On the uninodal quartic curve.

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1. In comparison with the general plane quartic on the one hand, and the curves having either two or three nodes on the other, the uninodal curve has been neglected.[1] Many of its properties may of course be deduced from those of the general quartic in the limiting case when an oval shrinks to a point or when two branches approach and ultimately unite. The modifications of properties of the bitangents are shewn more clearly by Geiser's method[2], in which these lines are obtained by projecting the lines of a cubic surface from a point on the surface. As the point moves up to and crosses a line on the surface, the quartic acquires a node and certain pairs of bitangents obviously coincide, viz. those obtained by projecting two lines coplanar with that on which the point lies. A nodal quartic curve and its double tangents may also be obtained by projecting a cubic surface which has a conical point from an arbitrary point on the surface. Each of these three methods leads us to the conclusion that, when a quartic acquires a node, twelve of the double tangents coincide two and two and become six tangents from the node, and the other sixteen remain as genuine bitangents: the twelve which coincide are six pairs of a Steiner complex. More precisely, if we adopt Hesse's notation for the 28 bitangents of a general quartic explained in Salmon, we may consider that the pairs:— 17, 18: 27, 28: 37, 38: 47, 48: 57, 58: 67, 68: —coincide and become six lines which pass through the node and touch the curve at a second point, while the other sixteen remain as genuine double tangents. But the symbols 7 and 8 have now become unnecessary. The six tangents from the node may be denoted by 1, 2, 3, 4, 5, 6, (1 being the line in which 17 and 18 coincide): fifteen of the sixteen genuine bitangents are denoted by a pair of these six symbols; while the last, originally denoted by 78, may be represented by some special symbol such as Ω.

[1] But see Hilton, *Plane Algebraic Curves*, pp. 298-303. Clebsch (*Vorlesungen*), Brill (*Math. Annalen*, 6 (1873), p. 66) and others have considered the curve in connection with hyperelliptic functions. But a treatment by the elementary methods which are applied successfully to the general quartic is wanting.

[2] *Math. Annalen*, 1 (1869) p, 129, Hilton, p. 345.

From the results tabulated in Salmon and elsewhere concerning sets of bitangents of the general quartic it is now easy to derive the corresponding properties of the uninodal quartic. To take one instance:—in the general quartic, four bitangents denoted by symbols such as 12, 34, 56, 78 or such as 12, 23, 34, 41 have their eight points of contact on a conic; other such sets are obtained by permuting the symbols 1, 2, 3, 4, 5, 6, 7, 8, in any way. Hence in the uninodal quartic, four bitangents such as Ω, 12, 34, 56, or four such as 12, 23, 34, 41, or any sets derived by permuting the symbols 1, 2, 3, 4, 5, 6, have their eight points of contact on a conic. And further the following sets of two bitangents and two tangents from the node:— Ω, 12, 1, 2: 12, 34, 5, 6: 12, 13, 2, 3:—have their points of contact on a conic which passes through the node. And so on.

2. The existence of a double point on a curve offers possibilities for the reduction of the equation to a standard form which are absent in the general curve. No one has discovered a satisfactory canonical form for a ternary quartic; there appears to be no point, or line, or triangle . . . which stands in a specially simple geometrical relation to the curve, and investigations are usually based upon the form $xyzw = f^2$, a form to which the equation of the curve can be reduced in 315 ways. But if the curve has a node, it and the two nodal tangents at once provide a point and two lines uniquely related to the curve, and on this foundation we may build further results.

In the case of a quartic curve having a cusp,[3] the equation of the curve can be reduced in one and only one way to the form

$$F \equiv x^2 z^2 + 2y^3 z - (Ax^4 + Bx^3 y + Cx^2 y^2 + Dxy^3 + Ey^4)$$

or

$$x^2 F \equiv (x^2 z + y^3)^2 - (y^6 + Ey^4 x^2 + Dy^3 x^3 + Cy^2 x^4 + Byx^5 + Ax^6)$$
$$\equiv (x^2 z + y^3)^2 - (y + ax)(y + bx)(y + cx)(y + dx)(y + ex)(y + fx)$$

where $$a + b + c + d + e + f = 0$$

Here $x = 0$, $y = 0$ is the cusp, and $x = 0$ the tangent there: the first polar of the cusp is a cusped cubic, and the triangle of reference is so chosen that this cubic has the standard form of equation $x^2 z + y^3 = 0$. The six lines

$$y + ax = 0, \; y + bx = 0, \; \ldots \; y + fx = 0,$$

[3] See *Quarterly Journal of Mathematics*, 26 (1893), 5–26.

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are the six tangents from the cusp, and the ten bitangents of the curve all have precisely similar equations such as

$$2(y^3 + x^2 z) = (y + ax)(y + dx)(y + ex) + (y + bx)(y + cx)(y + fx)$$

from which an extraneous factor x^2 may be removed. With these formulae properties of the curve may be discovered or verified; in particular a classification of cusped quartic curves with the number of branches, the reality and position of the bitangents, is easily effected: it depends solely upon the constants a, b, c, d, e, f. An analogous discussion of the nodal quartic will be attempted here, but the problem is complicated and only the forms of equations and one curious result will be given.

3. With a triangle of reference XYZ, the equation of a quartic curve which has a node at Z is $F = 0$, where

$$F \equiv z^2 u_2(x, y) - 2z\, u_3(x, y) + u_4(x, y),$$

or

$$u_2 F \equiv (z\, u_2 - u_3)^2 - (u_3{}^2 - u_2 u_4).$$

Now $u_2 = 0$ is the equation of the two tangents at Z, and $u_3{}^2 = u_2 u_4$ is the equation of six lines through Z which touch the quartic at six other points. The first polar of Z, which passes through the six points of contact[4] is the nodal cubic

$$zu_2 = u_3.$$

But if $x = 0$ and $y = 0$ are the tangents at the node—(they will be thought of as real in this paper)—of this cubic and also of the quartic, and $z = 0$ the line of inflexions of the cubic, its equation is reduced to the standard form

$$zxy = x^3 + y^3,$$

and the equation of the quartic $F = 0$ at the same time becomes

$$F \equiv xyz^2 - 2(x^3 + y^3)z + u_4(x, y)$$

or

$$xyF \equiv (xyz - x^3 - y^3)^2 - (x^3 + y^3)^2 + xy\, u_4(x, y)$$

With the factor xy introduced, the equation of the quartic may be written in the convenient shape

$$\begin{aligned}(zxy - x^3 - y^3)^2 &= (x^3 + y^3)^2 - xy\, u_4(x, y)\\ &= (x - ay)(x - by)(x - cy)(x - dy)(x - ey)(x - fy)\end{aligned}$$

where

$$abcdef = 1.$$

[4] The six points of contact lie also on a conic.

The lines $x = ay$, $x = by$, $\ldots$, $x = fy$, are the six tangents from the node Z. If a, b, c, d, e, f are the roots of a sextic equation

$$t^6 - s_1 t^5 + s_2 t^4 - s_3 t^3 + s_4 t^2 - s_5 t + 1 = 0,$$

$$F \equiv xyz^2 - 2(x^3 + y^3)z + s_1 x^4 - s_2 x^3 y + (s_3 + 2) x^2 y^2 - s_4 xy^3 + s_5 y^4$$

4. *Double tangents.* Suppose now that $z = hx + ky$ is a double tangent of the curve, and that its contacts lie on the two lines through Z given by

$$v \equiv lx^2 + mxy + ny^2 = 0.$$

Substituting for z in F we must have

$$(hx^2 y + kxy^2 - x^3 - y^3)^2 - (x - ay)(x - by) \ldots (x - fy) \equiv xyv^2$$

or $$(x^3 - hx^2 y - kxy^2 + y^3)^2 - xyv^2 \equiv (x - ay)(x - by) \ldots (x - fy).$$

The left hand member will factorize if we use the square root of xy; to avoid radicals we will replace

$$x, y, a, b, c, d, e, f \text{ by } X^2, Y^2, A^2, B^2, C^2, D^2, E^2, F^2.$$

Then $$(X^6 - hX^4 Y^2 - kX^2 Y^4 + Y^6)^2 - X^2 Y^2 V^2$$
$$\equiv (X^2 - A^2 Y^2)(X^2 - B^2 Y^2) \ldots (X^2 - F^2 Y^2)$$

where $$V \equiv lX^4 + mX^2 Y^2 + nY^4.$$

The left hand member is now the product of two factors

$$X^6 - hX^4 Y^2 - kX^2 Y^4 + Y^6 \pm XYV$$

which differ only in the sign of Y, and each bracket on the right is the product of two factors (*e.g.* $X \pm AY$) which differ in the same way. Each factor on the left must be equal to the product of six factors on the right, one from each bracket; the second factor on the left will then be equal to the product of the other six factors on the right. Now the quantities A, B, C, D, E, F, have not yet been exactly defined; $A^2 = a$, but A may be either square root of a, and there is little to guide us. If the values of A, B, C, D, E, F are chosen at random, the product $ABCDEF$ must be ± 1, since

$$A^2 B^2 C^2 D^2 E^2 F^2 = abcdef = 1,$$

and it is convenient here to suppose the values of A, B, C, D, E, F so chosen that

$$ABCDEF = 1$$

and further that A, B, C, D, E, F are roots of a sextic

$$T^6 - S_1 T^5 + S_2 T^4 - S_3 T^3 + S_4 T^2 - S_5 T + 1 = 0.$$

We may then satisfy the previous formulae by writing

$$X^6 - hX^4Y^2 - kX^2Y^4 + Y^6 + XYV$$
$$\equiv (X + AY)(X + BY)(X + CY)(X + DY)(X + EY)(X + FY),$$
$$X^6 - hX^4Y^2 - kX^2Y^4 + Y^6 - XYV$$
$$\equiv (X - AY)(X - BY)(X - CY)(X - DY)(X - EY)(X - FY);$$

for the coefficients of X^6 and Y^6 agree; and if in addition

$$-h = S_2; \quad -k = S_4; \quad V \equiv S_1 X^4 + S_3 X^2Y^2 + S_5 Y^4;$$

the identity is complete. Hence with these values of A, B, C, D, E, F, we have proved that

$$z + S_2 x + S_4 y = 0$$

is a double tangent, and that its points of contact lie on

$$S_1 x^2 + S_3 xy + S_5 y^2 = 0.$$

5. *The system of double tangents.* Other bitangents may be obtained by changes of sign of A, B, C, D, E, F, which leave the product equal to 1, *i.e.* by an even number of changes of sign. But to change the sign of all six does not affect S_2 and S_4, or the other results of § 4; and, as a consequence, a change of sign of four is equivalent to a change of sign of the other two. However since two out of six constants may be selected in fifteen ways, fifteen other bitangents are derivable from that of § 4, *i.e.* the complete system. The double tangent of § 4 being denoted by Ω, the others may be called (AB), (AC), ... (EF), (AB) denoting the double tangent obtained from Ω by a change of sign of A and B. We have thus arrived at a notation which closely resembles that of § 1, except that the figures 1, 2, 3, 4, 5, 6 have been superseded by letters A, B, C, D, E, F. But the letters have a definite significance: further *any* bitangent is derivable from *any* other by a change of sign of two of the constants A, B, C, D, E, F; thus there is a complete symmetry among the equations of the bitangents which is not suspected if the notation is regarded as a matter of symbols only.

These equations confirm many results obtained by the methods of § 1; on occasion they are to be preferred to the old methods. One problem which the new formulae answer most readily is that of classification of nodal quartic curves, especially those having a crunode (or node with real tangents) which present some novel features. If the nodal curve is reached as a degenerate form of the curve without a singular point, it is difficult to be certain that no case is overlooked; and neither of the methods of deriving the curve by projection from a cubic surface will give all its varieties. Projection from a point on a line of a non-singular cubic surface cannot give a quartic which has no real tangents from the node; and projection of a cubic surface with a conical point cannot give a nodal quartic which has no real bitangent. Yet both these types of nodal quartics exist.

6. *Classification of crunodal quartic curves.* The equation of a given curve can be reduced to the form used in § 3 in one way and only one; the classification therefore depends entirely upon the constants contained in that form: since the six constants a, b, c, d, e, f (whose product was shown to be unity), are roots of a sextic equation with real coefficients, crunodal quartic curves fall into four main classes, according as six, four, two or none of these constants are real.

If, in the equation of the curve in § 3, x/y is denoted by s, the coordinates of points are found as functions of s:—

$$x : y : z :: s^2 : s : s^3 + 1 \pm S^{\frac{1}{2}},$$

where $$S = (s-a)(s-b)(s-c)(s-d)(s-e)(s-f).$$

Hence all lines through the node Z given by values of s for which S is positive meet the curve in two real points, and those for which S is negative do not meet the curve. Further S is positive for values of s in the neighbourhood of 0 and ∞. Now if two, or four, or six, of the constants a, b, c, d, e, f are real, two, or four, or six lines drawn through Z touch the curve; and, being produced indefinitely in both directions, divide the plane into two, or four, or six compartments. Imaginary values of the constants occur in pairs and give quadratic factors of S which are always positive. Thus S changes sign only on passing from one compartment to the next, and the curve lies wholly in one, two or three of the two, four or six compartments, and not in any of the alternate intervening compartments. The curve therefore consists of three distinct parts, or is *tripartite* if the six constants are all real; it has two distinct parts, or is *bipartite* if four are real; it is unipartite if only two are real. Lastly, if none are real, S is positive for all values of s, and *every* line through Z meets the curve in two real points. The curve is unipartite.

7. *Reality of the double tangents.* The equations of the double tangents found in §§ 4 and 5 depend upon the square roots of a, b, c, d, e, f; these were denoted by A, B, C, D, E, F, the doubtful signs being so restricted that the product $ABCDEF = 1$, and it was proved that, with this restriction,

$$z + S_2 x + S_4 y = 0$$

is the equation of a double tangent. It is clear then that among real values of a, b, c, d, e, f, positive and negative values must be distinguished. Positive, negative, and complex values of a, b, c, d, e, f lead respectively to real, pure imaginary, and complex values of A, B, C, D, E, F. Since the product $abcdef = 1$, there must be an even number of each kind. An unexpected result presents itself if a, b, c, d, e, f are all real; viz., that if all are positive or all negative, all the sixteen double tangents are real; [5] if some are positive and some negative all the sixteen are imaginary. To carry out in detail the discussion of all the types of the curve would require more pages of algebraic work than can reasonably be asked for: but the geometrical explanation of this one rather surprising fact may not be superfluous. [All the double tangents must also be imaginary if two of the six constants a, b, c, d, e, f are positive, two negative and two complex.]

Branches of an algebraic curve are (as von Staudt first observed) of two sorts; those of even order which are met by any straight line in an even number of real points (possibly none) and those of odd order, met by every straight line in an odd number of real points. Any two branches of odd order intersect in at least one real point. Thus a curve without double points cannot have two branches of odd order, but a curve with a node, and in particular a nodal quartic (crunodal) may. If so, the quartic cannot have a real double tangent, because every line must meet each branch of odd order in one or three points and cannot touch the curve twice. But a line through the node may touch any branch of the curve at another point.

It is probable then that, in the exceptional curves which have no real bitangents at all, the node is formed by two distinct branches

[5] If all are negative, A, B, C, D, E, F are pure imaginary quantities, and S_2, S_4 are real.

of odd order intersecting; and this may be seen to be the case. For if at least two of the real constants $a, b, \ldots f$, are positive and at least two negative, the tangents ZX, ZY at the node lie in two different compartments into which the plane is divided by the tangents from Z. Hence the parts of the curve which touch ZX and ZY are distinct branches: they cross at Z at a finite angle, but a point moving continuously cannot pass from one to the other. Each of them is touched by the lines through Z which bound the compartment in which it lies, and such a line cuts each at a finite angle at Z. Each of the branches is therefore of odd order. In all other circumstances the lines ZX ZY lie in the same compartment, and the node is a point where a branch of even order crosses itself.

To discuss the other types of crunodal curves, and also acnodal curves (in which the node is isolated); the numbers of real bitangents, the reality of their points of contact and the branches on which they lie when real, etc., etc., would necessarily extend to many pages. As far as this has been done, no special difficulty has been met with, and the result scan as a rule be verified by other methods.