# SQUARE ROOTS IN BANACH *-ALGEBRAS

*by* F. F. BONSALL and D. S. G. STIRLING

(Received 23 April, 1970)

We give a simple proof of the lemma of Ford [1] on the existence of self-adjoint square roots in Banach *-algebras in which continuity of the involution is not assumed.

Let $A$ denote a real or complex Banach algebra and let

$$\rho(a) = \inf_n \| a^n \|^{1/n} \qquad (a \in A).$$

LEMMA 1. *Let $a \in A$ and $\rho(a) < 1$. Then there exists a unique element $x$ in $A$ for which $2x - x^2 = a$ and $\rho(x) < 1$.*

*Proof.* Since $\rho(a) = \inf |a|$, where the infimum is taken over all algebra-norms $|\cdot|$ equivalent to the given norm [2], we may choose such a norm $|\cdot|$ and real number $\eta$ with $|a| < \eta < 1$. Let $C$ denote the least closed subalgebra containing $a$, and let $E = \{x \in C : |x| \leqq \eta\}$. Then $T$, given by

$$Tx = \tfrac{1}{2}(a + x^2),$$

is a contraction mapping of $E$ into $E$, since the commutativity of $C$ gives

$$|Tx - Ty| = \tfrac{1}{2}|x^2 - y^2| \leqq \tfrac{1}{2}|x - y||x + y| \leqq \eta |x - y| \qquad (x, y \in E).$$

Therefore there exists $x \in E$ with $2x - x^2 = a$, $\rho(x) \leqq |x| < 1$.

Suppose now that $y \in A$, $2y - y^2 = a$, and $\rho(y) < 1$. Since $y$ commutes with $a$, and $x$ is a limit of polynomials in $a$, $y$ commutes with $x$. Therefore $\rho(x+y) < 2$ and we may again choose an equivalent algebra-norm $|\cdot|'$ with $|x+y|' < 2$. But then the inequality

$$|x - y|' = |\tfrac{1}{2}(a + x^2) - \tfrac{1}{2}(a + y^2)|' \leqq \tfrac{1}{2}|x + y|'|x - y|'$$

gives $|x - y|' = 0$.

LEMMA 2. (Ford [1]). *Let $B$ be a Banach *-algebra with $a \in B$, $a = a^*$ and $\rho(a) < 1$. Then there exists a unique $x \in B$ with $2x - x^2 = a$, $\rho(x) < 1$ and $x = x^*$.*

*Proof.* By Lemma 1, there is a unique $x \in B$ with $\rho(x) < 1$ and $2x - x^2 = a$. But $\mathrm{Sp}\, x^* = (\mathrm{Sp}\, x)^*$; so $\rho(x^*) = \rho(x) < 1$, and $a = a^* = (2x - x^2)^* = 2x^* - (x^*)^2$.

Therefore, by the uniqueness of $x$, $x = x^*$.

REMARK. The proof does not use all the axioms of an involution. It applies to any mapping $x \to x^*$ such that $\rho(x^*) \leqq \rho(x)$ and $(2x - x^2)^* = 2x^* - (x^*)^2$.

## REFERENCES

1. J. W. M. Ford, A square root lemma for Banach *-algebras, *J. London Math. Soc.* **42** (1967), 521–522.

2. R. B. Holmes, A formula for the spectral radius of an operator, *Amer. Math. Monthly* **75** (1968), 163–166.

UNIVERSITY OF EDINBURGH