MULTIPLE SOLUTIONS FOR SOME NEUMANN PROBLEMS IN EXTERIOR DOMAINS

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In this paper, we show that if \( q(x) \) satisfies suitable conditions, then the Neumann problem 
\[
-\Delta u + u = q(x)|u|^{p-2}u \quad \text{in } \Omega
\]
has at least two solutions of which one is positive and the other changes sign.

1. INTRODUCTION

Throughout this article, let \( N = m + n \), where \( m \) and \( n \) are nonnegative integers with \( m \geq 3 \). For \( x = (x_1, \ldots, x_N) = (x_1, \ldots, x_m, x_{m+1}, \ldots, x_N) \in \mathbb{R}^N \), let \( Px = (x_1, \ldots, x_m) \in \mathbb{R}^m \) and \( Qx = (x_{m+1}, \ldots, x_N) \in \mathbb{R}^n \). Consider the Neumann boundary value problem

\[
\begin{align*}
-\Delta u + u = q(x)|u|^{p-2}u & \quad \text{in } \Omega; \\
\frac{\partial u}{\partial \eta} & = 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \Omega = (\mathbb{R}^m \setminus \overline{\Omega^m}) \times \mathbb{R}^n \), \( \Omega^m \) is a smooth bounded domain in \( \mathbb{R}^m \), \( 2 < p < 2^* = \frac{(2N)/(N-2)}{2} \), \( \eta \) is the outward unit normal to \( \partial \Omega \) and \( q(x) \) is a bounded continuous function in \( \Omega \). Moreover, \( q(x) \) satisfies the following hypotheses:

\( (q1) \) \( q(x) \) is a positive function in \( \Omega \), \( \inf\{q(x)\,|\,x \in \Omega\} > 0 \) and \( q(x) = q(y) \) for any \( Px = Py \);

\( (q2) \) there exists a positive number \( q_\infty \) such that \( \lim_{|Px| \to \infty} q(x) = q_\infty \) and \( q(x) \neq q_\infty \) in \( \Omega \).

If \( \Omega^c \) is bounded (\( n = 0 \) in our case), Esteban [5, 6] proved the existence of the "ground state solution" of Equation (1) provided that \( q(x) \equiv 1 \). In the case \( q(x) \) is not a constant function, Cao [3] and Hsu and Lin [9] proved the multiplicity of the solutions of Equation (1). In this article, we assert that Equation (1) still has the same results of Hsu and Lin [9] even if \( \Omega^c \) is unbounded. First, we use the concentration-compactness argument of Lions [11, 12, 13, 14] to obtain the "ground state solution", and then combine it with some ideas of Zhu [16] to show the existence of another solution which changes sign.

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2. PRELIMINARIES

Associated with Equation (1), we consider the energy functionals $a$, $b$ and $J$, for $u \in H^1(\Omega)$

\[ a(u) = \int_{\Omega} (|\nabla u|^2 + u^2) \, dx; \]
\[ b(u) = \int_{\Omega} q(x)|u|^p \, dx; \]
\[ J(u) = \frac{1}{2} a(u) - \frac{1}{p} b(u). \]

Define

\[ \alpha = \inf_{u \in M(\Omega)} J(u), \]

where

\[ M(\Omega) = \left\{ u \in H^1(\Omega) \setminus \{0\} \mid a(u) = b(u) \right\}. \]

It is well known that there is a positive radially symmetric smooth solution $w$ of Equation (2)

\[ \begin{cases} - \Delta u + u = q_\infty |u|^{p-2} u & \text{in } \mathbb{R}^N; \\ u \in H^1(\mathbb{R}^N). \end{cases} \]

We also define

\[ a^\infty(u) = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \, dx; \]
\[ b^\infty(u) = \int_{\mathbb{R}^N} q_\infty |u|^p \, dx; \]
\[ J^\infty(u) = \frac{1}{2} a^\infty(u) - \frac{1}{p} b^\infty(u); \]
\[ \alpha^\infty = \inf_{u \in M^\infty(\mathbb{R}^N)} J^\infty(u), \]

where

\[ M^\infty(\mathbb{R}^N) = \left\{ u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid a^\infty(u) = b^\infty(u) \right\}. \]

Recall the fact that

\[ w(|x|)|x|^{(N-1)/2} \exp(|x|) \rightarrow \bar{c} > 0 \text{ as } |x| \rightarrow \infty, \]

where $\bar{c}$ is some constant. (See Bahri and Li [1], Bahri and Lions [2], Gidas, Ni and Nirenberg [7, 8] and Kwong [10].) In particular, we have

(i) there exists a constant $C_0 > 0$ such that

\[ w(x) \leq C_0 \exp(-|x|) \text{ for all } x \in \mathbb{R}^N; \]
(ii) for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that
\[ w(x) \geq C_\epsilon \exp(-(1 + \epsilon)|x|) \text{ for all } x \in \mathbb{R}^N. \]

We need the following definition and lemmas to prove the main theorems.

**DEFINITION 1:** For $\beta \in \mathbb{R}$, a sequence $\{u_k\}$ is a $(PS)_\beta$-sequence in $H^1(\Omega)$ for $J$ if $J(u_k) = \beta + o(1)$ and $J'(u_k) = o(1)$ strongly in $H^1(\Omega)$ as $k \to \infty$.

**Lemma 2.** Let $\beta \in \mathbb{R}$ and let $\{u_k\}$ be a $(PS)_\beta$-sequence in $H^1(\Omega)$ for $J$, then $\{u_k\}$ is a bounded sequence in $H^1(\Omega)$. Moreover,
\[ a(u_k) = b(u_k) + o(1) = \frac{2p}{p-2} \beta + o(1) \]
and $\beta \geq 0$.

**Proof:** For $n$ sufficiently large, we have
\[ |\beta| + 1 + \sqrt{a(u_k)} \geq J(u_k) - \frac{1}{p}\langle J'(u_k), u_k \rangle = \left(\frac{1}{2} - \frac{1}{p}\right)a(u_k). \]
It follows that $\{u_k\}$ is bounded in $H^1(\Omega)$. Since $\{u_k\}$ is a bounded sequence in $H^1(\Omega)$, then $\langle J'(u_k), u_k \rangle = o(1)$ as $k \to \infty$. Thus,
\[ \beta + o(1) = J(u_k) = \left(\frac{1}{2} - \frac{1}{p}\right)a(u_k) + o(1) = \left(\frac{1}{2} - \frac{1}{p}\right)b(u_k) + o(1), \]
that is, $a(u_k) = b(u_k) + o(1) = (2p/p - 2)\beta + o(1)$ and $\beta \geq 0$. \[
\]

**Lemma 3.**
(i) For each $u \in H^1(\Omega) \setminus \{0\}$, there exists an $s_u > 0$ such that $s_u u \in M(\Omega)$;
(ii) Let $\{u_k\}$ be a $(PS)_\beta$-sequence in $H^1(\Omega)$ for $J$ with $\beta > 0$. Then there is a sequence $\{s_k\}$ in $\mathbb{R}^+$ such that $\{s_k u_k\} \subset M(\Omega)$, $s_k = 1 + o(1)$ and $J(s_k u_k) = \beta + o(1)$. In particular, the statement holds for $J^\infty$.

**Proof:** See Chen, Lee and Wang [4]. \[
\]

**Lemma 4.** There exists a $c > 0$ such that $\|u\|_{H^1(\Omega)} \geq c > 0$ for each $u \in M(\Omega)$.

**Proof:** See Chen, Lee and Wang [4]. \[\]

**Lemma 5.** Let $u \in M(\Omega)$ satisfy $J(u) = \min_{v \in M(\Omega)} J(v) = \alpha$. Then $u$ is a nonzero solution of Equation (1).

**Proof:** We define $g(v) = a(v) - b(v)$ for $v \in H^1(\Omega) \setminus \{0\}$. Note that $\langle g'(u), u \rangle = (2 - p)a(u) \neq 0$. Since the minimum of $J$ is achieved at $u$ and is constrained on $M(\Omega)$, by the Lagrange multiplier theorem, there exists a $\lambda \in \mathbb{R}$ such that $J'(u) = \lambda g'(u)$ in $H^1(\Omega)$. Then we have
\[ 0 = \langle J'(u), u \rangle = \lambda \langle g'(u), u \rangle, \]
that is, \( \lambda = 0 \). Hence, \( J'(u) = 0 \) and \( u \) is a nonzero solution of Equation (1) in \( \Omega \) such that \( J(u) = \alpha \).

Define \( u^+ = \max\{u, 0\} \) and \( u^- = \max\{-u, 0\} \).

**Lemma 6.** Let \( u \) be a solution of Equation (1) that changes sign. Then \( J(u) \geq 2\alpha \). In particular, the result holds for \( J^\infty \).

**Proof:** Since \( u \) is a solution of Equation (1) that changes sign, then \( u^- \) is nonnegative and nonzero. Multiply Equation (1) by \( u^- \) and integrate to obtain

\[
\int_\Omega \nabla u \nabla u^+ + uu^- = \int_\Omega q(x)|u|^{p-2}uu^-,
\]

that is, \( u^- \in M(\Omega) \) and \( J(u^-) \geq \alpha \). Similarly, \( J(u^+) \geq \alpha \). Hence,

\[
J(u) = J(u^+) + J(u^-) \geq 2\alpha.
\]

**Lemma 7.** (Improved Decomposition Lemma) Let \( \{u_k\} \) be a \((PS)_\beta\)-sequence in \( H^1(\Omega) \) for \( J \). Then there are a subsequence \( \{u_k\} \), an integer \( l \geq 0 \), sequences \( \{x_k^i\}_{k=1}^\infty \) in \( \mathbb{R}^N \), functions \( v \in H^1(\Omega) \) and \( w_i \neq 0 \) in \( H^1(\mathbb{R}^N) \) for \( 1 \leq i \leq l \) such that

\[
-\Delta v + v = q(x)|v|^{p-2}v \text{ in } \Omega;
-\Delta w_i + w_i = q_{\infty}|w_i|^{p-2}w_i \text{ in } \mathbb{R}^N;
\]

\[
|P x_k^i| \to \infty \text{ for } 1 \leq i \leq l;
|x_k^i| \to \infty \text{ for } 1 \leq i \leq l;
\]

\[
u_k = v + \sum_{i=1}^l w_i(\cdot - x_k^i) + o(1) \text{ strongly in } H^1(\mathbb{R}^N);
\]

\[
J(u_k) = J(v) + \sum_{i=1}^l J^\infty(w_i) + o(1).
\]

In addition, if \( u_k \geq 0 \), then \( v \geq 0 \) and \( w_i \geq 0 \) for \( 1 \leq i \leq l \).

**Proof:** The proof can be obtained by using the arguments in Bahri and Lions [2] or see Tzeng and Wang [15].

3. Existence of the Ground State Solution

**Lemma 8.** If \( \alpha < \alpha^\infty \), then \( \alpha \) attains a minimiser \( v_1 \), that is, there exists a ground state solution \( v_1 \) of Equation (1).

**Proof:** See Cao [3].

Let \( w_k(x) = w(x + e_k) \mid_\Omega \), where \( e_k = (k, 0, \ldots, 0) \). Then we have the following lemmas.
**Lemma 9.** Let \( \Theta \) be a domain in \( \mathbb{R}^m \). If \( f : \Theta \to \mathbb{R} \) satisfies
\[
\int_{\Theta} |f(x) \exp(\sigma |x|)| \, dx < \infty \text{ for some } \sigma > 0,
\]
then
\[
\left( \int_{\Theta} f(x) \exp(-\sigma |x + e_k|) \, dx \right) \exp(\sigma k) = \int_{\Theta} f(x) \exp(-\sigma x_1) \, dx + o(1) \text{ as } k \to \infty,
\]
or
\[
\left( \int_{\Theta} f(x) \exp(-\sigma |x - e_k|) \, dx \right) \exp(\sigma k) = \int_{\Theta} f(x) \exp(\sigma x_1) \, dx + o(1) \text{ as } k \to \infty.
\]

**Proof:** We know \( \sigma |e_k| < \sigma |x| + \sigma |x + e_k| \), then
\[
\left| f(x) \exp(-\sigma |x + e_k|) \exp(\sigma |e_k|) \right| \leq \left| f(x) \exp(\sigma |x|) \right|.
\]
Since
\[
-\sigma |x + e_k| + \sigma |e_k| = -\sigma \frac{(x, e_k)}{|e_k|} + o(1) = -\sigma x_1 + o(1)
\]
as \( k \to \infty \), the lemma follows from the Lebesgue dominated convergence theorem. \( \square \)

**Lemma 10.** Assume that there are positive numbers \( \delta, R_0 \) and \( C \) such that
\[
(q) \quad q(x) \geq q_0 - C \exp(-2 + \delta |P_2|) \text{ for } |P_2| \geq R_0.
\]
Then there exists a \( k_0 \in \mathbb{N} \) such that for \( k \geq k_0 \), we have
\[
\sup_{s \geq 0} J(sw_k) < \alpha^\infty.
\]

**Proof:** Take a \( k_1 \in \mathbb{N} \) such that the \( N \)-ball \( B(-e_k; 1) \subset \Omega \) for \( k \geq k_1 \). Then we have
\[
J(sw_k) \leq \frac{s^2}{2} \int_{\mathbb{R}^N} \left( |\nabla w|^2 + w^2 \right) - c \frac{sp}{p} \int_{B(-e_k; 1)} w^p(x + e_k) \, dx
\]
\[
= \frac{s^2}{2} \int_{\mathbb{R}^N} \left( |\nabla w|^2 + w^2 \right) - c \frac{sp}{p} \int_{B(0, 1)} w^p \, dx.
\]
Therefore, there exists an \( s_1 > 0 \) such that
\[
J(sw_k) < 0 \text{ for } s \geq s_1 \text{ and } k \geq k_1.
\]
Since \( J \) is continuous in \( H^1(\Omega) \) and
\[
\int_\Omega \left[ |\nabla w_k|^2 + w_k^2 \right] \leq \int_{\mathbb{R}^N} |\nabla w|^2 + w^2 < \infty \text{ for any } k \in \mathbb{N},
\]
there exists an $s_0 > 0$ such that

$$J(sw_k) < \alpha^\infty$$

for $s < s_0$ and any $k \in \mathbb{N}$.

Then we only need to prove

$$\sup_{s_0 \leq s \leq s_1} J(sw_k) < \alpha^\infty$$

for $k$ sufficiently large.

For $k \geq k_1$ and $s_0 \leq s \leq s_1$, since

$$\sup_{s \geq 0} J^\infty(sw) = J^\infty(w) = \alpha^\infty,$$

$$J(sw_k) = \frac{s^2}{2} \int_{\Omega} \left[ |\nabla w(x + e_k)|^2 + |w(x + e_k)|^2 \right] - \frac{s^p}{p} \int_{\Omega} q(x)|w(x + e_k)|^p$$

$$= J^\infty(sw) - \frac{s^2}{2} \int_{\Omega \times \mathbb{R}^n} \left[ |\nabla w(x + e_k)|^2 + |w(x + e_k)|^2 \right]$$

$$+ \frac{s^p}{p} \int_{\Omega \times \mathbb{R}^n} q_\infty |w(x + e_k)|^p + \int_{\Omega} (q(x) - q_\infty) |w(x + e_k)|^p$$

$$\leq \alpha^\infty - \frac{s_0^2}{2} \int_{\Omega \times \mathbb{R}^n} \left[ |\nabla w(x + e_k)|^2 + |w(x + e_k)|^2 \right]$$

$$+ \frac{s^p}{p} \int_{\Omega \times \mathbb{R}^n} q_\infty |w(x + e_k)|^p + \int_{\Omega} (q(x) - q_\infty) |w(x + e_k)|^p.$$ 

(i) Let $B(0; 1) \subset \mathbb{R}^n$ be the unit $n$-ball, then

$$\int_{\Omega \times \mathbb{R}^n} |w(x + e_k)|^2 dx \geq \int_{\Omega \times B(0; 1)} C'_\varepsilon \exp(-2(1 + \varepsilon)|x + e_k|) dx$$

$$\geq C'_\varepsilon \exp(-2(1 + \varepsilon)k).$$

(ii) It is easy to see that the following inequality

$$\sqrt{(a^2 + b^2)} \geq \theta a + \sqrt{1 - \theta^2 b}$$

holds for any $a, b > 0$ and $0 \leq \theta \leq 1$. Take $\theta = 1$, and since $\Omega \times \mathbb{R}^n$ is unbounded, then for a small $\varepsilon > 0$, we have

$$\int_{\Omega \times \mathbb{R}^n} q_\infty |w(x + e_k)|^p dx \leq \int_{\Omega \times \mathbb{R}^n} q_\infty C_0^p \exp(-p|x + e_k|) dx$$

$$\leq \int_{\Omega \times \mathbb{R}^n} q_\infty C_0^p \exp\left(-p\theta|P + Pe_k|\right) dx$$

$$\leq C'_0 \exp(-(p - \varepsilon)k).$$

(iii) It is similar to (ii) we have

$$\int_{\Omega \cap \{|Pz| \leq R_0\}} (q_\infty - q(x))|w(x + e_k)|^p dx \leq M \exp(-(p - \varepsilon)k).$$
Since $q$ satisfies the condition $(Q)$, then by Lemma 9, there exists a $k_2 \geq k_1$ such that for $k \geq k_2$

$$
\int_{\Omega \cap \{|Pz| \geq R_0\}} (q_{\infty} - q(x)) |w(x + e_k)|^p \, dx
$$

$$
\leq \int_{\Omega \cap \{|Pz| \geq R_0\}} C \exp\left(- (2 + \delta)|Pz|\right) C_0 \exp\left(- p|x + e_k t|\right) \, dx
$$

$$
\leq \int_{\Omega \cap \{|Pz| \geq R_0\}} C_1 \exp\left(- (2 + \delta)|Pz|\right) \exp\left(- p\theta|\langle Pz + P e_k \rangle|\right) \, dx
$$

$$
\leq C' \exp\left(- \min\{2 + \frac{\delta}{2}, p\} k_2\right).
$$

By (i)-(iii) and $2 < p < 2^*$, choosing $\varepsilon > 0$, such that $2 + 2\varepsilon < p - \varepsilon$ and $2\varepsilon < \delta/2$, we can find a $k_0 \geq k_2$ such that for $k \geq k_0$

$$
\frac{s_0^2}{p} \left[ \int_{\Omega \times \mathbb{R}^n} q_{\infty} |w(x + e_k)|^p + \int_{\Omega} (q_{\infty} - q(x)) |w(x + e_k)|^p \right]
$$

$$
- \frac{s_0^2}{2} \int_{\Omega \times \mathbb{R}^n} |\nabla w(x + e_k)|^2 + |w(x + e_k)|^2 < 0.
$$

Hence, we have

$$
\sup_{s > 0} J(sw_k) < \alpha_\infty \text{ for } k \geq k_0.
$$

**Theorem 11.** Assume that $q$ satisfies $(q_1)$, $(q_2)$ and the condition $(Q)$, then Equation (1) has a positive solution $v_1$.

**Proof:** By Lemma 3 (i), there exists an $s_k > 0$ such that $s_k w_k \in M(\Omega)$, that is, $\alpha \leq J(s_k w_k)$. Applying Lemma 10, we have $\alpha < \alpha_\infty$. Thus, there exists a ground state solution $v_1$ of Equation (1). By the standard arguments and the maximum principle, $v_1 > 0$ in $\Omega$. 

**Remark 1.** $v_1(x) \leq C_1 \exp(-|x|)$ for $|x| \geq R_1$, where $C_1$ and $R_1$ are some positive constants.

**Proof:** See Cao [3].

**4. Existence of the Second Solution**

In this section, $q$ satisfies $(q_1)$, $(q_2)$ and the condition $(Q)$

$$(Q) \quad q(x) \geq q_{\infty} + C \exp(-\delta|Pz|) \text{ for } |Pz| \geq R_0$$

where $\delta < 1$, $C$ and $R_0$ are some positive constants. Let $h(u)$ be a functional in $H^1(\Omega)$ defined by

$$
h(u) = \begin{cases} 
    \frac{b(u)}{a(u)} & \text{for } u \neq 0; \\
    0 & \text{for } u = 0.
\end{cases}
$$
Denote by
\[ M_0 = \{ u \in H^1(\Omega) \mid h(u^+) = h(u^-) = 1 \}; \]
\[ N = \{ u \in H^1(\Omega) \mid |h(u^+) - 1| < \frac{1}{2} \}, \]
where \( u^+ = \max\{u, 0\} \) and \( u^- = \max\{-u, 0\} \).

**Lemma 12.**
(i) If \( u \in H^1(\Omega) \) changes sign, then there are positive numbers \( s^\pm(u) = s^\pm \) such that \( s^+u^+ \pm s^-u^- \in M(\Omega) \);
(ii) There exists a \( c' > 0 \) such that \( \|u^\pm\|_{H^1} \geq c' > 0 \) for each \( u \in N \).

**Proof:**
(i) Since \( u^+ \) and \( u^- \) are nonzero, by Lemma 3 (i), it is easy to obtain the result.
(ii) For each \( u \in N \), by Lemma 3 (i), there exist \( s^\pm(u) = s^\pm > 0 \) such that \( s^\pm u^\pm \in M(\Omega) \). Then we have
\[ \frac{1}{2} < (s^\pm)^{2-p} = \frac{b(u^\pm)}{a(u^\pm)} < \frac{3}{2} \text{ for each } u \in N. \]  
By Lemma 4, we have
\[ \|s^\pm u^\pm\|_{H^1} \geq c \text{ for some } c > 0 \text{ and each } u \in N. \]
Thus, by (3), we have \( \|u^\pm\|_{H^1} \geq c/s^\pm \geq c' > 0 \) for each \( u \in N \).  

Define
\[ \gamma = \inf_{u \in M_0} J(u). \]
By Lemma 12, \( \gamma > 0 \).

**Lemma 13.** There exists a sequence \( \{u_k\} \subset N \) such that \( J(u_k) = \gamma + o(1) \) and \( J'(u_k) = o(1) \) strongly in \( H^{-1}(\Omega) \).

**Proof:** It is similar to the proof of Zhu [16].

**Lemma 14.** Let \( f \) and \( g \) are real-valued functions in \( \Omega \). If \( g(x) > 0 \) in \( \Omega \), then we have the following inequalities.
\begin{itemize}
  \item[(i)] \( (f + g)^+ \geq f^+ \);
  \item[(ii)] \( (f + g)^- \leq f^- \);
  \item[(iii)] \( (f - g)^+ \leq f^+ \);
  \item[(iv)] \( (f - g)^- \geq f^- \).
\end{itemize}

**Lemma 15.** Let \( \{u_k\} \subset N \) be a \((PS)_\gamma\)-sequence in \( H^1(\Omega) \) for \( J \) satisfying
\[ \alpha < \gamma < \alpha + \alpha^\infty \text{(<2}\alpha^\infty). \]
Then there exists a \( v_2 \in M_0 \) such that \( u_k \) converges to \( v_2 \) strongly in \( H^1(\Omega) \). Moreover, \( v_2 \) is a higher energy solution of Equation (1) such that \( J(v_2) = \gamma \).
PROOF: By the definition of the (PS)$_J$-sequence in $H^1(\Omega)$ for $J$, it is easy to see that \( \{u_k\} \) is a bounded sequence in $H^1(\Omega)$ and satisfies
\[
\int_{\Omega} [\|\nabla u_k^+\|^2 + \|u_k^-\|^2] = \int_{\Omega} q(x)|u_k^\pm|^p + o(1).
\]
By Lemma 12 (ii), there exists a $C > 0$ such that
\[
C \leq \int_{\Omega} [\|\nabla u_k^+\|^2 + \|u_k^-\|^2] = \int_{\Omega} q(x)|u_k^\pm|^p + o(1).
\]
By the Decomposition Lemma 7, we have $\gamma = J(v_2) + \sum_{i=1}^{l} J^\infty(w_i)$, where $v_2$ is a solution of Equation (1) in $\Omega$ and $w_i$ is a solution of Equation (2) in $\mathbb{R}^N$. Since $J^\infty(w_i) \geq \alpha^\infty$ for each $i \in \mathbb{N}$ and $\alpha < \alpha^\infty$, we have $l \leq 1$. Now we want to show that $l = 0$. On the contrary, suppose $l = 1$.

(i) $w_1$ is a changed sign solution of Equation (2): by Lemma 6, we have $\gamma \geq 2\alpha^\infty$, which is a contradiction.

(ii) $w_1$ is a constant sign solution of Equation (2): we may assume $w_1 > 0$. By the Decomposition Lemma 7, there exists a sequence $\{x_k\}$ in $\mathbb{R}^N$ such that $|x_k| \to \infty$, and
\[
\|u_k - v_2 - w_1(\cdot - x_k^1)\|_{H^1(\Omega)} = o(1) \text{ as } k \to \infty.
\]
By the Sobolev continuous embedding inequality, we obtain
\[
\|u_k - v_2 - w_1(\cdot - x_k^1)\|_{L^p(\Omega)} = o(1) \text{ as } k \to \infty.
\]
Since $w_1 > 0$, by Lemma 14, then
\[
\|(u_k - v_2)^-\|_{L^p(\Omega)} = o(1) \text{ as } k \to \infty.
\]
Suppose $v_2 \equiv 0$, we obtain $\|u_k^-\|_{L^p(\Omega)} = o(1) \text{ as } k \to \infty$. Then
\[
0 < C \leq \int_{\Omega} q(x)|u_k^-|^p = o(1),
\]
which is a contradiction. Hence, $v_2 \neq 0$. So we have $\gamma = J(v_2) + J^\infty(w_1) \geq \alpha + \alpha^\infty$, which a contradiction.

By (i) and (ii), then $l = 0$. Thus, $\|u_k - v_2\|_{H^1(\Omega)} = o(1) \text{ as } k \to \infty$ and $J(v_2) = \gamma$. Similarly, by Lemma 14, we obtain that $v_2$ is a changed sign solution of Equation (1) in $\Omega$. By Lemma 6, $2\alpha \leq \gamma$. 

Recall that $u_k(x) = w(x + e_k) |_{\Omega}$, where $e_k = (k, 0, \ldots, 0)$ and $w$ is a positive ground state solution of Equation (2) in $\mathbb{R}^N$. Then we have the following results.

\[\]
LEMMA 16. There are $k_0 \in \mathbb{N}$, real numbers $t_1^*$ and $t_2^*$ such that for $k \geq k_0$

$$t_1^* v_1 - t_2^* w_k \in M_0$$

and $\gamma \leq J(t_1^* v_1 - t_2^* w_k)$, where $1/2 \leq t_1^*, t_2^* \leq 2$.

PROOF: The proof is similar to Zhu [16] or see Cao [3].

LEMMA 17. There exists a $k_0^* \in \mathbb{N}$ such that for $k \geq k_0^* \geq k_0$

$$\gamma \leq \sup_{1/2 \leq t_1, t_2 \leq 2} J(t_1 v_1 - t_2 w_k) < \alpha + \alpha^\infty,$$

where $v_1$ is a ground state solution of Equation (1) in $\Omega$.

PROOF: Since $v_1$ is a positive solution of Equation (1) in $\Omega$ and $w_k > 0$ for each $k \in \mathbb{N}$, we have

$$J(t_1 v_1 - t_2 w_k) = \frac{1}{2} a(t_1 v_1) + \frac{1}{2} a(t_2 w_k) - t_1 t_2 \left( \int_\Omega \nabla v_1 \nabla w_k + v_1 w_k \right) - \frac{1}{p} b(t_1 v_1 - t_2 w_k)$$

$$\leq J(t_1 v_1) + J^\infty(t_2 w) - \frac{1}{p} b(t_1 v_1 - t_2 w_k) + \frac{1}{p} b(t_1 v_1) + \frac{1}{p} b^\infty(t_2 w_k)$$

We use the inequality

$$(c_1 - c_2)^p > c_1^p + c_2^p - K(c_1^{p-1} c_2 + c_1 c_2^{p-1}),$$

for any $c_1, c_2 > 0$ and some positive constant $K$, then

$$\sup_{1/2 \leq t_1, t_2 \leq 2} J(t_1 v_1 - t_2 w_k) \leq \sup_{t_1 \geq 0} J(t_1 v_1) + \sup_{t_2 \geq 0} J^\infty(t_2 w) - \frac{1}{2p} \int_\Omega (q(x) - q_\infty) w_k^p$$

$$+ K \left( \int_\Omega v_1^{p-1} w_k + w_k^{p-1} v_1 \right) + \frac{2p}{p} \int_{\Omega \times \mathbb{R}^n} q_\infty w_k^p.$$

The following estimates is similar to Lemma 10.

(i) \[ \int_{\Omega \times \mathbb{R}^n} q_\infty w_k^p = \int_{\Omega \times \mathbb{R}^n} q_\infty |w(x + e_k)|^p dx \leq C_0^p \exp(-(p-\varepsilon)k). \]

(ii) By the Hölder inequality,

$$\int_{\Omega \cap \{|x| \leq R_1\}} v_1^{p-1} w_k \leq \left( \int_{\Omega \cap \{|x| \leq R_1\}} v_1^p \right)^{(p-1)/p} \left( \int_{\Omega \cap \{|x| \leq R_1\}} w_k^p \right)^{1/p} \leq M \exp(-k).$$

Applying Lemma 9, there exists a $k_1 \geq k_0$ such that for $k \geq k_1$

$$\int_{\Omega \cap \{|x| \geq R_1\}} v_1^{p-1} w_k \leq C_1 \int_{\Omega \cap \{|x| \geq R_1\}} \exp(-(p-1)|x|) \exp(-|x + e_k|) dx \leq C_1^\prime \exp(-k).$$
Similarly, we also obtain
\[\int_{|x| \leq R_1} u_k^{p-1} v_1 \leq M' \exp(-(p-1)k),\]
\[\int_{|x| \leq R_0} |q(x) - q_\infty| u_k^p \leq M'' \exp(-(p-\varepsilon)k),\]
and there exists a \(k_2 \geq k_1\) such that for \(k \geq k_2\)
\[\int_{|x| \geq R_1} u_k^{p-1} v_1 \leq C' \exp(-k).\]

(iii) Since \(q\) satisfies the condition \((\overline{Q})\) and \(0 < \delta < 1\), by Lemma 9, there exists a \(k_3 \geq k_2\) such that for \(k \geq k_3\)
\[\int_{|x| \geq R_0} (q(x) - q_\infty) u_k^p \geq C'' \exp(-\delta k).\]

By (i)–(iii) and \(2 < p < 2^*\), for small \(\varepsilon < 1\) we can find a \(k^*_0 \geq k_3 \geq k_0\) such that for \(k \geq k^*_0\)
\[K'' \left(\int_\Omega v_1^{p-1} w_k + u_k^{p-1} v_1 \right) + \frac{2p}{p} \int_{\Omega \times \mathbb{R}^n} q_\infty u_k^p - \frac{1}{2p} \int_\Omega (q(x) - q_\infty) u_k^p \, dx < 0.\]
Since \(J(v_1) = \sup_{t \geq 0} J(tv_1)\) and \(J^\infty(w) = \sup_{t \geq 0} J^\infty(tw)\), we have for \(k \geq k^*_0\)
\[\sup_{1/2 \leq t_1, t_2 \leq 2} J(t_1 v_1 - t_2 w_k) < J(v_1) + J^\infty(w) = \alpha + \alpha^\infty.\]

**Theorem 18.** Assume that \(q\) satisfies \((q_1), (q_2)\) and the condition \((\overline{Q})\), then Equation (1) has a positive solution \(v_1\) and a solution \(v_2\) which changes sign.

**Proof:** By Lemmas 13, 15 16 17 and Theorem 11.

**References**


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