathrm{d}h)}{\Lambda([C_i \cap V])} \\ &\quad = \gamma\,\mathrm{d}t\mathbf{1}_{\{C_i \cap V\}}(h)\Lambda(\mathrm{d}h).\end{aligned}$$

On the other hand, with respect to the construction inside W,

$$\begin{aligned}&\mathbb{P}(\tau(C_i) \leq t + \mathrm{d}t, h(C_i) \in \mathrm{d}h, h(C_i) \cap (C_i \cap V) \neq \varnothing) + o(\mathrm{d}t) \\ &\quad = \gamma\Lambda([C_i])\,\mathrm{d}t\mathbf{1}_{\{C_i \cap V\}}(h)\frac{\Lambda(\mathrm{d}h)}{\Lambda([C_i])} \\ &\quad = \gamma\,\mathrm{d}t\mathbf{1}_{\{C_i \cap V\}}(h)\Lambda(\mathrm{d}h),\end{aligned}$$

which yields the proposition.

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