A COMMUTATIVITY THEOREM FOR RINGS
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We prove the following theorem: Let \( R \) be a ring, \( l \) a positive integer, and \( n \) a non-negative integer. If for each \( x, y \in R \), either \( xy = yx \) or \( xy = x^n f(y)x^l \) for some \( f(X) \in X^2Z[X] \), then \( R \) is commutative.

Throughout, \( R \) will represent a ring with centre \( C = C(R) \), and \( D = D(R) \) the commutator ideal of \( R \). Let \( l, m \) be positive integers, and \( n \) a non-negative integer. We consider the following conditions:

\[
\begin{align*}
(\ast)_{(l,m,n)} & \text{ For each } x, y \in R, \text{ either } [x, y] = xy - yx = 0 \text{ or } x^m y = x^n f(y) x^l \text{ for some } f(X) \in X^2Z[X]. \\
(\ast\ast)_{(l,m,n)} & \text{ For each } x, y \in R, \text{ either } [x, y] = 0 \text{ or } x^m y - x^n f(y)x^l \in C \text{ for some } f(X) \in X^2Z[X]. \\
(\dagger)_{(l,m,n)} & \text{ For each } x, y \in R, \text{ there exists } f(X) \in X^2Z[X] \text{ such that } [x, x^m y - x^n f(y)x^l] = 0. \\
(S) & \text{ For each } x, y \in R, \text{ there exists } f(X, Y) \in Z(X, Y)[X, Y]Z(X, Y) \text{ each of whose monomial terms is of length } \geq 3 \text{ such that } [x, y] = f(x, y).
\end{align*}
\]

As is easily seen, \((\ast)_{(l,m,n)}\) implies \((\ast\ast)_{(l,m,n)}\), and \((\ast\ast)_{(l,m,n)}\) implies \((\dagger)_{(l,m,n)}\). Recently, Bell [1] announced that every ring \( R \) satisfying \((\ast)_{(1,1,0)}\) is commutative. The next result has been proved in [3, Theorem 1].

**Proposition 1.** Let \( R \) be a ring with 1. If \( R \) satisfies \((\dagger)_{(l,m,n)}\) then \( R \) is commutative.

Our present objective is to prove the following theorem, by making use of Proposition 1.

**Theorem 1.** If a ring \( R \) satisfies \((\ast)_{(l,1,n)}\), then \( R \) is commutative.

We start our preparation for proving Theorem 1 with the following proposition.

**Proposition 2.** Let \( R \) be a ring generated by two elements such that \( D \) is the heart of \( R \) and \( RD = DR = 0 \). Then \( R \) is nilpotent.

**Proof:** Obviously, \( D \) is \( Z \)-isomorphic to \( Z/pZ \) for some prime \( p \). Noting that \( R/D \) is a homomorphic image of the subring \( \langle X, Y \rangle \) of \( Z[X, Y] \) and every ideal of
\((X, Y)\) is an ideal of \(\mathbb{Z}[X, Y]\), we see that \(R/D\) is Noetherian. Accordingly, \(R\) is right Noetherian.

Now, let \(x\) be an arbitrary element of \(R\), and \(k\) a positive integer such that \(r(x^k) = r(x^{k+1})\), where \(r(\ast)\) denotes the right annihilator of \(\ast\) in \(R\). Since \(Rx^k \subseteq (x^kR + D)R \subseteq x^kR\), \(x^kR\) is an ideal of \(R\). Further, if \(x^ka \in x^kR \cap D\) then \(x^{k+1}a = x(x^ka) = 0\), and so \(x^ka = 0\). Hence \(x^kR \cap D = 0\), whence \(x^kR = 0\) follows. We have thus seen that \(R\) is nil. Now, it is easy to see that \(R\) is nilpotent.

Combining Proposition 2 with [2, Theorem 5], we see that if \(R\) is not commutative then there exists a factorRING of \(R\) which is of type (a)\(_1\), (a)\(_r\), (b), (c), (d) or (e):

\[
(a)\_1 \quad \begin{pmatrix} GF(p) & GF(p) \\ 0 & 0 \end{pmatrix}, \text{ } p \text{ a prime.}
\]

\[
(a)\_r \quad \begin{pmatrix} 0 & GF(p) \\ 0 & GF(p) \end{pmatrix}, \text{ } p \text{ a prime.}
\]

\[
(b) \quad M_\sigma(K) = \{ \begin{pmatrix} \alpha & \beta \\ 0 & \sigma(\alpha) \end{pmatrix} \} | \alpha, \beta \in K\}, \text{ where } K \text{ is a finite field with a non-trivial automorphism } \sigma.
\]

\[
(c) \quad \text{A non-commutative division ring.}
\]

\[
(d) \quad \text{A simple radical ring with no non-zero divisors of zero.}
\]

\[
(e) \quad \text{A finite nilpotent ring } S \text{ such that } D(S) \text{ is the heart of } S \text{ and } SD(S) = D(S)S = 0.
\]

In particular, if \(R\) is non-commutative and satisfies \((S)\) then there exists a factorRING of \(R\) which is of type (a)\(_1\), (a)\(_r\), (b), (c) or (d) (see [2, Corollary S.1]).

This result gives the following Meta-Theorem.

**META-Theorem.** Let \(P\) be a ring property which is inherited by factorsubrings. If no rings of type (a)\(_1\), (a)\(_r\), (b), (c), (d) or (e) satisfy \(P\), then every ring satisfying \(P\) is commutative. (If no rings of type (a)\(_1\), (a)\(_r\), (b), (c) or (d) satisfy \(P\), then every ring satisfying \((S)\) and \(P\) is commutative.)

We are now ready to complete the proof of Theorem 1.

**Proof of Theorem 1:** In view of the Meta-Theorem, it suffices to show that \(R\) cannot be of type (a)\(_1\), (a)\(_r\), (b), (c), (d) or (e).

Noting that \(e_{12} = e_{11}e_{12} \neq e_{11}^p f(e_{12}) e_{11}' = 0\) and \(e_{12} = e_{12}e_{22} \neq e_{12}f(e_{22})e_{12}' = 0\) for any \(f(X) \in X^2Z[X]\), we see that \(R\) cannot be of type (a)\(_1\) or (a)\(_r\). Further, by Proposition 1, no rings of type (b) or (c) satisfy \((\ast)_{(1,1,n)}\).

Now, suppose that \(R\) is of type (d), and choose \(x, y \in R\) with \([x, y] \neq 0\). Then there exists \(p(X) \in XZ[X]\) such that \(xy = x^np(y)yz\). If \([x, y'] \neq 0\) and \([x', y'] \neq 0\), there exist \(f(X), g(X) \in X^2Z[X]\) such that \(xy' = x^nf(y')z^l\) and \(yz' = y^ng(x')y'^l\). Putting \(f(y') = f_0(y)y\) and \(g(x') = g_0(x)x\) with some \(f_0(X), g_0(X) \in XZ[X]\), we
obtain $xy^l = z^n f_0(y)y^n g_0(x^l)xy^l$. Since $R$ is a radical ring, this forces a contradiction $xy^l = 0$. Next, if $[x^l, y] = 0$ then $xy = x^n p(y)x^{l-1}xy$, which implies a contradiction $xy = 0$. Similarly, $[x, y^l] = 0$ forces a contradiction. We have thus seen that $R$ cannot be of type (d).

Finally, suppose that $R$ is of type (e). Then $R^2 \subseteq C$. Given $x, y \in R$ with $[x, y] \neq 0$, we can take $p(X) \in XZ[X]$ such that $xy = x^n yp(y)x^l = xyp(y)x^{l+n-1}$, whence $xy = 0$ follows; similarly $yx = 0$. But this is impossible.

**Corollary 1.** If $R$ satisfies $(S)$ and $(**)_{(l,1,n)}$, then $R$ is commutative.

**Proof:** In view of Proposition 1 and the Meta-Theorem, it suffices to show that $R$ cannot be of type (a) or (d). It is easy to see that $R$ is not of type (a) or (d). If $R$ is of type (d), then $C = 0$ and $R$ satisfies $(*)_{(l,1,n)}$. Thus $R$ is commutative by Theorem 1. But this is impossible.

Finally, we remark that a ring with $(*)_{(l,m,n)}$ for $m > 1$ need not be commutative. Actually, there exists a non-commutative ring $R$ with $R^3 = 0$.

**References**

