

THE SPECTRUM OF SKEW ROOM SQUARES

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Abstract

We give a short proof that skew Room squares exist for all odd sides s exceeding 5.

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1. Introduction

A *Room square* of side s is a square array R of side s , satisfying:

- (1) each cell of R is either empty or contains a 2-subset of a set S of size $s + 1$,
- (2) each symbol occurs exactly once in each row and each column of R ,
- (3) every 2-subset of S occurs in a unique cell of R .

The existence question for Room squares has been solved. The following is shown by Mullin and Wallis in [4].

THEOREM 1.1. *A Room square of side s exists if and only if s is an odd positive integer other than 3 or 5.*

A Room square R , on symbol set S , is said to be *standardized* with respect to $\infty \in S$ provided that the rows and columns of R have been permuted so that ∞ occurs in the cells on the main diagonal of R . We will index the rows and

columns of such a standardized Room square by $S \setminus \{\infty\}$, so that $\{\infty, x\}$ occurs in cell (x, x) , for all $x \in S \setminus \{\infty\}$.

A standardized Room square is *skew* provided that, of any two cells (i, j) and (j, i) with $i \neq j$, precisely one is empty. Skew Room squares have been studied extensively: see the list of references.

We show here that a skew Room square of side s exists if and only if s is an odd positive integer other than 3 or 5, that is, skew Room squares exist for precisely the same set of sides as Room squares.

2. Frames and Room squares

Let S be a set of size s , and partition $S = \cup_{j=1}^n S_j$.

An $\{S_1, \dots, S_n\}$ -*frame* is a square array F of side s , having rows and columns indexed by S , which satisfies:

- (1) each cell is either empty or contains a 2-subset of S ,
- (2) the subsquares S_j^2 of F are empty, for $1 \leq j \leq n$,
- (3) row (or column) x contains precisely the symbols $S \setminus S_j$, where $x \in S_j$,
- (4) the 2-subsets occurring in F are precisely those $\{x, y\}$ such that $(x, y) \in S^2 \setminus \cup_{j=1}^n S_j^2$.

A frame is *skew* provided that for all $(i, k) \in S^2 \setminus \cup_{j=1}^n S_j^2$, exactly one of cell (i, k) or (k, i) is empty.

A Room square of side s (standardized with respect to some element $\infty \notin S$) can be constructed from an $\{S_i: s \in S\}$ -frame by placing $\{\infty, x\}$ in cell (x, x) , for all $x \in S$. Similarly, a skew Room square can be constructed from a skew frame of that type.

The study of Room squares predates that of frames, of course. However, the above construction indicates the fact that frames are a generalization of Room squares. The study of frames seems to be of interest in its own right, as well for applications in the construction of Room squares and Howell designs. The reader is referred to [1], [2], [5], [6], [10], [12], [15],[16], [17], and [18].

We make use of frames as follows.

LEMMA 2.1. *Suppose a skew $\{S_1, \dots, S_n\}$ -frame exists, and suppose also that skew Room squares of side $|S_i| + 1$ exist, for $1 \leq i \leq n$. Then a skew Room square of side $1 + \sum_{i=1}^n |S_i|$ exists.*

PROOF. See [16, Theorem 3.1]

We will describe a construction for frames which uses group-divisible designs (GDDs). A GDD is a triple $(X, \mathcal{G}, \mathcal{A})$ where X is a set (of *points*), \mathcal{G} is a set of subsets of X (called *blocks*), such that every 2-subset of X not contained in a group is contained in a unique block, and such that a group and a block contain at most one common point. A *weighting* of a GDD is a map $w: X \rightarrow \mathbf{Z}^+ \cup \{0\}$. The following construction is a generalization of [16, Theorem 2.1], where all weights are equal. The reader will note the similarity between this construction and Wilson's fundamental construction for GDDs [20].

CONSTRUCTION 2.2. Suppose $(X, \mathcal{G}, \mathcal{A})$ is a GDD and w is a weighting. For every $x \in X$, let S_x be a set of size $w(x)$. Suppose these S_x 's are disjoint, and for any $T \subseteq S$, define $S_T = \bigcup_{x \in T} S_x$. For every block $A \in \mathcal{A}$, suppose F_A is a skew $\{S_x: x \in A\}$ -frame. Let F be the array obtained by superimposing all these F_A 's. Then F is a skew $\{S_G: G \in \mathcal{G}\}$ -frame.

In the next section, we derive a corollary to the above construction which will enable us to close the spectrum of skew Room squares, given an initial segment.

We close this section by indicating four frames we will use in the next section. We say that an $\{S_1, \dots, S_n\}$ -frame has *type* $t_1^{u_1} \cdots t_k^{u_k}$ if there are u_i S_j 's of size t_i , for $1 \leq i \leq k$.

LEMMA 2.3. *There exist skew frames of type 4^4 , $4^4 2^1$, 4^5 , and $4^4 6^1$.*

PROOF. See [17].

3. The spectrum

LEMMA 3.1. *Suppose there exist three MOLS (mutually orthogonal Latin squares) of order m , and skew Room squares of sides $4m + 1$ and $2t + 1$, where $0 < t \leq 3m$. Then a skew Room square of side $16m + 2t + 1$ exists.*

PROOF. Since three MOLS of order m exist, there is a GDD $(X, \mathcal{G}, \mathcal{A})$ with five groups of size m and m^2 blocks of size 5.

Define a weighting $w: X \rightarrow \{0, 2, 4, 6\}$ by setting $w(x) = 4$ if $x \in X \setminus G_5$, and defining $w(x)$ for $x \in G_5$ so that $\sum_{x \in G_5} w(x) = 2t$.

Now apply Construction 2.2. A block A requires one of the four frames of Lemma 2.3, and the result is a skew frame of type $4m^4 2t^1$. Apply Lemma 2.1 to obtain a skew Room square of side $16m + 2t + 1$.

Denote $\text{SRS} = \{s \geq 7: \text{a skew Room square of side } s \text{ exists}\}$.

LEMMA 3.2 *Suppose $\{7, 9, \dots, 117\} \subseteq \text{SRS}$. Then $\text{SRS} = \{s \geq 7: s \text{ odd}\}$.*

PROOF. The proof is by induction on s . Let $s \geq 119$ be odd, and write $s = 16m + 2t + 1$ with m odd and $3 \leq t \leq 18$. Then $m \geq 7$, so $t \leq 3m$. Since m is odd and exceeds 5, there exist three MOLS of order m by [19]. Lemma 3.1 implies the result.

Thus we have only to construct skew Room squares of sides 7 through 117. We will do this mainly by using results which appear in the literature.

LEMMA 3.3. *Suppose $q = 2^nt + 1$ is a prime power with t odd, $t > 1$. Then $q \in \text{SRS}$.*

PROOF. Mullin and Nemeth [8].

LEMMA 3.4. *Suppose $s = 16t^2 + 1$. Then $s \in \text{SRS}$.*

PROOF. Dinitz [4].

LEMMA 3.5.

- (i) *If $\{u, v\} \subseteq \text{SRS}$, then $uv \in \text{SRS}$.*
- (ii) *If $\{u, v\} \subseteq \text{SRS}$ and $v \neq 7$, then $u(v - 1) + 1 \in \text{SRS}$.*
- (iii) *If $v \in \text{SRS}$, $v \neq 13$, and $u \equiv 1 \pmod{4}$ is a prime power, then $u(v - 1) + 1 \in \text{SRS}$.*

PROOF. For (i) and (ii), see [9]; for (iii), see [10].

The other small skew Room squares have been constructed by various methods, some of which are generalizations of the constructions of Lemma 3.5. In Table 1 below, we list sides from 7 to 117 which are not constructed by Lemmas 3.3 and 3.4, together with a reference describing their construction.

TABLE 1

<u>Order</u>	<u>Construction</u>
9	Beaman and Wallis [3]
15	Mullin, Schellenberg, Stinson and Vanstone [9]
21	[9]
33	Mullin and Wallis [13]
35	[9]
39	[9]
45	[9]
51	Lemma 3.5(iii), $51 = 5(11 - 1) + 1$
55	Lemma 3.5(iii), $55 = 9(7 - 1) + 1$
57	Lemma 3.5(ii), $57 = 7(9 - 1) + 1$
63	Lemma 3.5(i), $63 = 7 \cdot 9$
69	Stinson [17]
75	Anderson, Mullin and Stinson [1]
77	Lemma 3.5(i), $77 = 7 \cdot 11$
85	Lemma 3.5(ii), $85 = 7(13 - 1) + 1$
87	Lemma 3.1, $m = 5, t = 3$
91	Lemma 3.5(i), $91 = 7 \cdot 13$
93	Stinson [16]
95	Lemma 3.1, $m = 5, t = 7$
99	Lemma 3.5(i), $99 = 9 \cdot 11$
105	Lemma 3.5(i), $105 = 7 \cdot 15$
111	Lemma 3.5(ii), $111 = 11(11 - 1) + 1$
115	[1]
117	Lemma 3.5(i), $117 = 9 \cdot 13$

THEOREM 3.6. $SRS = \{s: s \geq 7 \text{ is odd}\}$.

PROOF. Lemmata 3.1–3.5 and Table 1.

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