ON DEVELOPMENT OF FORMAL SYSTEMS
STARTING FROM PRIMITIVE LOGIC

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Introduction

It has been my program to develop fundamental theories of mathematics starting from TABOOS and standing on the primitive logic LO at first instead of starting from AXIOMS and standing on the fairly brought up logic, the lower classical logic LK. This was proposed in my work [1].

To put this program into practice, however, we have to state TABOOS in LO. The logical vocabulary of LO is very scanty since it has originally no logical constants other than IMPLICATION and UNIVERSAL QUANTIFICATION.

In fact, we can interpret faithfully LK as well as the intuitionistic predicate logic LJ in LO as has been shown in my work [2], but we can not define logical constants such as CONJUNCTION, DISJUNCTION, and EXISTENTIAL QUANTIFICATION for propositions of every formal system. For most formal systems, I must admit that TABOOS are hard to be stated in terms of the scanty logical vocabulary of LO only.

Recently, I was able to show in my work [3] that, in any formal system having just one primitive notion and standing on the primitive logic LO, CONJUNCTION, DISJUNCTION, and EXISTENTIAL QUANTIFICATION can be so defined in terms of the original logical constants of LO that propositions of LO behave with respect to the newly defined logical constants together with the original ones just as propositions of the positive logic LP.

We can further define NEGATION so that propositions behave with respect to this and the above mentioned logical constants just as propositions of LJ.

In the present paper, I will show that also for any formal system standing on LO, we can introduce a substitute system having just one primitive notion

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1) The primitive logic LO was introduced in my work [1] as the primitive system of positive logic. The terminology PRIMITIVE LOGIC together with its reference notation LO was introduced in my work [2]. As for the TABOO notion, see my work [1].

2) As for LP, see my work [2] and [3]. See also LORENZEN [1] and CURRY [1].
which serves as a substitute for the \textit{whole class of primitive notions} of the original system. Here we assume naturally that the number of primitive notions of any formal system is \textit{finite}. I will illustrate the device \textit{informally} in (1) and \textit{formally} in (2).

In (3), I will explain that this method contributes to development of formal systems under my program, because it makes description of TABOOS in \textit{LO easier}. This method enables us also to unify all the TABOOS of a TABOO system into a single TABOO, if the TABOO system can be stated in a finite number of TABOOS. AXIOM systems consisting of a single AXIOM show \textit{no essential superiority} over systems consisting of a finite number of AXIOMS. However, single-TABOO TABOO-systems have an \textit{essential superiority} over other TABOO systems. I will discuss the matter also in the same section.

\textbf{(1) Informal illustration}

As a typical example of formal systems, I will take up a theory of \textit{natural numbers} having the primitive notions $\#$, $\emptyset$, and $\$$. ($\#$ is a predicate and $\#(x)$ means that $x$ is a \textit{natural number}, $\emptyset$ is also a predicate and $\emptyset(x)$ means that $x$ is \textit{zero}, and $\$ is a binary relation and $x\$y$ means that $y$ is the \textit{successor} of $x$).

Let us now define a relation $\mathcal{R}$ by

$$\mathcal{R}(x, y, u, v) \iff \#(x) \lor \emptyset(y) \lor u \$ v.$$ 

If we adopt $\mathcal{R}$ as a \textit{primitive notion} instead of the \textit{class} of $\#$, $\emptyset$, and $\$$, then these notions can be defined in terms of $\mathcal{R}$ by

\begin{align*}
\#(x) & \iff (y)(u)(v)\mathcal{R}(x, y, u, v), \\
\emptyset(y) & \iff (x)(u)(v)\mathcal{R}(x, y, u, v), \\
u \$ v & \iff (x)(y)\mathcal{R}(x, y, u, v).
\end{align*}

Thus the single relation $\mathcal{R}$ can be regarded as a substitute for the whole class of the original primitive notions $\#$, $\emptyset$, and $\$$. In \textit{LO}, we can not define the relation $\mathcal{R}$ in terms of $\#$, $\emptyset$, and $\$$. However, the device above described would suggest a method of development of formal systems in \textit{LO}.

\textbf{(2) Formal description}

Let us assume the case where a formal system having the primitive notions
$P_1, \ldots, P_m$ should be developed in LO. Let $P_i$ be an $n(i)$-ary relation for every $i = 1, \ldots, m$. Put

$$s(k) \equiv n(1) + \cdots + n(k - 1) \text{ for every } k = 1, \ldots, m + 1.$$  

For this formal system, we take the single $s(m + 1)$-ary relation $R$ as a substitute for the whole class of the $m$ primitive notions. Now, I will define the substitutes of the $m$ original primitive notions in terms of $R$ by

$$\exists (x_1) \cdots (x_{s(i - 1)}) (x_{s(i - 1) + 1}) \cdots (x_{s(m + 1)}) R(x_1, \ldots, x_{s(m + 1)})$$  

for every $i = 1, \ldots, m$.

Corresponding to the original formal system, we can thus introduce another formal system having just one primitive notion $R$ and behaving quite similarly as the original system with respect to the substitutes of the original primitive notions. This corresponding system will be called SUBSTITUTE SYSTEM.

Since the substitute system has just one primitive notion $R$, we can define by virtue of my work [3], CONJUNCTION, DISJUNCTION, and EXISTENTIAL QUANTIFICATION in such a way that propositions in the substitute system behave with respect to these newly defined logical constants together with the original ones just as propositions in the positive logic LP. Hence, for the substitute system, we can start as if standing on LP from the beginning.

Considering in the inverse direction, we can make for any formal system standing on LP its substitute system standing of LO by a natural correspondence of elementary formulas to their substitutes of the form $(u) \cdots (v) R(x, \ldots, z)$, logical constants of LP correspond partly to the original logical constants and partly to the newly defined logical constants of LO. This gives an embedding of any LP-system into an LO-system.\(^{3)}\) If we further define NEGATION by

$$\neg \exists \equiv \exists \neg(x) \cdots (z) R(x, \ldots, z),$$  

then we would have an embedding of any LJ-system into an LO-system. We can prove these facts by making use of the device mentioned above and the result

\(^{3)}\) I call any formal system $\mathfrak{S}$-SYSTEM (logical system standing on $\mathfrak{S}$) if and only if it has a definite class of its primitive notions and standing on the logic $\mathfrak{S}$ but assuming nothing more. As for embedding of a logical system into another, see my work [2].
Remark. I have occasionally employed $R(p, x, \ldots, z, d, \ldots, d)$ in my work [2] as a substitute for $P(x, \ldots, z)$. If we make substitute system by this method, we are forced to use some variables (such as $p$ in the above example) as denoting object constants. Our device introduced here has a strong point that it does not force to do anything of this kind. In reality, my former proof of faithful interpretability of J- and K-series logics\(^4\) in LO can be improved by the device of the present paper and the result of my recent work [3].

(3) On description of TABOOS

According to my program proposed in my work [1], we have to describe TABOOS of formal systems in LO. The logical vocabulary of LO looks like too scanty to do this.

By adopting the device described in (2) for any formal system, however, we can define for its substitute system, CONJUNCTION, DISJUNCTION, and EXISTENTIAL QUANTIFICATION so as to satisfy all the inference rules of LP. Hence, for the substitute system, we have only to describe TABOOS in LP. As the logical vocabulary of LP is much richer than that of LO, it looks like much easier to state TABOOS for the substitute system than to state them for the original system in LO.

If the TABOO system of the substitute system consists of a finite number of TABOOS $\exists_1, \ldots, \exists_n$, we can further unify them into a single TABOO $\exists_1 \lor \cdots \lor \exists_n$. Although AXIOM systems consisting of a single AXIOM show no essential superiority over AXIOM systems consisting of a finite number of AXIOMS, TABOO systems consisting of a single TABOO has an essential superiority over other TABOO systems.

If any TABOO system contains more than one TABOOS, we are forced to add the assumption (as has been done in my work [1]) that members in the TABOO system are mutually equivalent, when we wish to develop nicely a formal theory standing on a brought up logic such as LJ or LK. Since such assumption can never be called agreeable, we are eager to develop fundamental theories of mathematics without assuming such kind of things. Only if we could have

\(^4\) The notion of J- and K-series logics is introduced in my work [2].
a single-TABOO TABOO-system for each theory, we would surely be able to get rid of such assumption. It must be an interesting task to find out a fundamental theory of mathematics such as set theory which is developable from a single-TABOO TABOO-system, if any.

The Bernays-Gödel set-theory\textsuperscript{5) having an AXIOM system consisting of a finite number of AXIOMS seems to suggest that such pursuit could be successful. Naturally, we can not expect to obtain a \textit{single unified TABOO }$\mathcal{I}_1 \lor \cdots \lor \mathcal{I}_n$trivially by taking the substitutes $\mathcal{I}_i$ of $\neg \mathcal{I}_i$ of the finite AXIOM system $\mathcal{I}_1, \ldots, \mathcal{I}_n$ of the Bernays-Gödel set-theory, for example, because NEGATION can only be interpreted by TABOOS. Only by assuming that \textit{TABOOS are equivalent to }$(x) \cdots (z)R(x, \ldots, z)$, we can describe TABOOS of the substitute system in LO as if we describe them in LJ. Accordingly, $(x) \cdots (z)R(x, \ldots, z)$ should be regarded as an additional TABOO, unless the substitute system has a TABOO logically equivalent to it.

\textbf{References}


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\textsuperscript{5) As for Bernays-Gödel AXIOM-system of set theory, see BERNAYS [1] and GÖDEL [1].}