Appendix C

Strings in $\mathcal{N} = 2$ SQED

In this Appendix we briefly review the Abelian Abrikosov–Nielsen–Olesen strings in $\mathcal{N} = 2$ supersymmetric QED in four dimensions. The BPS strings in this theory were first considered in [148, 35].

C.1 $\mathcal{N} = 2$ supersymmetric QED

$\mathcal{N} = 2$ supersymmetric QED is discussed in Section 8.1. Here we summarize basic features of this theory for convenience. The field content of $\mathcal{N} = 2$ supersymmetric QED consists of a U(1) vector $\mathcal{N} = 2$ multiplet as well as N_f matter hypermultiplets. The mass terms are introduced via the superpotential

$$W = \sum_A \left(m_A Q^A \tilde{Q}_A + \frac{1}{\sqrt{2}} \mathcal{A} Q^A \tilde{Q}_A \right). \tag{C.1}$$

For the definition of the Fayet–Iliopoulos term see Eq. (3.2.1). In this form it is the same in $\mathcal{N} = 1$ and $\mathcal{N} = 2$, cf. Eq. (4.1.5). The bosonic part of the action of this theory is

$$\begin{aligned} S = \int d^4x \Bigg\{ & \frac{1}{4g^2} F_{\mu\nu}^2 + \frac{1}{g^2} |\partial_\mu a|^2 + \bar{\nabla}_\mu \bar{q}_A \nabla_\mu q^A + \bar{\nabla}_\mu \tilde{q}_A \nabla_\mu \bar{\tilde{q}}^A \\ & + n_e^2 \frac{g^2}{2} \left(|q^A|^2 - |\tilde{q}_A|^2 - \xi \right)^2 + 2 n_e^2 g^2 \left| \tilde{q}_A q^A \right|^2 \\ & + \frac{1}{2} (|q^A|^2 + |\tilde{q}^A|^2) \left| a + \sqrt{2} m_A \right|^2 \Bigg\}, \end{aligned} \tag{C.2}$$

where

$$\nabla_\mu = \partial_\mu - i n_e A_\mu, \quad \bar{\nabla}_\mu = \partial_\mu + i n_e A_\mu. \tag{C.3}$$

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Here ξ is the coefficient in front of the Fayet–Iliopoulos term; we consider the FI D-term here while g is the U(1) gauge coupling and n_e is the electric charge. It can be integer or half integer. The index $A = 1, \ldots, N_f$ is the flavor index. Below we consider the case $N_f = 1$. This is the simplest case which admits BPS string solutions.

The FI term triggers the squark condensation. The vacuum of this theory is given by

$$a = -\sqrt{2}m, \quad q = \sqrt{\xi}, \quad \tilde{q} = 0. \tag{C.4}$$

Hereafter in search for string solutions we will stick to the *ansatz* $\tilde{q} = 0$.

Now let us discuss the mass spectrum of the light fields in this vacuum. The spectrum can be obtained by diagonalizing the quadratic form in (C.2). This is done in Ref. [35]; the result is as follows: one real component of the field q is eaten up by the Higgs mechanism to become the third component of the massive photon. Three components of the massive photon, one remaining component of q and four real components of the fields $\tilde{q}$ and a form one long $\mathcal{N} = 2$ multiplet (8 boson states + 8 fermion states), with mass

$$m_\gamma^2 = 2n_e^2\, g^2\, \xi. \tag{C.5}$$

C.2 String solutions

As soon as fields a and $\tilde{q}$ play no role in string solutions we can look for these solutions using the reduced theory with these fields set to zero. The bosonic action (C.2) reduces to

$$S = \int \mathrm{d}^4x \left\{ \frac{1}{4g^2} F_{\mu\nu}^2 + |\nabla_\mu q|^2 + \frac{g^2}{2}\, n_e^2 \left(|q|^2 - \xi \right)^2 \right\}. \tag{C.6}$$

Since the U(1) gauge group is spontaneously broken, the model supports conventional ANO strings [36]. The topological stability of the ANO string is due to the fact that $\pi_1(\mathrm{U}(1)) = Z$.

Let us derive the first-order equations which determine the string solution making use of the Bogomol'nyi representation [5] of the model (C.6). We have for the string tension

$$\begin{aligned} T = \int d^2x \Bigg\{ \left[\frac{1}{\sqrt{2}g} F_3^* + \frac{g}{\sqrt{2}}\, n_e\big(|q|^2 - \xi \big) \right]^2 \\ + \left| \nabla_1\, q + i \nabla_2\, q \right|^2 + n_e\, \xi\, F_3^* \Bigg\}, \end{aligned} \tag{C.7}$$

where $F_3^* = F_{12}$ and we assume that the fields in this expression depend only on the coordinates x_i, $i = 1, 2$.

The Bogomol'nyi representation (C.7) leads us to the following first-order equations:

$$F_3^* + gn_e\big(|q|^2 - \xi\big) = 0,$$
$$(\nabla_1 + i\nabla_2)q = 0. \tag{C.8}$$

Once these equations are satisfied the energy of the BPS object is given by the last surface term in (C.7). Note that representation (C.7) can be written also with different sign in front of the terms proportional to the gauge fluxes. This would give first-order equations for the anti-string, with negative values of gauge fluxes.

For the topologically stable string solution, the scalar field winds n times in U(1) gauge group when we move around the string along a large circle in the (x, y) plane (we assume that the string stretches along the z-axis),

$$q \sim \sqrt{\xi}\, e^{in\alpha},$$
$$A_i \sim \frac{n}{n_e}\,\partial_i\alpha, \quad r \to \infty, \tag{C.9}$$

where r and α are the polar coordinates in the (x, y) plane (see Fig. 3.6) and $i = 1, 2$. This ensures that the flux of the string is

$$\int d^2x\, F_3^* = \frac{2\pi n}{n_e}. \tag{C.10}$$

The tension of the string with winding number n is determined by the surface term in (C.7),

$$T_n = 2\pi n\,\xi. \tag{C.11}$$

For the elementary $n = 1$ string the solution can be found using the standard *ansatz* [5]

$$q(x) = \phi(r)\, e^{i\,\alpha}\,, \quad A_i(x) = \frac{1}{n_e}\,\partial_i\alpha\,[1 - f(r)]\,, \tag{C.12}$$

where we introduced two profile functions ϕ and f for the scalar and gauge fields, respectively.

The *ansatz* (C.12) goes through the set of equations (C.8), and we get the following two equations for the profile functions:

$$-\frac{1}{r}\frac{df}{dr} + n_e^2 g^2(\phi^2 - \xi) = 0\,, \quad r\,\frac{d\phi}{dr} - f\,\phi = 0. \tag{C.13}$$

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The boundary conditions for the profile functions are the following. At large distances we have

$$\phi(\infty)=\sqrt{\xi}, \quad f(\infty)=0. \tag{C.14}$$

At the origin the smoothness of the field configuration at hand requires

$$\phi(0)=0, \quad f(0)=1. \tag{C.15}$$

These boundary conditions are such that the scalar field reaches its vacuum value at infinity. The same first-order equations arise for the BPS vortex in $\mathcal{N}=1$ QED in $(2+1)$ dimensions, see Chapter 3. The fermion zero modes for the BPS vortices in $(3+1)$ and $(2+1)$ dimensions are different, however. Equations (C.13) have a unique solution for the profile functions, which can be found numerically [4], see Fig. 3.7. The string transverse size is $\sim 1/m_\gamma$.

First-order equations (C.13) can also be obtained using supersymmetry. We start from the supersymmetry transformations for the fermion fields in the theory (C.2),

$$\begin{aligned}
\delta\lambda^{\alpha f} &= \frac{1}{2}(\sigma_\mu\bar{\sigma}_\nu\epsilon^{f})^{\alpha}F_{\mu\nu}+\epsilon^{\alpha p}F^{m}(\tau^{m})_{p}^{f} \;+\ldots,\\
\delta\bar{\tilde{\psi}}_{\dot{\alpha}}^{A} &= i\sqrt{2}\,\bar{\nabla}_{\dot{\alpha}\alpha}q_{f}^{A}\epsilon^{\alpha f} \;+\cdots,\\
\delta\bar{\psi}_{\dot{\alpha}A} &= i\sqrt{2}\,\bar{\nabla}_{\dot{\alpha}\alpha}\bar{q}_{fA}\epsilon^{\alpha f} \;+\cdots
\end{aligned} \tag{C.16}$$

Here $f=1,2$ is the $\mathrm{SU}(2)_R$ index so $\lambda^{\alpha f}$ are the fermions from the $\mathcal{N}=2$ vector supermultiplet, while q^{Af} denotes the $\mathrm{SU}(2)_R$ doublet of the squark fields q^A and $\bar{\tilde{q}}^A$ in the quark hypermultiplets. The parameters of the SUSY transformations are denoted as $\epsilon^{\alpha f}$. Furthermore, the F terms in Eq. (C.16) are

$$F^3=-i\,n_e\,g^2\,(\mathrm{Tr}\,|q|^2-\xi), \quad F^1+iF^2=0. \tag{C.17}$$

The dots in (C.16) stand for terms involving the a field which vanish on the string solution because it is given by its vacuum expectation value (C.4).

In Ref. [35] it was shown that four (real) supercharges generated by

$$\epsilon^{12}, \quad \epsilon^{21} \tag{C.18}$$

act trivially on the BPS string. Namely imposing conditions $\epsilon^{11}=\epsilon^{22}=0$ and requiring that the left-hand sides of Eqs. (C.16) are zero we get the first-order equations (C.13) upon substitution of the *ansatz* (C.12).[1]

[1] If we instead of (C.18) impose that different combinations of SUSY transformation parameters act trivially we get the equations for anti-string with the opposite directions of gauge fluxes.

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C.3 The fermion zero modes

The string is half-critical, so 1/2 of the supercharges (related to the SUSY transformation parameters ϵ^{12} and ϵ^{21}), act trivially on the string solution. The remaining four (real) supercharges parametrized by ϵ^{11} and ϵ^{22} generate four (real) supertranslational fermion zero modes. They have the form [35]

$$\begin{aligned}
\bar{\psi}_{\dot{2}} &= -2\sqrt{2}\,\frac{x_1 + i x_2}{r^2}\, f\, \phi\, \alpha^{11}, \\
\bar{\tilde{\psi}}_{\dot{1}} &= 2\sqrt{2}\,\frac{x_1 - i x_2}{r^2}\, f\, \phi\, \alpha^{22}, \\
\bar{\psi}_{\dot{1}} &= 0, \quad \bar{\tilde{\psi}}_{\dot{2}} = 0, \\
\lambda^{11} &= -i n_e g^2 (\phi^2 - \xi) \alpha^{11}, \\
\lambda^{22} &= i n_e g^2 (\phi^2 - \xi) \alpha^{22}, \\
\lambda^{12} &= 0, \quad \lambda^{21} = 0,
\end{aligned} \tag{C.19}$$

where the modes proportional to complex Grassmann parameters α^{11} and α^{22} are generated by ϵ^{11} and ϵ^{22} transformations, respectively.

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