ERGODIC MEASURES FOR THE IRRATIONAL ROTATION ON THE CIRCLE

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Abstract

Riesz products are employed to give a construction of quasi-invariant ergodic measures under the irrational rotation of $T$. By suitable choice of the parameters such measures may be required to have Fourier-Stieltjes coefficients vanishing at infinity. We show further that these are the unique quasi-invariant measures on $T$ with their associated Radon-Nikodym derivative.


1. Introduction

Fix an irrational number $\alpha \in (0,1)$ and consider the action $T_\alpha(x) = x + \alpha \ (\text{mod } 1)$ on the circle group $T$. Lebesgue measure is well known to be the unique invariant measure for this action and it is ergodic. The question arises as to whether there exist singular quasi-invariant measures which are ergodic for the action of $T_\alpha$. Such measures allow the construction of non-monomial representations of the discrete Heisenberg group (see [1]):

$$H = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{Z} \right\}.$$
Measures of this kind were first constructed by Michael Keane in [5]. His proof used the continued fraction expansion of \( \alpha \) to identify the action of \( T_\alpha \) with the odometer action on a subset of an infinite product space. For this action it is clear that appropriate infinite product measures have the desired properties. Roughly the same end can be obtained by choosing a sequence \((n_k)\) of positive integers such that the fractional part \( \{n_k \alpha\} \) satisfies \( 0 < \{n_k \alpha\} < 2^{-k} \) and letting \( \nu \) be the infinite convolution measure

\[
\nu = \lim_{k \to \infty} \frac{1}{2} (\delta(-\{n_k \alpha\}) + \delta(\{n_k \alpha\}))
\]

where \( \delta(x) \) denotes the unit point mass at \( x \). The measure \( \nu \) is not itself quasi-invariant but it is ergodic. This is easily seen from a result of Gavin Brown and the author [3, Proposition 1]. Replacing \( \nu \) by a convex combination of its translates by all numbers of the subgroup \( D_\alpha = \{j \alpha: j \in \mathbb{Z}\} \) of \( \Pi \), we achieve a measure of the desired type.

A more general method for obtaining quasi-invariant ergodic measures which works for suitable actions on compact spaces \( X \) has been given by Katznelson and Weiss in [4]. Specifically, they show that a homeomorphism of a compact metric space has a quasi-invariant ergodic measure if and only if it has a recurrent point, and in these circumstances it has uncountably inequivalent quasi-invariant ergodic measures. Their technique is akin to those of [3] and [5] in that it employs a Cantor-type construction.

In some recent work of Larry Baggett, Alan Carey, Arlan Ramsay and the author on non-monomial representations of nilpotent groups [1], there arose the problem of finding measures quasi-invariant and ergodic for the action of \( T_\alpha \), and with Fourier-Stieltjes coefficients vanishing at infinity. Ultimately, we were able to complete the work without finding a solution to this problem. Nonetheless, it remains of some interest to construct such measures, and we do this here.

For very special irrationals it is possible to follow a method analogous to that of Keane. Let \( \alpha \) be an irrational number between 0 and 1 satisfying a quadratic equation of the form \( \alpha^2 + b\alpha + c = 0 \) where \( b, c \in \mathbb{Z} \) and \( c \neq \pm 1 \). Then \( \alpha^{-1} \) is not a Pisot number so that by a result of Salem [7, Chapter IV, Theorem II]

\[
\nu = \lim_{n \to \infty} \frac{1}{2} (\delta(-\alpha^n) + \delta(\alpha^n))
\]

has Fourier-Stieltjes coefficients vanishing at infinity. Moreover \( \alpha^n \equiv k_n \alpha \pmod{1} \) for some \( k_n \in \mathbb{Z} \), so that invoking the result of [3] again we can see that \( \nu \) is ergodic under \( T_\alpha \).

This method fails, however, when \( c = \pm 1 \), since \( \alpha^{-1} \) is then a Pisot number, and so the Fourier-Stieltjes coefficients do not vanish at infinity in this case. We have no information when \( \alpha \) is not a quadratic irrational as to the ergodicity of \( \nu \) under \( T_\alpha \).
Singular measures with Fourier-Stieltjes coefficients vanishing at infinity are most easily obtained by using a Riesz product construction. Let \((n_k)\) be a sequence of positive integers which is lacunary in the sense that \(n_{k+1}n_k^{-1} > 3\) for all \(k\), and let \((a_k)\) be a sequence of real numbers with \(0 \leq a_k \leq 1\). Then if \(P_N(t) = \prod_{k=1}^{N} (1 + a_k \cos 2\pi n_k t)\)

we have \(\int_0^1 P_n(t) \, dt = 1\) and \(P_n(t) \geq 0\) for all \(t\). The measure \(\mu^{(N)}\) given by \(d\mu^{(N)}(t) = P_N(t) \, dt\) is then a probability measure. Moreover, the sequence \((\mu^{(N)})\) converges in the weak* topology to a probability measure \(\mu\). This is called a Riesz product measure (see [8, Chapter V, §7]). We write

\[
d\mu(t) = \prod_{k=1}^{\infty} (1 + a_k \cos 2\pi n_k t) \cdot dt,
\]

though, in general, \(\mu\) will be singular. It is a theorem of Zygmund that \(\mu\) is absolutely continuous if and only if \(\sum_{k=1}^{\infty} a_k^2 < \infty\) (loc cit.). Its Fourier Stieltjes coefficients can be written explicitly

\[
\hat{\mu}(\varepsilon_1 n_1 + \varepsilon_2 n_2 + \cdots + \varepsilon_k n_k) = a_1^{|\varepsilon_1|} \cdot a_2^{|\varepsilon_2|} \cdots a_k^{|\varepsilon_k|}
\]

provided \(\varepsilon_i \in \{-1,0,1\} (i = 1, 2, \ldots, k)\) and

\[
\hat{\mu}(r) = 0 \quad \text{otherwise}.
\]

It follows that \(\hat{\mu}(r) \to 0\) as \(|r| \to \infty\) if and only if \(a_k \to 0\) as \(k \to \infty\).

The ergodic properties of certain Riesz products have been investigated by Gavin Brown in [2]. He showed that if \(n_k\) divides \(n_{k+1}\) for all \(k\) then \(\mu\) is quasi-invariant and ergodic for the action of the subgroup \(D\) of \(T\) generated by the set \(\{n_k^{-1}: k = 1, 2, 3, \ldots\}\). His technique appears to rely heavily on the rigid arithmetical properties enjoyed by the \(n_k\)'s and on the existence of appropriate finite subgroups of \(D\). Of course, there are no such subgroups of \(D_\alpha = \{j\alpha: j \in \mathbb{Z}\}\). However, by an appropriate modification of Brown's argument we are able to produce Riesz products with the desired properties.

We will show that these measures \(\mu\) are uniquely defined as quasi-invariant probability measures by the Radon-Nikodym derivative \(d\mu T / d\mu\). Krieger [6] has shown that Radon-Nikodym derivatives of measures with this property form a dense \(G_\delta\) in all such Radon-Nikodym derivatives.

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2. Construction

We fix, in advance, a sequence \((a_k)\) of real numbers between 0 and \(\rho\) where \(0 < \rho < 1\). The sequence \((\tau_k)\) will be defined inductively along with a sequence \((\lambda_k)\) of probability measures with supports in \(D_\alpha\). Then we shall take \(\mu\) to be the Riesz product

\[
d\mu(t) = \prod_{k=1}^{\infty} (1 + a_k \cos 2\pi n_k t) \cdot dt.
\]

To facilitate the argument we make the following definitions

\[
P_N(t) = \prod_{k=1}^{N} (1 + a_k \cos 2\pi n_k t);
\]

\[
d\mu_N(t) = \prod_{R=N+1}^{\infty} (1 + a_k \cos 2\pi n_k t) \cdot dt;
\]

\[
\Omega_N = \{\varepsilon_1 n_1 + \varepsilon_2 n_2 + \cdots + \varepsilon_N n_N : \varepsilon_i \in \{-1,0,1\}\};
\]

\[
T_N = \sup\{r : r \in \Omega_N\};
\]

\[
b_N = \sup\{|j| : j\alpha \in \text{supp} \lambda_N\}.
\]

It will be convenient also to write \(e_k(t) = \exp 2\pi ikt\). Observe that \(\mu = P_N \cdot \mu_N\), and

\[
P_N(t) = \sum_{r \in \Omega_N} P_N(r)e_r(t) = \sum_{r \in \Omega_N} \hat{\mu}(r)e_r(t).
\]

Let \((\varepsilon_N)\) be a sequence of real numbers with \(0 < \varepsilon_N < 1\) and \(\varepsilon_N \to 0\). We shall require \((n_k)\) and \((\lambda_k)\) to have the following properties,

(a) \(|\lambda_N^\wedge(m)| < \frac{1}{2}6^{-N}\varepsilon_N\) \((0 < |m| \leq 2T_N)\);

(b) for \(|j| \leq b_N\), \(\delta(j\alpha) * \mu_n\) is equivalent to \(\mu_N\) and

\[||\delta(j\alpha) * \mu_N - \mu_N|| < \frac{1}{2}\varepsilon_N6^{-N}.
\]

Assume, for the moment, that this has been done. We shall show that (a) and (b) together yield the ergodicity of \(\mu\).

**Lemma 1.** \(||P_N \cdot (\lambda_N * \mu) - \mu|| < \varepsilon_n\).**

**Proof.** Let \(\nu_N = \lambda_N * \mu - \mu_N\). Then, by (7),

\[
\nu_N = \sum_{r \in \Omega_N} P_N^\wedge(r) \lambda_N * \mu(e_r \cdot \mu_N) - \mu_N
\]

\[
= \sum_{r \in \Omega_N} P_N^\wedge(r) \sum_{|j| \leq b_N} \beta_j e_r(j\alpha)e_r \cdot \delta(j\alpha) * \mu_N - \mu_N
\]
where $\lambda_N = \sum_{|j| \leq b_N} \beta_j \delta(j\alpha)$. It follows that

$$
\|\nu_N\| \leq \left\| \sum_{r \in \Omega_N} P_N^\alpha(r) \sum_{|j| \leq b_N} \beta_j \overline{e_r(j\alpha)} e_r \cdot \mu_N - \mu_N \right\|
+ \left\| \sum_{r \in \Omega_N} P_N^\alpha(r) \sum_{|j| \leq b_N} \beta_j \overline{e_r(j\alpha)} e_r \cdot (\delta(j\alpha) \ast \mu_N - \mu_N) \right\|
\leq \left\| \sum_{r \in \Omega_N} P_N^\alpha(r) \lambda_N^\wedge(r) e_r \cdot \mu_N - \mu_N \right\| + 3^N \sup_{|j| \leq b_N} \|\delta(j\alpha) \ast \mu_N - \mu_N\|
\leq \left\| \sum_{r \in \Omega_N} P_N^\alpha(r) \lambda_N^\wedge(r) e_r \cdot \mu_N \right\| + 2^{-N-1} \varepsilon_N
$$

by property (b), and by (a) this is less than $2^{-N} \varepsilon_N$. Now $P_N \cdot (\lambda_N \ast \mu) - \mu = P_N \cdot \nu_N$ and $\sup_t |P_N(t)| \leq 2^n$ so the result follows.

We are now able to state and prove our main result, subject of course, to having found $(\nu_k)$ and $(\lambda_k)$ satisfying (a) and (b).

**Theorem.** The measure $\mu$ of (1) is ergodic for the action of $T_\alpha$.

**Proof.** Fix a Borel set $E$ which is invariant for the action of $T_\alpha$, and let $r \in \mathbb{Z}$. By Lemma 1, we have, for any $N$,

$$
\left| \int_E e_r(t) d\mu(t) - \int_E e_r(t) P_N(t) d\lambda_N \ast \mu(t) \right| < \varepsilon_N.
$$

Moreover, by (7),

$$
\int_E e_r(t) P_N(t) d\lambda_N \ast \mu(t) = \sum_{s \in \Omega_N} \hat{\mu}(s) \int_E e_{s-r}(t) d\lambda_N \ast \mu(t)
= \sum_{s \in \Omega_N} \hat{\mu}(s) \lambda_N^\wedge(r-s) \int_E e_{s-r}(t) d\mu(t)
$$

If $N$ is sufficiently large, then $|s| < T_N$ and so $|r-s| < 2T_N$ for all $s \in \Omega_N$. Thus we have

$$
\int_E e_r(t) P_N(t) d\lambda_N \ast \mu(t) = \hat{\mu}(r) \mu(E) + \sum_{s \in \Omega_N} \hat{\mu}(s) \lambda_N^\wedge(r-s) \int_E e_{s-r}(t) d\mu(t)
$$

and the final term is less in absolute value than $2^{-(N+1)} \varepsilon_N$. Combining this with (8) and letting $N$ tend to infinity we see that

$$
\int_E e_r(t) d\mu(t) = \hat{\mu}(r) \mu(E)
$$

for all $r \in \mathbb{Z}$, and hence that $\mu(E)$ is 0 or 1.
Now we turn to the definition of \((n_k)\) and \((\lambda_k)\) to satisfy (a) and (b). Assume that \(\lambda_1, \lambda_2, \ldots, \lambda_k\) and \(n_1, n_2, \ldots, n_k\) have been defined to satisfy (a) and (b). Choose \(n_{k+1} \geq 3n_k\) such that

\[
0 < \{n_{k+1}\alpha\} < \frac{(1 - \rho)}{24\pi} b_{k}^{-1} 6^{-k} \min \left( \frac{\varepsilon_1}{2^{k}}, \frac{\varepsilon_2}{2^{k-1}}, \ldots, \frac{\varepsilon_k}{2}\right).
\]

Define \(\Omega_{k+1}\) and \(P_{k+1}\) according to (2) and (4) and choose \(\lambda_{k+1}\) to be a probability measure with finite support contained in \(D_{\alpha}\) which is a weak * approximation to Lebesgue measure, sufficiently close for (a) to hold.

We shall use the following lemma to achieve (b).

**Lemma 2.** For \(|j| \leq b_N\), the infinite product

\[
Q_j^N(t) = \prod_{k=N+1}^{\infty} \left( \frac{1 + \cos 2\pi n_k(t - j\alpha)}{1 + \cos 2\pi n_k t} \right)
\]

converges uniformly and

\[
|Q_j^N(t) - 1| < \frac{1}{2} 6^{-N} \varepsilon_N.
\]

**Proof.** We observe by (9) that if \(k > N\),

\[
\left| \left( \frac{1 + \alpha_k \cos 2\pi n_k(t - j\alpha)}{1 + \alpha_k \cos 2\pi n_k t} \right) - 1 \right| \leq \frac{1}{1 - \rho} \left| \cos 2\pi n_k(t - j\alpha) - \cos 2\pi n_k t \right| \\
\leq \frac{2}{1 - \rho} \left| \sin 2\pi n_k j\alpha \right| \leq \frac{4\pi}{1 - \rho} j\{n_k\alpha\} \\
\leq \frac{1}{6} 6^{-N} \varepsilon_N 2^{-k+N-1}
\]

Now the result follows from the general inequality

\[
\left| \prod_{k=1}^{\infty} (1 + c_k) - 1 \right| \leq \exp \left( \sum_{k=1}^{\infty} |c_k| \right) - 1, \text{ and } \varepsilon_N < 1.
\]

**Proposition.** The measure \(\mu\) is quasi-invariant for the action of \(T_\alpha\). Furthermore, for \(|j| \leq b_N\), \(\delta(j\alpha) * \mu_N\) is equivalent to \(\mu_N\) and

\[
\|\delta(j\alpha) * \mu_N - \mu_N\| < \frac{1}{2} 6^{-N} \varepsilon_N.
\]
PROOF. We prove first that \( \delta(j\alpha) \ast \mu_N \) is equivalent to \( \mu_N \). To see this, note that, for \( f \in C(T) \),

\[
\int f(t) \, d\delta(j\alpha) \ast \mu_N(t) = \lim_{R \to \infty} \int_0^1 f(t) \prod_{k=N+1}^R \frac{1 + a_k \cos 2\pi n_k(t - j\alpha)}{1 + a_k \cos 2\pi n_k t} \, dt
\]

\[
= \lim_{R \to \infty} \int_0^1 f(t) \prod_{k=N+1}^R \left( \frac{1 + a_k \cos 2\pi n_k(t - j\alpha)}{1 + a_k \cos 2\pi n_k t} \right) \prod_{k=N+1}^R (1 + \cos 2\pi n_k t) \, dt
\]

\[
= \int_0^1 f(t) Q_j^N(t) \, d\mu(t)
\]

by Lemma 2. It follows that

\[
\frac{d(\delta(j\alpha) \ast \mu_N)}{d\mu_N} = Q_j^N(t),
\]

and hence that

\[
\|\delta(j\alpha) \ast \mu_N - \mu_N\| < \frac{1}{2} 6^{-N} \varepsilon_N
\]

also by Lemma 2.

Finally taking \( j = N = 1 \) and noting that \( (1 + a_k \cos 2\pi n_k(t - \alpha))(1 + a_k \cos 2\pi n_1 t)^{-1} \) is a continuous function bounded away from 0, we obtain the quasi-invariance of \( \mu \).

The proof that \( \mu \) is quasi-invariant and ergodic is complete. If in addition we assume that \( \sum_{k=1}^\infty a_k^2 = \infty \) then \( \mu \) is singular, and if \( a_k \to 0 \) as \( k \to \infty \) then \( \mu \) vanishes at infinity. Thus a choice of \( a_k = (1 + k)^{-1/2} \) will produce a quasi-invariant ergodic singular measure with Fourier-Stieltjes coefficients vanishing at infinity. A more austere choice of \( (a_k) \), say \( a_k = (\log(k + 1))^{-1} \), will produce a measure every convolution power of which is singular, yet \( \mu \) is still ergodic and quasi-invariant for the action of \( T_\alpha \) and has Fourier-Stieltjes coefficients vanishing at infinity.

3. Uniqueness

Here we show that \( \mu \) is the unique quasi-invariant probability measure \( \tau \) for the action of \( T \) such that \( (d\delta(j\alpha) \ast \tau)/d\tau = Q_j \) \((= Q_j^0)\). Observe that both sides of the equation are 1-cocycles, that is, they satisfy \( \phi_{j+k}(t) = \phi_{j}(t - k) + \varphi_k(t) \) for \( j, k \in \mathbb{Z}, t \in T \).

Suppose then that \( \tau \) is such a measure and fix \( r \in \mathbb{Z} \). Consider the integral

\[
I = \int e^{-r}(t) P_M(t) \, d\lambda_M \ast \nu(t)
\]

for suitably large \( M \). On the one hand this is equal to

\[
\sum_{s \in \Omega_M} \mu(s) \lambda_M^*(r - s) \hat{\nu}(r - s);
\]
on the other, writing \( \lambda_M = \sum_{|j| \leq b_M} \beta_j \delta(j\alpha) \), we obtain

\[
I = \sum_{|j| \leq b_M} \beta_j \int e^{-r(t)} P_M(t) \, d\delta(j\alpha) \ast \nu(t)
\]

\[
= \sum_{|j| \leq b_M} \beta_j \int e^{-r(t)} P_M(t) Q_j(t) \, d\nu(t)
\]

\[
= \sum_{|j| \leq b_M} \beta_j \int e^{-r(t)} P_M(t - j\alpha) Q_j^M(t) \, d\nu(t)
\]

by (10). It follows from Lemma 2 that

\[
\left| I - \sum_{|j| \leq b_M} \beta_j \int e^{-r(t)} P_M(t - j\alpha) \, d\nu(t) \right|
\leq 2^M \sup_t |Q_j^M(t) - 1| \leq \frac{1}{2} 3^{-M} \varepsilon_M.
\]

Now we observe that

\[
J = \sum_{|j| \leq b_M} \beta_j \int e^{-r(t)} P_M(t - j\alpha) \, d\nu(t)
\]

\[
= \sum_{|j| \leq b_M} \beta_j \sum_{s \in \Omega_M} \hat{\mu}(s) \int e^{-r(t)} e_s(t - j\alpha) \, d\nu(t)
\]

\[
= \sum_{s \in \Omega_M} \hat{\mu}(s) \hat{\nu}(\tau - s) \sum_{|j| \leq b_M} \beta_j e_s(-j\alpha).
\]

Therefore,

\[
J = \sum_{s \in \Omega_M} \hat{\mu}(s) \hat{\nu}(\tau - s) \lambda_M^\wedge(s).
\]

By (a), \( |\lambda_M^\wedge(m)| \leq \frac{1}{2} 6^{-M} \varepsilon_M \) provided \(|m| < 2T_M\), so that, by (14), if \(|r| < T_M\),

\[
|J - \hat{\nu}(\tau)| < \sum_{s \in \Omega_M} \hat{\mu}(s) \hat{\nu}(\tau - s) \lambda_M^\wedge(s)
\]

\[
\leq 2^{-(M+1)} \varepsilon_M,
\]

whereas, by (12)

\[
|I - \hat{\mu}(r)| \leq 2^{-(M+1)} \varepsilon_M.
\]

Now (13), (15) and (16) combine to give \(|\hat{\nu}(\tau) - \hat{\mu}(r)| \leq 2^{-(M+1)} \varepsilon_M\). Therefore \( \hat{\mu}(r) = \hat{\nu}(\tau) \) for all \( r \in \mathbb{Z} \) and so \( \mu = \nu \).
References


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