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HYPERBOLIC LINEAR INVARIANCE AND HYPERBOLIC k-CONVEXITY

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Abstract

Pommerenke initiated the study of linearly invariant families of locally schlicht holomorphic functions defined on the unit disk $\mathbb{D}$. The concept of linear invariance has proved fruitful in geometric function theory. One aspect of Pommerenke's work is the extension of certain results from classical univalent function theory to linearly invariant functions. We propose a definition of a related concept that we call hyperbolic linear invariance for locally schlicht holomorphic functions that map the unit disk into itself. We obtain results for hyperbolic linearly invariant functions which generalize parts of the theory of bounded univalent functions. There are many similarities between linearly invariant functions and hyperbolic linearly invariant functions, but some new phenomena also arise in the study of hyperbolic linearly invariant functions.

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1. Introduction

This paper is a sequel to [6], in which we considered euclidean linearly invariant functions, an idea due to Pommerenke ([14, 15]). The concept of euclidean linear invariance is defined for locally schlicht holomorphic functions from the unit disk $\mathbb{D}$ with hyperbolic geometry to the complex plane $\mathbb{C}$ with euclidean geometry; let $\mathscr{L}$ denote the family of locally univalent holomorphic functions defined on $\mathbb{D}$. (We shall employ the words 'univalent' and 'schlicht' interchangeably). The euclidean linear invariant order of a function f in $\mathscr{L}$ is defined by

$$(1)\quad \alpha(f) = \sup\left\{ \frac{|\tilde{f}''(0)|}{2|\tilde{f}'(0)|} : \tilde{f} = R \circ f \circ S, \text{ where } R \in \mathrm{Euc}(\mathbb{C}),\ S \in \mathrm{Aut}(\mathbb{D}) \text{ and } \tilde{f}(0) = 0 \right\}.$$

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Here $\mathrm{Aut}(\mathbb{D})$ is the group of conformal automorphisms of $\mathbb{D}$ and $\mathrm{Euc}(\mathbb{C})$ is the group of euclidean motions of $\mathbb{C}$. Also, $\mathscr{F}(\alpha)$ is the family of f in $\mathscr{L}$ such that $\alpha(f) \le \alpha$.

In the present paper we shall consider bounded locally univalent functions. More precisely, we shall consider locally schlicht functions $f : \mathbb{D} \to \mathbb{D}$; let $\mathscr{L}_h$ denote the family of all such functions. (We often use the subscript 'h' to distinguish a concept in this paper from the analogous concept in [6].) From a geometric perspective we are considering functions with hyperbolic geometry on $\mathbb{D}$ for both the domain and the range. It is not immediate how to extend the concept of linear invariance to functions in $\mathscr{L}_h$. Any definition of hyperbolic linear invariance should lead to analogs for some of the important known results for euclidean linearly invariant functions. In particular, we should obtain generalizations of results for bounded univalent functions just as euclidean linear invariance extends much of the classical theory of univalent functions. We propose the following definition for the hyperbolic linearly invariant order for functions in $\mathscr{L}_h$:

$$(2)\qquad \alpha_h(f) = \sup\left\{\frac{|\tilde{f}''(0)|}{2|\tilde{f}'(0)|(1-|\tilde{f}'(0)|)} : \tilde{f} = R \circ f \circ S,\ \text{where } R,\ S \in \mathrm{Aut}(\mathbb{D}) \text{ and } \tilde{f}(0) = 0\right\}.$$

A function f in $\mathscr{L}_h$ is called *hyperbolically linearly invariant* provided its order $\alpha_h(f)$ is finite. Let $\mathscr{F}_h(\alpha)$ be the class of hperbolically linearly invariant functions f such that $\alpha_h(f) \le \alpha$.

In this paper we establish analogs for many of the results for euclidean linearly invariant functions in [6]. The analogies are not always obvious since we are now dealing with hyperbolic rather than euclidean geometry on the range of the functions. We give a purely geometric interpretation for the linear invariant order in terms of uniform local hyperbolic k-convexity. We also relate hyperbolic linear invariance to uniform local univalence in the hyperbolic sense. Growth, distortion and covering theorems for the class $\mathscr{F}_h(\alpha)$ are obtained, which we know are sharp in certain cases.

We would like to thank the referee for carefully reading the paper and making a number of useful suggestions which helped to clarify the exposition.

2. Preliminaries

In this section, our purpose is to define certain concepts employed in this paper and recall some basic facts that will be needed.

We begin with a brief discussion of hyperbolic geometry on $\mathbb{D}$ and related ideas. The density for the hyperbolic metric on $\mathbb{D}$ is $\lambda_{\mathbb{D}}(z) = 1/(1-|z|^2)$. The hyperbolic length of a curve γ in $\mathbb{D}$ is

$$\text{length}_h(\gamma) = \int_\gamma \frac{|dz|}{1-|z|^2}.$$

The hyperbolic distance function on $\mathbb{D}$ induced by this metric is

$$d_h(a,b) = \frac{1}{2}\log\frac{1+|(a-b)/(1-a\bar{b})|}{1-|(a-b)/(1-a\bar{b})|}.$$

The hyperbolic disk in $\mathbb{D}$ with hyperbolic center $a \in \mathbb{D}$ and hyperbolic radius ρ, $0 < \rho \le \infty$, is defined by $D_h(a,\rho) = \{z \in \mathbb{D} : d_h(a,z) < \rho\}$.

The hyperbolic curvature of a path $\gamma : z = z(t)$ in $\mathbb{D}$ is

$$\kappa_h(z,\gamma) = (1-|z|^2)\kappa_e(z,\gamma) + \text{Im}\left\{\frac{2\overline{z(t)}z'(t)}{|z'(t)|}\right\}, \tag{3}$$

where $\kappa_e(z,\gamma)$ denotes the euclidean curvature,

$$\kappa_e(z,\gamma) = \frac{1}{|z'(t)|}\,\text{Im}\left\{\frac{z''(t)}{z'(t)}\right\}.$$

Note that $\kappa_e(0,\gamma) = \kappa_h(0,\gamma)$; in words, the hyperbolic curvature and euclidean curvature coincide at the origin. Recall that hyperbolic curvature is invariant under the group $\text{Aut}(\mathbb{D})$, that is $\kappa_h(S(z), S\circ\gamma) = \kappa_h(z,\gamma)$ for any $S \in \text{Aut}(\mathbb{D})$. For more details see [2, 4, 10]. Later, we shall need to know that if γ is the positively oriented boundary of the hyperbolic disk $D_h(a,\rho)$, then $\kappa_h(z,\gamma) = 2\coth(2\rho)$. Also, the formula (3) for hyperbolic curvature readily yields that $\kappa_e(z,\gamma) \ge 0$ when $\kappa_h(z,\gamma) \ge 2$.

Next we recall the notion of hyperbolic k-convexity. A region Ω in $\mathbb{D}$ is *hyperbolically k-convex* (*relative to* $\mathbb{D}$) if for any pair of distinct points $a, b \in \Omega$ there exist two shortest arcs of constant hyperbolic curvature k in $\mathbb{D}$ connecting a and b. Note that curves in $\mathbb{D}$ of constant hyperbolic curvature are circular arcs. This notion was introduced independently by Flinn and Osgood [2] and Mejia [8]. If Ω is a region in $\mathbb{D}$ such that $\partial\Omega$ is a closed Jordan curve of class C^2, then Ω is hyperbolically k-convex if and only if $\kappa_h(z,\partial\Omega) \ge k$ for all $z \in \partial\Omega$ (see [2, 8, 9]). In particular, $D_h(a,\rho)$ is hyperbolically k-convex if and only if $2\coth(2\rho) \ge k$, a consequence of which is that the unit disk is hyperbolically 2-convex. A univalent function f in $\mathscr{L}_h$ is called *hyperbolically k-convex* provided the image $f(\mathbb{D})$ is hyperbolically k-convex. Hyperbolically k-convex functions and related topics were investigated in [5, 9].

We now turn to hyperbolic linear invariance and related topics. First, we point out an apparent problem with the definition of the hyperbolic linearly invariant order. In the definition (2) of the quantity $\alpha_h(f)$, the expression $|\tilde{f}''(0)|/2|\tilde{f}'(0)|(1-|\tilde{f}'(0)|)$

does not make sense if f is a conformal automorphism of $\mathbb{D}$. In fact, if f is a conformal automorphism of $\mathbb{D}$, then the function $\tilde{f}$ in (2) is a rotation of $\mathbb{D}$ so that $|\tilde{f}'(0)| = 1$ and $\tilde{f}''(0) = 0$. We define $\alpha_h = 1$ when $f \in \mathrm{Aut}(\mathbb{D})$. This convention is made plausible by the fact that $\alpha_h(cz) = 1$ for any constant c with $0 < |c| < 1$. In addition, we later show (Theorem 3) that $\alpha_h(f) \geq 1$ and $\alpha_h(f) = 1$ if and only if f is hyperbolically 2-convex which further justifies our convention for the hyperbolic linearly invariant order of a conformal automorphism of $\mathbb{D}$.

Note that $\alpha_h(f) = \alpha_h(R \circ f \circ S)$ for all R and S in $\mathrm{Aut}(\mathbb{D})$.

EXAMPLE 1. Let $f \in \mathscr{L}_h$ be univalent and suppose $f(\mathbb{D})$ is hyperbolically 2-convex. Then $\alpha_h(f) \leq 1$. This is an immediate comsequence of the inequality

$$|f''(0)| \leq 2|f'(0)|(1 - |f'(0)|),$$

which holds for a normalized ($f(0) = 0$) hyperbolically 2-convex function f [10, Theorem 5]. Here the only extremal functions are the rotations of

$$k_{1,\beta}(z) = \frac{\beta z}{1 - (1 - \beta)z} \qquad (0 < \beta \leq 1). \tag{4}$$

Thus $\mathscr{F}_h(1)$ contains all hyperbolically 2-convex functions, which is analogous to $\mathscr{F}(1)$ consisting of all euclidean convex functions. We point out that hyperbolic 2-convexity, rather than hyperbolic convexity, seems to be the appropriate analog for euclidean convexity in the setting of hyperbolic linear invariance. We note that $\alpha(k_{1,\beta}) = 1$, independent of β.

EXAMPLE 2. If $f \in \mathscr{L}_h$ is univalent, then $\alpha_h(f) \leq 2$. This follows directly from an inequality of Pick [13]

$$|f''(0)| \leq 4|f'(0)|(1 - |f'(0)|),$$

which holds for a normalized ($f(0) = 0$) univalent function mapping $\mathbb{D}$ into itself with equality if and only if f is a rotation of

$$k_{2,\beta}(z) = \frac{\sqrt{(1-z)^2 + 4\beta z} - (1-z)}{\sqrt{(1-z)^2 + 4\beta z} + (1-z)} \qquad (0 < \beta \leq 1). \tag{5}$$

See [2] for an elementary proof of Pick's inequality. Hence, $\mathscr{F}_h(2)$ contains all univalent functions in $\mathscr{L}_h$, just as $\mathscr{F}(2)$ contains all univalent functions in $\mathscr{L}$. We remark that $\alpha(k_{2,\beta}) = 2$, independent of β.

The similarity of the coefficient bounds for hyperbolically 2-convex functions and bounded univalent functions in Examples 1 and 2 was a strong motivation for the definition of the hyperbolic linear invariant order.

DEFINITION 1. For a holomorphic function f in $\mathbb{D}$ and a point a in $\mathbb{D}$, we let $\rho(a, f)$ be the hyperbolic radius of the largest hyperbolic disk in $\mathbb{D}$ centred at a in which f is univalent. Note that $\rho(a, f)$ can be zero or infinite. Define $\rho(f) = \inf\{\rho(a, f) : a \in \mathbb{D}\}$; we say f is *uniformly locally univalent* (*in the hyperbolic sense*) if $\rho(f) > 0$.

Next we define uniform local hyperbolic k-convexity.

DEFINITION 2. Let $f \in \mathscr{L}_h$. For $a \in \mathbb{D}$, the radius of hyperbolic k-convexity, $k \geq 0$, at a is given by

$$\rho_{hk}(a, f) = \sup\{\rho \leq \rho(a, f) : f(D_h(a, \rho)) \text{ is hyperbolically } k\text{-convex}\}.$$

The *uniform radius of hyperbolic k-convexity*, $k \geq 0$, of f is

$$\rho_{hk}(f) = \inf\{\rho_{hk}(a, f) : a \in \mathbb{D}\};$$

a function f is called *uniformly locally hyperbolically k-convex* provided $\rho_{hk}(f) > 0$. When no confusion results, we simply write $\rho_k(f)$ in place of $\rho_{hk}(f)$.

We observe that $\rho_{hk}(f) = \rho_{hk}(S \circ f \circ T)$ for all S and $T \in \text{Aut}(\mathbb{D})$. Note that $k_1 \leq k_2$ implies that $\rho_{k_1}(f) \geq \rho_{k_2}(f)$. In particular, $\rho_k(f) \leq \rho_2(f)$ for $k \geq 2$. Observe that $\rho_{hk}(f) \leq \rho(f)$.

DEFINITION 3. Let $f \in \mathscr{L}_h$. We say that f is *hyperbolic k-convexity preserving* if f maps every hyperbolically k-convex subset of $\mathbb{D}$ injectively onto a hyperbolically k-convex set. Define

$$k_h(f) = \inf\{k \geq 2 : f \text{ is hyperbolic } k\text{-convexity preserving}\}.$$

The reason that we can restrict k to be at least two in this definition will be made clear at the end of the next section. In fact, the only functions in $\mathscr{L}_h$ which are hyperbolic k-convexity preserving for $0 \leq k < 2$ are the conformal automorphisms of $\mathbb{D}$. Obviously, $k_h(f) = k_h(R \circ f \circ S)$ for all R and S in $\text{Aut}(\mathbb{D})$.

3. Hyperbolically invariant differential operators

We introduce two invariant differential operators for holomorphic functions mapping the unit disk into itself.

DEFINITION 4. Suppose f is holomorphic in $\mathbb{D}$ and $f(\mathbb{D}) \subset \mathbb{D}$. Set

$$D_{h1} f(z) = \frac{(1 - |z|^2) f'(z)}{1 - |f(z)|^2} \tag{6}$$

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and

$$(7)\quad D_{h2}f(z) = \frac{(1-|z|^2)^2 f''(z)}{1-|f(z)|^2} + \frac{2(1-|z|^2)^2\overline{f(z)}f'(z)^2}{(1-|f(z)|^2)^2} - \frac{2\bar{z}(1-|z|^2)f'(z)}{1-|f(z)|^2}.$$

Note that the Schwarz-Pick Lemma asserts that either $|D_{h1}f(z)| < 1$ for all $z \in \mathbb{D}$, or else f is a conformal automorphism of $\mathbb{D}$ and $|D_{h1}f(z)| \equiv 1$. Also, if $f(0) = 0$, then $D_{h1}f(0) = f'(0)$ and $D_{h2}f(0) = f''(0)$. For $a \in \mathbb{D}$, set $T_a(z) = (z-a)/(1-\bar{a}z)$. Then T_a is a conformal automorphism of $\mathbb{D}$ which sends a to 0, $T_{-a} = T_a^{-1}$, $T_a'(0) = 1-|a|^2$, and $T_a'(a) = 1/(1-|a|^2)$. If $\tilde{f} = T_{f(a)} \circ f \circ T_{-a}$, then $\tilde{f}$ is a holomorphic function mapping $\mathbb{D}$ into itself with $\tilde{f}(0) = 0$. It is straightforward to verify that $D_{h1}f(a) = \tilde{f}'(0)$ and $D_{h2}f(a) = \tilde{f}''(0)$. Therefore, the hyperbolic linearly invariant order of a function f in $\mathscr{L}_h$ can be expressed concisely in terms of these two differential operators as

$$(8)\qquad \alpha_h(f) = \sup\left\{\frac{|D_{h2}f(z)|}{2|D_{h1}f(z)|(1-|D_{h1}f(z)|)} : z \in \mathbb{D}\right\},$$

with the restriction that $f \notin \mathrm{Aut}(\mathbb{D})$.

We next establish an invariance property of these differential operators under the group $\mathrm{Aut}(\mathbb{D})$.

THEOREM 1. *Suppose f is holomorphic in $\mathbb{D}$ and $f(\mathbb{D}) \subset \mathbb{D}$. If $R, S \in \mathrm{Aut}(\mathbb{D})$, then*

$$|D_{hj}(R \circ f \circ S)| = |D_{hj}(f)| \circ S \qquad (j = 1,\ 2).$$

PROOF. We establish this result when $j = 1$; the case $j = 2$ is basically the same. Fix $a \in \mathbb{D}$ and set $b = S(a)$, $g = R \circ f \circ S$. We have $D_{h1}f(b) = \tilde{f}'(0)$, where $\tilde{f} = T_{f(b)} \circ f \circ T_{-b}$ and $D_{h1}g(a) = \tilde{g}'(0)$, where $\tilde{g} = T_{g(a)} \circ g \circ T_{-a}$. Note that both functions $\tilde{f}$ and $\tilde{g}$ fix the origin. It suffices to show that $|\tilde{f}'(0)| = |\tilde{g}'(0)|$. This is elementary because $\tilde{g} = \tilde{R} \circ \tilde{f} \circ \tilde{S}$, where $\tilde{R} = T_{g(a)} \circ R \circ T_{-f(b)}$ and $\tilde{S} = T_b \circ S \circ T_{-a}$ are both rotations of the unit disk about the origin.

There is a close connection between these invariant differential operators and hyperbolic curvature. We derive a formula for the change in hyperbolic curvature under a function $\mathscr{L}_h$ which establishes the connection.

THEOREM 2. *Suppose $f \in \mathscr{L}_h$. Then*

$$\kappa_h(f(z), f\circ\gamma)|D_{h1}f(z)| = \kappa_h(z,\gamma) + \mathrm{Im}\left\{\frac{D_{h2}f(z)}{D_{h1}f(z)}\frac{z'(t)}{|z'(t)|}\right\}.$$

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PROOF. Fix $a \in \mathbb{D}$. Set $\tilde{f} = T_{f(a)} \circ f \circ T_{-a}$ and $\tilde{\gamma} = T_a \circ \gamma$. If γ is parametrized by $z = z(t)$ with $z(t_0) = a$, then $\tilde{\gamma}$ is parametrized by $\tilde{z} = \tilde{z}(t)$ and $\tilde{z}(t_0) = 0$. Also, we note that the unit tangent to γ at a and the unit tangent to $\tilde{\gamma}$ at 0 are the same since $T_a'(a) > 0$; in symbols, $z'(t_0)/|z'(t_0)| = \tilde{z}'(t_0)/|\tilde{z}'(t_0)|$. Because hyperbolic curvature is invariant under $\mathrm{Aut}(\mathbb{D})$ and $\tilde{f} \circ \tilde{\gamma} = T_{f(a)} \circ f \circ \gamma$, it suffices to show that

$$\kappa_h(\tilde{f}(0), \tilde{f} \circ \tilde{\gamma})|D_{h1}\tilde{f}(0)| = \kappa_h(0, \tilde{\gamma}) + \mathrm{Im}\left\{\frac{D_{h2}\tilde{f}(0)}{D_{h1}\tilde{f}(0)} \frac{\tilde{z}'(t_0)}{|\tilde{z}'(t_0)|}\right\},$$

or equivalently,

$$\kappa_e(\tilde{f}(0), \tilde{f} \circ \tilde{\gamma})|\tilde{f}'(0)| = \kappa_e(0, \tilde{\gamma}) + \mathrm{Im}\left\{\frac{\tilde{f}''(0)}{\tilde{f}'(0)} \frac{\tilde{z}'(t_0)}{|\tilde{z}'(t_0)|}\right\}.$$

But this latter identity is just the formula for the change of euclidean curvature under a locally schlicht holomorphic function [2].

THEOREM 3. *Let $f \in \mathscr{L}_h$. Then $1 \le \alpha(f) \le \alpha_h(f)$, where $\alpha(f)$ denotes the euclidean linearly invariant order of f. Also, $\alpha_h(f) = 1$ if and only if f is univalent and $f(\mathbb{D})$ is hyperbolically 2-convex.*

PROOF. First, we show that $1 \le \alpha_h(f) = \alpha$. There is no harm in assuming $f(0) = 0$. Suppose that $\alpha < 1$. Then f is not a conformal automorphism of $\mathbb{D}$ and

$$\frac{|D_{h2}f(z)|}{|D_{h1}f(z)|} \le 2\alpha(1 - |D_{h1}f(z)|).$$

This gives

$$\left|(1-|z|^2)\frac{f''(z)}{f'(z)} - 2\bar{z} + \frac{2(1-|z|^2)\overline{f(z)}f'(z)}{1-|f(z)|^2}\right| \le 2\alpha\left(1 - \frac{(1-|z|^2)|f'(z)|}{1-|f(z)|^2}\right),$$

$$\left|\frac{zf''(z)}{f'(z)} + 1 - \frac{1+|z|^2}{1-|z|^2}\right| \le \frac{2\alpha|z|}{1-|z|^2}\left(1 - \frac{(1-|z|^2)|f'(z)|}{1-|f(z)|^2}\right) + \frac{2|zf(z)f'(z)|}{1-|f(z)|^2},$$

$$\left|\frac{zf''(z)}{f'(z)} + 1 - \frac{1+|z|^2}{1-|z|^2}\right| \le \frac{2|z|}{1-|z|^2}\left(\alpha + (|z|-\alpha)\frac{(1-|z|^2)|f'(z)|}{1-|f(z)|^2}\right).$$

In the final step we used the inequality $|f(z)| \le |z|$ which is valid because $f(0) = 0$ and f maps $\mathbb{D}$ into itself. Hence, if $\alpha < |z| < 1$, then

$$\left|\frac{zf''(z)}{f'(z)} + 1 - \frac{1+|z|^2}{1-|z|^2}\right| < \frac{2|z|}{1-|z|^2}(\alpha + (|z|-\alpha)) = \frac{2|z|^2}{1-|z|^2}$$

since $|D_{h1}f(z)| < 1$. For $z \in \mathbb{D}$ define

$$u(z) = 1 + \operatorname{Re} \frac{zf''(z)}{f'(z)}.$$

The function u is harmonic in $\mathbb{D}$ and $u(0) = 1$. For $\alpha < |z| < 1$, we have

$$u(z) > \frac{1+|z|^2}{1-|z|^2} - \frac{2|z|^2}{1-|z|^2} = 1.$$

The preceding inequality implies that $u(z) > 1$ for $z \in \mathbb{D}$ by the minimum principle for harmonic functions. This contradicts $u(0) = 1$, so we must have $\alpha \geq 1$.

Next we show that $1 \leq \alpha(f) \leq \alpha_h(f)$. Recall that $1 \leq \alpha(f)$ and that [14]

$$\alpha(f) = \sup \left\{ \left| (1-|z|^2)\frac{f''(z)}{2f'(z)} - \bar{z} \right| : z \in \mathbb{D} \right\}.$$

We have

$$\begin{aligned}
\left| (1-|z|^2)\frac{f''(z)}{2f'(z)} - \bar{z} \right| &\leq \left| \frac{D_{h2}f(z)}{2D_{h1}f(z)} \right| + |f(z)||D_{h1}f(z)| \\
&\leq \alpha_h(f)(1 - |D_{h1}f(z)|) + |f(z)||D_{h1}f(z)| \\
&\leq \alpha_h(f) + (|f(z)| - \alpha_h(f))|D_{h1}f(z)| \leq \alpha_h(f)
\end{aligned}$$

since $\alpha_h(f) \geq 1$ and $|f(z)| < 1$. This yields the desired result.

From Example 1 it now follows that $\alpha_h(f) = 1$ if f is univalent in $\mathbb{D}$ and $f(\mathbb{D})$ is hyperbolically 2-convex. All that remains is to show that if $\alpha_h(f) = 1$, then f is univalent in $\mathbb{D}$ and $f(\mathbb{D})$ is hyperbolically 2-convex. If $\alpha_h(f) = 1$, then we also have $\alpha(f) = 1$ so f is univalent in $\mathbb{D}$ and $f(\mathbb{D})$ is convex [14] (also see [6]). In order to show that $f(\mathbb{D})$ is hyperbolically 2-convex, it is sufficient to prove that $f(D_h(0, \rho))$ is hyperbolically 2-convex for all $\rho > 0$, or equivalently, $\partial f(D_h(0, \rho))$ has hyperbolic curvature at least 2 for all $\rho > 0$ (see [2, 8, 9]). Let γ be the positively oriented circle of hyperbolic radius ρ centered at the origin. Then $\kappa_h(z, \gamma) = 2\coth(2\rho) > 2$ and from Theorem 2

$$\begin{aligned}
\kappa_h(f(z), f \circ \gamma)|D_{h1}f(z)| &\geq \kappa_h(z, \gamma) - \frac{|D_{h2}f(z)|}{|D_{h2}f(z)|} \\
&> 2 - 2\alpha_h(f)(1 - |D_{h1}f(z)|) = 2|D_{h1}f(z)|.
\end{aligned}$$

Hence, $\kappa_h(f(z), f \circ \gamma) > 2$. This completes the proof.

REMARK. Now we can readily explain why the restriction $k \geq 2$ in the definition of the quantity $k_h(f)$ can be imposed without loss of generality. The reason is that for

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$0 \le k < 2$ there are no functions in $\mathscr{L}_h$, except conformal automorphisms of $\mathbb{D}$, which map every hyperbolically k-convex subset of $\mathbb{D}$ injectively onto a hyperbolically k-convex set. Recall that by convention $\alpha_h(f) = 1$ for $f \in \mathrm{Aut}(\mathbb{D})$. Suppose f maps every hyperbolically k-convex subset of $\mathbb{D}$ injectively onto a hyperbolically k-convex set for some $k \in [0, 2)$ and $f \notin \mathrm{Aut}(\mathbb{D})$. Then for every $a \in \mathbb{D}$ there exists a circular arc $\gamma : z = z(t)$ of constant hyperbolic curvature k, which is part of the boundary of a hyperbolically k-convex subset of $\mathbb{D}$, such that $z(0) = a$ and

$$\mathrm{Im}\left\{\frac{D_{h2}f(a)}{D_{h1}f(a)}\frac{z'(0)}{|z'(0)|}\right\} = -\left|\frac{D_{h2}f(a)}{D_{h1}f(a)}\right|.$$

Hence, by making use of Theorem 2, we have

$$\begin{aligned} k|D_{h1}f(a)| &\le \kappa_h(f(a), f\circ\gamma)|D_{h1}f(a)| \\ &= \kappa_h(a,\gamma) + \mathrm{Im}\left\{\frac{D_{h2}f(a)}{D_{h1}f(a)}\frac{z'(0)}{|z'(0)|}\right\} = k - \left|\frac{D_{h2}f(a)}{D_{h1}f(a)}\right|, \end{aligned}$$

so that

$$\frac{|D_{h2}f(a)|}{|D_{h1}f(a)|(1-|D_{h1}f(a)|)} \le k.$$

Because $a \in \mathbb{D}$ is arbitrary, this, in conjunction with (8), implies that $\alpha_h(f) \le k/2 < 1$, a contradiction to Theorem 3.

4. Relationships between hyperbolic linear invariance and hyperbolic k-convexity

We now investigate connections between the quantities $\alpha_h(f)$, $k_h(f)$ and $\rho_{hk}(f)$; the latter quantity we shall write simply as $\rho_k(f)$.

THEOREM 4. *Suppose* $f \in \mathscr{L}_h$. *Then* $k_h(f) = 2\coth(2\rho_{k_h(f)}(f))$.

PROOF. In this proof we will write $k(f)$ in place of $k_h(f)$. First, we show that $2\coth(2\rho_{k(f)}(f)) \le k(f)$. We may assume that $k(f) < \infty$. For any $\epsilon > 0$, we determine $\rho = \rho(\epsilon)$ from the equation $2\coth(2\rho) = k(f) + \epsilon$. Then $D_h(a, \rho)$ is hyperbolically $(k(f) + \epsilon)$-convex since $\partial D_h(a, \rho)$ has constant hyperbolic curvature $k(f)+\epsilon$. It follows from Definition 3 that f is injective in the hyperbolic disk $D_h(a, \rho)$ and $f(D_h(a, \rho))$ is hyperbolically $k(f)$-convex. This implies that $\rho_{k(f)}(f) \ge \rho$, which is equivalent to the inequality $2\coth(2\rho_{k(f)}(f)) \le k(f) + \epsilon$. Because $\epsilon > 0$ is arbitrary, this establishes the desired inequality.

Next we show that $k(f) \le 2\coth(2\rho_{k(f)}(f))$. We may assume that $\rho_{k(f)}(f) > 0$. Set $k = 2\coth(2\rho_{k(f)}(f)) \ge 2$. Suppose Ω is any subset of $\mathbb{D}$ which is hyperbolically

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k-convex. Since Ω is hyperbolically k-convex ($k \geq 2$), Ω is contained in a hyperbolic disk of radius $\rho_{k(f)}(f)$ (see [8, 9]). In particular, f is univalent on Ω. We shall show that $f(\Omega)$ is also hyperbolically k-convex. Because f is injective on Ω, it suffices to show that for any $a, b \in \Omega$, the points $f(a)$ and $f(b)$ lie in a hyperbolically k-convex set which is contained in $f(\Omega)$. Consider distinct points $a, b \in \Omega$. Then there exist two closed hyperbolic disks Δ_1 and Δ_2 with hyperbolic radii $\rho_{k(f)}(f)$ such that a, $b \in \partial\Delta_j$ ($j = 1, 2$) and $H_k[a, b] \equiv \Delta_1 \cap \Delta_2 \subset \Omega$. As $f(\Delta_j)$ is hyperbolically $k(f)$-convex, the hyperbolic lens-shaped region $H_k[f(a), f(b)]$ is contained in $f(\Delta_j)$ ($j = 1, 2$). Let Γ be any path in $H_k[f(a), f(b)]$ from $f(a)$ to $f(b)$. Then there is a path γ_j in Δ_j from a to b such that $f \circ \gamma_j = \Gamma$ ($j = 1, 2$). Since f is a local homeomorphism and γ_1, γ_2 are both paths from a to b, the condition $f \circ \gamma_1 = f \circ \gamma_2$ implies that $\gamma_1 = \gamma_2 = \gamma$. Thus, γ is contained in $\Delta_1 \cap \Delta_2$, so $\Gamma = f \circ \gamma$ lies in $f(\Delta_1 \cap \Delta_2)$. It follows that $H_k[f(a), f(b)] \subset f(\Delta_1 \cap \Delta_2) \subset f(\Omega)$. This shows that $f(\Omega)$ is hyperbolically $k(f)$-convex. Hence $k(f) \leq k$.

THEOREM 5. *Suppose $f \in \mathscr{L}_h$. Then $\alpha_h(f) = \coth(2\rho_{k_h(f)}(f))$.*

PROOF. In this proof we again simply write $k_h(f)$ as $k(f)$. From Theorem 3 the desired equality holds if $\alpha_h(f) = 1$. Henceforth, we assume $\alpha_h(f) > 1$. In particular, $f \notin \mathrm{Aut}(\mathbb{D})$.

First, we show that $\alpha_h(f) \leq \coth(2\rho_{k(f)}(f))$. Set $m = 2\coth(2\rho_{k(f)}(f))$ and fix $z_0 \in \mathbb{D}$. From (8), it suffices to show that

$$\frac{|D_{h2}f(z_0)|}{|D_{h1}f(z_0)|(1 - |D_{h1}f(z_0)|)} \leq 2\coth(2\rho_{k(f)}(f)).$$

We need only consider the case in which $D_{h2}f(z_0) \neq 0$. Then there is a unique point $a \in \mathbb{D}$ such that $z_0 \in \partial D_h(a, \rho_{k(f)}(f))$ and

$$\mathrm{Im}\left\{\frac{D_{h2}f(z_0)}{D_{h1}f(z_0)}\frac{z'(t_0)}{|z'(t_0)|}\right\} = -\left|\frac{D_{h2}f(z_0)}{D_{h1}f(z_0)}\right|,$$

where $\gamma : z = z(t)$ is a parametrization of $\partial D_h(a, \rho_{k(f)}(f))$ and $z(t_0) = z_0$. By making use of Theorem 4, we see that $f \circ \gamma$ is a hyperbolically m-convex curve of class C^2, so that [9] $\kappa_h(f(z_0), f \circ \gamma) \geq m$. Hence, by using Theorem 2,

$$\begin{aligned} m|D_{h1}f(z_0)| &\leq \kappa_h(f(z_0), f \circ \gamma)|D_{h1}f(z_0)| \\ &= \kappa_h(z_0, \gamma) + \mathrm{Im}\left\{\frac{D_{h2}f(z_0)}{D_{h1}f(z_0)}\frac{z'(t_0)}{|z'(t_0)|}\right\} = m - \left|\frac{D_{h2}f(z_0)}{D_{h1}f(z_0)}\right| \end{aligned}$$

or

$$\frac{|D_{h2}f(z_0)|}{|D_{h1}f(z_0)|(1 - |D_{h1}f(z_0)|)} \leq m = 2\coth(2\rho_{k(f)}(f)).$$

Next, we establish $\coth(2\rho_{k(f)}(f)) \leq \alpha_h(f)$. Determine ρ from $\coth(2\rho) = \alpha_h(f) \geq 1$ and set $\gamma = \partial D_h(a, \rho)$. Then for $z \in \gamma$, Theorem 2 and (8) yield

$$\begin{aligned}\kappa_h(f(z), f \circ \gamma)|D_{h1} f(z)| &= \kappa_h(z, \gamma) + \operatorname{Im}\left\{\frac{D_{h2} f(z)}{D_{h1} f(z)} \frac{z'(t)}{|z'(t)|}\right\} \\ &\geq 2\coth(2\rho) - \left|\frac{D_{h2} f(z)}{D_{h1} f(z)}\right| \\ &\geq 2(\coth(2\rho) - \alpha_h(f)(1 - |D_{h1} f(z)|)) \\ &= 2\alpha_h(f)|D_{h1} f(z)|.\end{aligned}$$

Thus, $\kappa_h(f(z), f \circ \gamma) \geq 2\alpha_h(f)$, so $f(D_h(a, \rho))$ is a hyperbolically $2\alpha_h(f)$-convex set. Because $a \in \mathbb{D}$ is arbitrary, f maps every hyperbolic disk of radius ρ onto a hyperbolically $2\alpha_h(f)$-convex set. Now, proceeding exactly as we did in the second part of the proof of Theorem 4, we can show that the set $f(\Omega)$ is hyperbolically $2\alpha_h(f)$-convex for each hyperbolically $2\alpha_h(f)$-convex subset Ω in $\mathbb{D}$. This implies, by Definition 3, the inequality $\kappa_h(f) \leq 2\alpha_h(f)$, which is equivalent to $\coth(2\rho_{k(f)}(f)) \leq \alpha_h(f)$ by Theorem 4.

COROLLARY 1. *Suppose $f \in \mathscr{L}_h$. Then $2\alpha_h(f) = k_h(f)$.*

This should be viewed as an analog of the result for euclidean linearly invariant functions which asserts that the euclidean linearly invariant order is simply related to the uniform local hyperbolic radius of euclidean convexity [6, Theorem 3]. It provides a simple hyperbolic geometric interpretation for the order of a hyperbolic linearly invariant function.

COROLLARY 2. *Let $f \in \mathscr{L}_h$. Then $\alpha_h(f) \geq \coth(2\rho_2(f))$. In particular, f is hyperbolically 2-convex in the disk $\left\{z : |z| < \alpha_h(f) - \sqrt{\alpha_h(f)^2 - 1}\right\}$.*

PROOF. As $k_h(f) \geq 2$, we have $\rho_2(f) \geq \rho_{k_h(f)}(f)$. Therefore,

$$\coth(2\rho_2(f)) \leq \coth(2\rho_{k_h(f)}(f)) = \alpha_h(f).$$

5. Uniform local univalence and hyperbolic linear invariance

It is known that a function f in $\mathscr{L}$ has finite linear invariant order if and only if it is uniformly locally univalent ([6, 12, 14]). This is a consequence of the following inequalities [14]

$$\coth(\rho(f)) \leq \alpha(f) \leq 2\coth(\rho(f)), \tag{9}$$

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where $\rho(f)$ is the hyperbolic radius of uniform local univalence of f given in Definition 1 and the euclidean linear invariant order $\alpha(f)$ is defined in (1). Improved bounds were given in [6, Theorem 6]. We now show that there is an analogous result for the hyperbolic linearly invariant order. In this section we relate the quantities $\rho(f)$ and $\alpha_h(f)$ in a similar fashion when f is in $\mathscr{L}_h$. We also connect these quantifies to the notion of hyperbolic starlikeness.

DEFINITION 5. A set $\Omega \subset \mathbb{D}$ is called *hyperbolically starlike* with respect to $a \in \Omega$ if for each $z \in \Omega$ the hyperbolic geodesic connecting a and z lies in Ω. When $a = 0$ this is equivalent to euclidean starlikeness. For $f \in \mathscr{L}_h$, let

$$\rho^*(a, f) = \sup\{\rho < \rho(a, f) : f(D_h(a, \rho)) \text{ is hyperbolically starlike with respect to } f(a)\}$$

and

$$\rho^*(f) = \inf\{\rho^*(a, f) : a \in \mathbb{D}\}.$$

A function $f \in \mathscr{L}_h$ is called *uniformly locally hyperbolically starlike* if $\rho^*(f) > 0$.

THEOREM 6. *Suppose* $f \in \mathscr{L}_h$. *Then* $\rho^*(f) \geq 2\rho_2(f)$.

PROOF. Set $\rho = \rho_2(f)$. There is nothing to prove if $\rho = 0$, so we may assume $\rho > 0$. It suffices to show that $\rho^*(a, f) \geq 2\rho$ for all $a \in \mathbb{D}$. Because of the invariance of both $\rho^*(a, f)$ and $\rho_2(f)$ under conformal automorphisms of $\mathbb{D}$, we need only consider the case in which $a = 0$ and $f(0) = 0$. Then we must show that $f(D_h(0, 2\rho))$ is starlike (in the euclidean sense) with respect to the origin. Actually, we show that the curve $f(\partial D_h(0, 2r))$ is starlike with respect to the origin for any $r \in (\rho/2, \rho)$. Fix any $z_0 \in \mathbb{D}$ with $d_h(0, z_0) = 2r$. It is enough to establish the inequality $\text{Re}\{z_0 f'(z_0)/f(z_0)\} \geq 0$ [1, pp. 41–42]. Select a on the radial segment $[0, z_0]$ with $d_h(0, a) = d_h(a, z_0)$ so that 0 and z_0 lie on $\partial D_h(a, r) = \gamma$. Because $r < \rho$ we know that $f(\overline{D_h(a, r)})$ is hyperbolically 2-convex. This implies that it is also euclidean convex, so the image of the circle γ is starlike with respect to $0 = f(0)$. If c is the euclidean center and R the euclidean radius of γ, then $z(t) = c + Re^{it}$, $0 \leq t \leq 2\pi$, is a parametrization of this circle. Since $f \circ \gamma$ is starlike with respect to the origin,

$$0 \leq \frac{d}{dt} \arg f(c + Re^{it}) = \text{Re}\left\{\frac{Re^{it} f'(c + Re^{it})}{f(c + Re^{it})}\right\}.$$

Select $t_0 \in [0, 2\pi]$ with $z(t_0) = z_0$. Then we have $Re^{it_0} = z_0 - c$. But $c = uz_0$ for some $u \in (0, 1)$, so $Re^{it_0} = (1 - u)z_0$. The preceding inequality implies that

$$\text{Re}\left\{\frac{z_0 f'(z_0)}{f(z_0)}\right\} \geq 0.$$

Now we give a quantitative version of the fact that a function f in $\mathscr{L}_h$ has finite hyperbolic linearly invariant order if and only if it is uniformly locally univalent.

THEOREM 7. *For* $f \in \mathscr{L}_h$, $\coth(\rho(f)) \le \alpha_h(f) \le 2\coth(\rho(f))$.

PROOF. The lower bound follows from Corollary 2 of Theorem 5 and Theorem 6 since $2\rho_2(f) \le \rho^*(f) \le \rho(f)$ implies that $\coth(\rho(f)) \le \coth(2\rho_2(f))$. Now we establish the upper bound. In view of (8), it suffices to show $|D_{h2}f|/(2|D_{h1}f|(1 - |D_{h1}f|)) \le 2\coth(\rho(f))$. Since the quantity $|D_{h2}f|/(2|D_{h1}f|(1 - |D_{h1}f|))$ is invariant when f is replaced by $S \circ f \circ T$ for any $S, T \in \operatorname{Aut}(\mathbb{D})$, it is enough to prove that

$$\frac{|f''(0)|}{2|f'(0)|(1 - |f'(0)|)} \le 2\coth(\rho(f))$$

when $f(0) = 0$ and f is univalent in $\{z : |z| < R\}$, where $R = \tanh(\rho(f))$. Define $g(z) = f(Rz)/R$. Schwarz's Lemma implies that g maps $\mathbb{D}$ into itself. Because g is holomorphic and univalent in $\mathbb{D}$, a result of Pick [13] (also see Example 2) gives

$$|g''(0)| \le 4|g'(0)|(1 - |g'(0)|),$$

or equivalently,

$$\frac{|f''(0)|}{2|f'(0)|(1 - |f'(0)|)} \le \frac{2}{R}.$$

This is the desired result.

COROLLARY 3. *Let* $f \in \mathscr{L}_h$. *Then* $1 \le \alpha(f) \le \alpha_h(f) < 2\alpha(f)$. *In words, a function in* $\mathscr{L}_h$ *is hyperbolically linearly invariant if and only if it is euclidean linearly invariant.*

PROOF. This is an immediate consequence of Theorem 3, Theorem 7 and the inequality $\coth(\rho(f)) \le \alpha(f)$ stated in (9), where $\alpha(f)$ is defined in (1).

REMARK. By making use of the inequality [10, Theorem 5] (also see Example 1)

$$|g''(0)| \le 2|g'(0)|(1 - |g'(0)|)$$

for hyperbolically 2-convex functions, we can show that

$$\coth(2\rho_2(f)) \le \alpha_h(f) \le \coth(\rho_2(f))$$

when $f \in \mathscr{L}_h$.

6. Growth, distortion and covering theorems

Before we derive various growth, distortion and covering theorems for the family $\mathscr{F}_h(\alpha)$, we introduce functions that we will prove to be extremal for some functionals over the class $\mathscr{F}_h(\alpha)$ in the special cases that $\alpha = 1, 2$. These functions are exact hyperbolic analogs of the basic extremal functions for the class $\mathscr{F}(\alpha)$ and include the functions in Examples 1 and 2 as special cases. For $\alpha \geq 1$ and $\beta \in (0, 1]$, we set

$$\mathscr{F}_h(\alpha, \beta) = \{f \in \mathscr{F}_h(\alpha) : f(0) = 0 \text{ and } f'(0) = \beta\}.$$

EXAMPLE 3. For $1 \leq \alpha \leq 2$ and $\beta \in (0, 1]$, let $k_{\alpha,\beta}$ be the normalized ($k_{\alpha,\beta}(0) = 0$ and $k'_{\alpha,\beta}(0) = \beta$) holomorphic function on $\mathbb{D}$ determined from the identity

$$\left(\frac{1 + k_{\alpha,\beta}(z)}{1 - k_{\alpha,\beta}(z)}\right)^{\alpha} - 1 = \beta\left[\left(\frac{1+z}{1-z}\right)^{\alpha} - 1\right]. \tag{10}$$

Clearly the function $k_{\alpha,\beta}$ is univalent. Note that $k_{1,\beta}$ and $k_{2,\beta}$ are the functions of Examples 1 and 2, respectively. From Examples 1 and 2, we know that $\alpha_h(k_{\alpha,\beta}) = \alpha$ in case α is 1 or 2. For $1 < \alpha < 2$, it is straightforward to verify that $\alpha_h(k_{\alpha,\beta}) \geq \alpha$. We believe that $\alpha_h(k_{\alpha,\beta}) = \alpha$ for $1 < \alpha < 2$, but have been unable to prove this.

For $\alpha > 2$, we cannot define a holomorphic function $k_{\alpha,\beta}$ on $\mathbb{D}$ by the same functional equation (10) since it does not have a solution which is holomorphic on all of $\mathbb{D}$ in this case. But for real numbers $x \in (-1, 1)$, the real-valued function $k_{\alpha,\beta}(x)$ is well defined by

$$\left(\frac{1 + k_{\alpha,\beta}(x)}{1 - k_{\alpha,\beta}(x)}\right)^{\alpha} - 1 = \beta\left[\left(\frac{1+x}{1-x}\right)^{\alpha} - 1\right]. \tag{11}$$

Thus, for $\alpha > 2$ and $\beta \in (0, 1]$ we regard the function $k_{\alpha,\beta}$ as defined only for real values $x \in (-1, 1)$.

THEOREM 8. *Suppose $f \in \mathscr{F}_h(\alpha)$ but f is not a conformal automorphism of $\mathbb{D}$. Then for any path γ in $\mathbb{D}$ from a to b,*

$$\begin{aligned}\frac{|D_{h1}f(a)|}{1 - |D_{h1}f(a)|}\exp[-2\alpha\,\text{length}_h(\gamma)] &\leq \frac{|D_{h1}f(b)|}{1 - |D_{h1}f(b)|}\\ &\leq \frac{|D_{h1}f(a)|}{1 - |D_{h1}f(a)|}\exp[2\alpha\,\text{length}_h(\gamma)].\end{aligned}$$

PROOF. It suffices to establish the upper bound because the lower bound then follows by simply interchanging the roles of a and b. Let $\gamma : z = z(s), 0 \leq s \leq L = \text{length}_h(\gamma)$, be a parametrization of γ by hyperbolic arclength. This implies that

$$z'(s) = \left[1 - z(s)\overline{z(s)}\right]e^{i\theta(s)}.$$

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If we make use of the preceding expression for $z'(s)$, then direct calculation results in

$$\frac{d}{ds}\log D_{h1}f(z) = \frac{f''(z)}{f'(z)}(1-z\bar{z})e^{i\theta} + \operatorname{Re}\left\{\left[\frac{2(1-z\bar{z})\overline{f(z)}f'(z)}{1-f(z)\overline{f(z)}} - 2\bar{z}\right]e^{i\theta}\right\}.$$

Here we have suppressed the variable s. Thus,

$$\begin{aligned}\frac{\frac{d}{ds}|D_{h1}f(z)|}{|D_{h1}f(z)|} &= \frac{d}{ds}\log|D_{h1}f(z)| \\ &= \operatorname{Re}\left\{\left[\frac{f''(z)}{f'(z)}(1-z\bar{z}) + \frac{2(1-z\bar{z})\overline{f(z)}f'(z)}{1-f(z)\overline{f(z)}} - 2\bar{z}\right]e^{i\theta}\right\} \\ &= \operatorname{Re}\left\{\frac{D_{h2}f(z)}{D_{h1}f(z)}e^{i\theta}\right\} \le \left|\frac{D_{h2}f(z)}{D_{h1}f(z)}\right|,\end{aligned}$$

so that

$$\frac{d}{ds}|D_{h1}f(z)| \le |D_{h2}f(z)|.$$

Because $f \in \mathscr{F}_h(\alpha)$, we obtain from (8) and the preceding inequality that

$$\frac{d}{ds}|D_{h1}f(z)| \le 2\alpha|D_{h1}f(z)|(1-|D_{h1}f(z)|),$$

or

$$\frac{\frac{d}{ds}|D_{h1}f(z)|}{|D_{h1}f(z)|(1-|D_{h1}f(z)|)} \le 2\alpha.$$

By integrating this differential inequality over the interval [0, L], we have

$$\log\left[\frac{|D_{h1}f(b)|}{1-|D_{h1}f(b)|}\left(\frac{|D_{h1}f(a)|}{1-|D_{h1}f(a)|}\right)^{-1}\right] \le 2\alpha L.$$

This is equivalent to the desired result.

REMARK. We make the following observations about the case in which equality holds in Theorem 8. We assume that $\text{length}_h(\gamma) > 0$. If equality holds, then we must have

$$(12)\quad \frac{d}{ds}\log|D_{h1}f(z)| = \operatorname{Re}\left\{\frac{D_{h2}f(z)}{D_{h1}f(z)}e^{i\theta}\right\} = \left|\frac{D_{h2}f(z)}{D_{h1}f(z)}\right| = 2\alpha(1-|D_{h1}f(z)|)$$

along γ. In particular,

$$\operatorname{Im}\left\{\frac{D_{h2}f(z)}{D_{h1}f(z)}e^{i\theta}\right\} = 0$$

along γ. Since the identity $z'/|z'| = e^{i\theta}$ holds along γ, we obtain

$$\kappa_h(f(z), f \circ \gamma)|D_{h1} f(z)| = \kappa_h(z, \gamma) \tag{13}$$

along γ from Theorem 2. Recall that a path γ in $\mathbb{D}$ is a subarc of a hyperbolic geodesic if and only if it has vanishing hyperbolic curvature. From the preceding identity we see that in the case of equality, γ is a hyperbolic geodesic if and only if $f \circ \gamma$ is.

COROLLARY 4. *Let $f \in \mathscr{F}_h(\alpha, \beta)$. Then for any point z in $\mathbb{D}$*

$$D_{h1} k_{\alpha,\beta}(-|z|) \le |D_{h1} f(z)| \le D_{h1} k_{\alpha,\beta}(|z|).$$

These bounds are best possible for $\alpha = 1, 2$.

PROOF. If $f \in \mathscr{F}_h(\alpha, \beta)$ and f is a conformal automorphism of $\mathbb{D}$, then $\alpha = 1$, $\beta = 1$ and the bounds are trivial. Otherwise, the choices $b = z$, $a = 0$ and $\gamma = [0, z]$ in the theorem give

$$\frac{\beta}{1-\beta}\left(\frac{1-|z|}{1+|z|}\right)^{\alpha} \le \frac{|D_{h1} f(z)|}{1 - |D_{h1} f(z)|} \le \frac{\beta}{1-\beta}\left(\frac{1+|z|}{1-|z|}\right)^{\alpha}.$$

Because the function $h(t) = t/(1-t)$ is increasing on the interval $[0, 1)$ with inverse function $h^{-1}(s) = s/(1+s)$, we deduce that

$$\frac{\beta(1-|z|)^{\alpha}}{\beta(1-|z|)^{\alpha} + (1-\beta)(1+|z|)^{\alpha}} \le |D_{h1} f(z)| \le \frac{\beta(1+|z|)^{\alpha}}{\beta(1+|z|)^{\alpha} + (1-\beta)(1-|z|)^{\alpha}}.$$

All that remains is to calculate $D_{h1} k_{\alpha,\beta}(x)$ for $-1 < x < 1$. By differentiating the functional equations (10) and (11) in Example 3 that were used to define $k_{\alpha,\beta}$, we find that

$$\frac{2\alpha k'_{\alpha,\beta}(x)[1 + k_{\alpha,\beta}(x)]^{\alpha-1}}{[1 - k_{\alpha,\beta}(x)]^{\alpha+1}} = \frac{2\alpha\beta[1+x]^{\alpha-1}}{[1-x]^{\alpha+1}}.$$

Therefore,

$$D_{h1} k_{\alpha,\beta}(x) = \frac{\beta(1+x)^{\alpha}}{\beta(1+x)^{\alpha} + (1-\beta)(1-x)^{\alpha}}.$$

The proof is now complete.

EXAMPLE 4. Let us explicitly state the bounds in the Corollary to Theorem 8 in the special cases $\alpha = 1$ and $\alpha = 2$.

If $f \in \mathscr{F}_h(1, \beta)$, then

$$\frac{\beta(1-|z|)}{1 - (2\beta - 1)|z|} \le |D_{h1} f(z)| \le \frac{\beta(1+|z|)}{1 + (2\beta - 1)|z|}.$$

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Equality holds for $k_{1,\beta}$ and its rotations. This is a new distortion theorem for the class of hyperbolically 2-convex functions. See [5] for other distortion theorems for hyperbolically 2-convex functions.

If $f \in \mathscr{F}_h(2, \beta)$, then

$$\frac{\beta(1-|z|)^2}{(1+|z|)^2 - 4\beta|z|} \le |D_{h1} f(z)| \le \frac{\beta(1+|z|)^2}{(1-|z|)^2 + 4\beta|z|}.$$

Equality holds for $k_{2,\beta}$ and its rotations. In particular, these latter inequalities are valid for any univalent function mapping $\mathbb{D}$ into itself.

REMARK. We investigate the implication of equality in the Corollary to Theorem 8 when $\beta \le 1$. For $\beta = 1$, the function f is a rotation of $\mathbb{D}$. Assume $\beta < 1$. We suppose equality holds at $z_0 \neq 0$. By performing a rotation of the function if necessary, we may assume that $z_0 = x_0 > 0$. Then γ in the proof of the Corollary to Theorem 8 is the radial segment $[0, x_0]$ and so a hyperbolic geodesic. Since we are assuming equality holds, $f \circ \gamma$ is also a hyperbolic geodesic from (13) and therefore must be the radial segment $[0, f(x_0)]$. Because $f'(0) = \beta$ is positive, it follows that $[0, f(x_0)]$ must lie along the nonnegative real axis. In particular, $f(t) \ge 0$ for $t \in [0, x_0]$ and f, f' and f'' are all real valued on $(-1, 1)$. That $D_{h2} f / D_{h1} f > 0$ and $D_{h1} f > 0$ hold on $[0, x_0]$ follows from (12) and the fact that $D_{h1} f(0) = \beta > 0$. Hence, the identity (12) yields that

$$\frac{f''(t)}{f'(t)} + \frac{2f(t)f'(t)}{1-f(t)^2} - \frac{2t}{1-t^2} = 2\alpha\left[\frac{1}{1-t^2} - \frac{f'(t)}{1-f(t)^2}\right]$$

for $t \in [0, x_0]$. By integrating this identity over the interval $[0, y]$ for any fixed $y \in (0, x_0]$, we obtain

$$\log\frac{(1-y^2)f'(y)}{1-f(y)^2} = \alpha\log\frac{(1+y)/(1-y)}{(1+f(y))/(1-f(y))} + \log\beta,$$

or

$$\frac{(1+f(y))^{\alpha-1}f'(y)}{(1-f(y))^{\alpha+1}} = \beta\frac{(1+y)^{\alpha-1}}{(1-y)^{\alpha+1}}.$$

Integration of this equality over the interval $[0, x]$ for any $x \in (0, x_0]$ results in

$$\left[\frac{1+f(x)}{1-f(x)}\right]^\alpha - 1 = \beta\left[\left(\frac{1+x}{1-x}\right)^\alpha - 1\right],$$

which is exactly the same as (10) when $1 \le \alpha \le 2$ and (11) when $\alpha > 2$. This final identity is valid for all $x \in (0, x_0]$, so the Identity Theorem implies that, up to rotation, there is at most one extremal function. It does imply that for $1 \le \alpha \le 2$, the function $k_{\alpha,\beta}$ is extremal provided $\alpha(k_{\alpha,\beta}) = \alpha$, which we know to be true by Example 3 in the cases $\alpha = 1$ and $\alpha = 2$. We suspect that $\alpha(k_{\alpha,\beta}) = \alpha$ remains valid for $1 < \alpha < 2$ but have not been able to verify this.

THEOREM 9. *Let $f \in \mathscr{F}_h(\alpha, \beta)$. Then for any point z in $\mathbb{D}$*

$$|f(z)| \le k_{\alpha,\beta}(|z|).$$

Equality holds for the function $k_{\alpha,\beta} \in \mathscr{F}_h(\alpha, \beta)$ in case $\alpha = 1, 2$.

PROOF. Set $\gamma = [0, z]$ and $\Gamma = f \circ \gamma$. Then from the Corollary of Theorem 8,

$$\begin{aligned} d_h(0, f(z)) &\le \int_\Gamma \frac{|dw|}{1-|w|^2} = \int_\gamma \frac{|f'(\zeta)||d\zeta|}{1-|f(\zeta)|^2} = \int_\gamma |D_{h1} f(\zeta)| \frac{|d\zeta|}{1-|\zeta|^2} \\ &\le \int_\gamma \left|D_{h1} k_{\alpha,\beta}(\zeta)\right| \frac{|d\zeta|}{1-|\zeta|^2} = \int_0^{|z|} \left|D_{h1} k_{\alpha,\beta}(t)\right| \frac{dt}{1-t^2} = \int_0^{|z|} \frac{k'_{\alpha,\beta}(t)}{1-k^2_{\alpha,\beta}(t)} dt \\ &= \frac{1}{2} \log \frac{1+k_{\alpha,\beta}(|z|)}{1-k_{\alpha,\beta}(|z|)} = d_h(0, k_{\alpha,\beta}(|z|)). \end{aligned}$$

This implies the desired inequality.

For $\alpha = 2$ and f univalent, Theorem 9 is due to Pick [13], while the case $\alpha = 1$ is contained in [5].

Finally, we obtain a covering theorem for hyperbolic linearly invariant functions. For a holomorphic function f defined on $\mathbb{D}$ with $f(0) = 0$, let $r(0, f)$ denote the radius of the largest schlicht disk centered at the origin which is contained in the image Riemann surface $f(\mathbb{D})$. In more classical terms, $r(0, f)$ is the radius of the largest disk centered at the origin such that the branch f^{-1} of the inverse function that satisfies $f^{-1}(0) = 0$ can be continued analytically to this disk with values in $\mathbb{D}$.

We recall the analogous covering theorem for euclidean linearly invariant functions. Let

$$\mathscr{F}(\alpha, \beta) = \{f \in \mathscr{F}(\alpha) : f(0) = 0 \text{ and } f'(0) = \beta\},$$

and

$$r(\alpha, \beta) = \inf\{r(0, f) : f \in \mathscr{F}(\alpha, \beta)\},$$

for any $\beta > 0$. Then $r(\alpha, \beta) = \beta/(2\alpha)$ [14]. Actually, the result in [14] is only stated in case $\beta = 1$, but the extension to arbitrary positive β is elementary. This bound is attained for the function

$$f_{\alpha,\beta}(z) = \beta \frac{((1+z)/(1-z))^\alpha - 1}{2\alpha}$$

and its rotations. In particluar, note that $r(\alpha, \beta) \to 0$ as $a \to \infty$ for each fixed β.

The situation for the classes $\mathscr{F}(\alpha, \beta)$ is quite different. Set

$$r_h(\alpha, \beta) = \inf\{r_h(0, f) : f \in \mathscr{F}_h(\alpha, \beta)\}.$$

There is a general covering theorem for normalized holomorphic self-maps of the unit disk. If f is holomorphic in $\mathbb{D}$ with $f(0) = 0$, $f'(0) = \beta$ and $f(\mathbb{D}) \subset \mathbb{D}$, then Landau (see [3, p. 38]) proved that

$$r(0, f) \geq \frac{\beta^2}{\left[1 + \sqrt{1-\beta}\right]^2}.$$

This bound is sharp for the finite Blaschke product $F(z) = z(\beta - z)/(1 - \beta z)$. Note that this extremal function is not locally schlicht in $\mathbb{D}$. Landau's result implies that

$$r_h(\alpha, \beta) \geq \frac{\beta^2}{\left[1 + \sqrt{1-\beta}\right]^2}.$$

This lower bound is independent of α, so $r_h(\alpha, \beta)$ cannot approach 0 when $\alpha \to \infty$ for any fixed β. The following result improves this lower bound on $r_h(\alpha, \beta)$ for $\alpha \in [1, 2 + \epsilon]$ for some $\epsilon > 0$.

THEOREM 10. *For $\alpha \geq 1$ and $\beta \in (0, 1]$,*

$$r_h(\alpha, \beta) \geq |k_{\alpha,\beta}(-1)| = \frac{1 - (1-\beta)^{1/\alpha}}{1 + (1-\beta)^{1/\alpha}}.$$

The lower bound is sharp for $\alpha = 1, 2$.

PROOF. Suppose $f \in \mathscr{F}_h(\alpha, \beta)$ and $r = r_h(0, f)$. Because f is locally schlicht, there is a point ω with $|\omega| = r$ such that the branch of f^{-1} that satisfies $f^{-1}(0) = 0$ can be continued analytically along the radial segment $[0, \omega) = \Gamma$ with values in $\mathbb{D}$, but cannot be continued analytically to $[0, \omega]$ and still have values in $\mathbb{D}$. Assume $\Gamma : w = w(t)$, $0 \leq t < 1$, and set $\gamma = f^{-1} \circ \Gamma : z = z(t)$, $0 \leq t < 1$. Then γ is a path in $\mathbb{D}$ with $|z(t)| \to 1$ as $t \to 1$. If $r = 1$, we are done. Otherwise, from the Corollary of Theorem 8, we have

$$\begin{aligned} d_h(0, \omega) &= \int_\Gamma \frac{|dw|}{1 - |w|^2} = \int_\gamma \frac{|f'(z)||dz|}{1 - |f(z)|^2} = \int_\gamma |D_{h1} f(z)| \frac{|dz|}{1 - |z|^2} \\ &\geq \int_\gamma \left|D_{h1} k_{\alpha,\beta}(-|z|)\right| \frac{|dz|}{1 - |z|^2} \geq \int_0^1 \left|D_{h1} k_{\alpha,\beta}(-|z|)\right| \frac{d|z|}{1 - |z|^2} \\ &= \int_0^1 \frac{k_{\alpha,\beta}(-|z|)}{1 - k_{\alpha,\beta}^2(-|z|)} d|z| = d_h(0, k_{\alpha,\beta}(-1)). \end{aligned}$$

This implies that $r = |\omega| \geq |k_{\alpha,\beta}(-1)|$. The explicit value of $k_{\alpha,\beta}(-1)$ can be determined from the functional equations (10) and (11) in Example 3.

For $\alpha = 1$ we have $r_h(1, \beta) = \beta/(2 - \beta)$, which was established in [9] by a completely different method, while for $\alpha = 2$, Theorem 10 becomes $r_h(2, \beta) = \beta/(1 + \sqrt{1 - \beta})^2$. This latter covering result was established for the subclass of $\mathscr{F}_h(2, \beta)$ consisting of univalent functions by Pick [13]. By using the same method as that employed in the proof of Theorem 10, we can show that if $f \in \mathscr{F}_h(\alpha, \beta)$ is univalent, which forces $1 \le \alpha \le 2$, then

$$|k_{\alpha,\beta}(-|z|)| \le |f(z)|.$$

For $\alpha = 2$, this is due to Pick [13], while for $\alpha = 1$ it is a special case of a growth theorem in [5].

7. Concluding comments

There are a number of open questions for hyperbolic linearly invariant functions; we mention a few directly related to this paper. First of all, is $\alpha(k_{\alpha,\beta}) = \alpha$ for $1 < \alpha < 2$? Next, what are plausible extremal functions for the family $\mathscr{F}_h(\alpha, \beta)$ when $\alpha > 2$? How can the Corollary to Theorem 8 plus Theorems 9 and 10 be improved for $\alpha > 2$?

Now that euclidean and hyperbolic linearly invariant functions have been considered, it is natural to try to consider 'spherical linear invariance'. In this context one would investigate locally univalent meromorphic functions $f : \mathbb{D} \to \mathbb{P}$, where $\mathbb{P}$ denotes the Riemann sphere. Here $\mathbb{D}$ is endowed with hyperbolic geometry while $\mathbb{P}$ is considered with spherical geometry. A definition for the spherical linearly invariant order has been proposed in [7]:

$$\alpha_s(f) = \sup\left\{\frac{|\tilde{f}''(0)|}{2|\tilde{f}'(0)|} : \tilde{f} = R \circ f \circ S, \text{ where } R \in \operatorname{Rot}(\mathbb{P}),\ S \in \operatorname{Aut}(\mathbb{D}) \text{ and } \tilde{f}(0) = 0\right\}.$$

Here $\operatorname{Rot}(\mathbb{P})$ is the group of rotations of the sphere. Note that this parallels the definition of the euclidean linearly invariant order (see [1]). However, the study of spherically linearly invariant functions does not exhibit a simple parallelism with the theory of euclidean linearly invariant functions. New difficulties arise in this context, partly because there is no uniform upper bound on the second coefficient of normalized ($f(0) = 0$ and $f'(0) = 1$) meromorphic univalent functions defined on the unit disk. However, there is still an analogous close connection with uniform local spherical convexity. Also, there is an interesting connection with normal functions: every spherically linearly invariant function is normal. There is no euclidean analog for this result; there are euclidean linearly invariant functions that are not Bloch functions. Also, in [7] a definition of 'spherical linear invariance' for locally schlicht meromorphic functions $f : \mathbb{C} \to \mathbb{P}$ is proposed. In this latter context it turns out that the class of spherically linearly invariant functions is rather uninteresting; the only such functions are Möbius transformations and compositions of Möbius transformations with the exponential function.

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