Finally, it is of interest to compare our results for $x(t)$ with the corresponding situation when the resistive force is proportional to the particle speed and is represented in the equation of motion by $-k\dot{x}$. This latter is a well-known problem with solution corresponding to damped simple harmonic motion whose general appearance graphically is very similar to the initial part of Figure 1; in both cases the amplitude of the motion decreases continually due to the particle's energy being dissipated by the frictional force. There are, however, two significant qualitative differences between the two situations. In our problem the period of the damped SHM is $2\pi/\omega$, independent of the strength of the damping, that is, independent of $K$, while when the damping is of the form $-k\dot{x}$, the period of the damped SHM depends on $k$. The second difference is that in our situation motion of the particle finally terminates after a finite value of $t$ (corresponding to equation (6)) and with $x$ in general non-zero. On the other hand, for damping proportional to the particle speed, motion finally stops only after an infinite time and with $x = 0$.

**Acknowledgement**

I should like to express my thanks to the referee for suggestions on improving the article, together with bringing reference [1] to my attention.

**Reference**


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**95.68 The mechanics of sliding down curves**

**Introduction**

This note was conceived when the first-named author discussed the problems below with a Sixth Form class. For a particle of mass $m$ sliding from rest down a smooth plane inclined at angle $\psi$ to the horizontal, the reaction force $R$ is given by $R = mg \cos \psi$. If the plane is bent into a smooth parabola and the particle starts from rest at the vertex then, as we shall see below, $R = mg \cos^3 \psi$ where, as usual, $\psi$ denotes the angle of inclination of the tangent to the horizontal: see Figure 1. Note that here, and throughout this note, it is convenient to have the $y$-axis vertically downwards and, for simplicity, particles will always start from rest at the origin: it may or may not be the case that the origin is an equilibrium position.
In this note, we discuss general problems of this type; in particular we deal with the natural "inverse problem": what shape of curve has \( R = mg \cos^2 \psi \)? \textit{En route}, we shall also look at curves (such as the circle) where the particle loses contact because \( R = 0 \) at some point. We reprise below sufficient theory for our needs: further background and exercises in this area may be found in older mechanics texts such as [1, 2, 3, 4].

Some general theory

With the usual notation as in Figure 1(iii) and with \( \rho = \frac{ds}{d\psi} \) denoting the radius of curvature, we have \( \mathbf{v} = \dot{s} \begin{pmatrix} \cos \psi \\ \sin \psi \end{pmatrix} \) so that

\[
\mathbf{a} = \dot{s} \begin{pmatrix} \cos \psi \\ \sin \psi \end{pmatrix} + \ddot{s} \begin{pmatrix} -\sin \psi \\ \cos \psi \end{pmatrix} \quad \text{or} \quad \mathbf{a} = \dot{s} \hat{\mathbf{t}} + \frac{\ddot{s}^2}{\rho} \hat{\mathbf{n}}
\]

since \( \ddot{s} = \frac{d\psi}{ds} \frac{ds}{dt} = \dot{s} \).

Resolving in the directions of \( \hat{\mathbf{t}} \) and \( \hat{\mathbf{n}} \) then gives

\[
ms = mg \sin \psi \quad (1)
\]

\[
\frac{m\ddot{s}^2}{\rho} = mg \cos \psi - R \quad (2)
\]

and the energy equation (with the particle starting from rest at the origin) gives

\[
\frac{1}{2}ms^2 = mgy. \quad (3)
\]

We also need the curvature formula

\[
\rho = \frac{(1 + y'')^3}{y'''} = \frac{\sec^3 \psi}{y''}. \quad (4)
\]

Substituting (3), (4) into (2) yields

\[
2mgyy'' \cos^3 \psi = mg \cos \psi - R \quad (5)
\]

and gives our first result: the particle loses contact with the curve if there is a point on the curve satisfying \( R = 0 \) or \( 2mgyy'' \cos^3 \psi = mg \cos \psi \); that is

\[
2yy'' = \sec^2 \psi = 1 + y'^2. \quad (6)
\]
Applications

To verify our initial observation about the parabola, let
\[ y = kx^2 = \frac{1}{2k} \tan^2 \psi \] since \( \tan \psi = y' = 2kx \). Then (5) gives
\[
R = mg \cos \psi - 2mg \left( \frac{1}{4k} \tan^2 \psi \right) 2k \cos^3 \psi
\]
\[
= mg \cos \psi \left( 1 - \sin^2 \psi \right) = mg \cos \psi.
\]

Next we consider the inverse problem: given \( R(\psi) \), what is the curve?

First note that \( y' = \tan \psi \) gives \( y'' = \sec^2 \psi \frac{dy}{d\psi} \), \( y' = \sin \psi \sec^3 \psi \frac{dy}{d\psi} \) which in (5) yields
\[
2mg \sin \psi \frac{dy}{dy} = mg \cos \psi - R(\psi) \text{ or }
\]
\[
\int \frac{dy}{y} = \int \frac{2 \sin \psi}{\cos \psi - \frac{R}{mg}} d\psi. \tag{7}
\]

This gives \( y \) in terms of \( \psi \): we may or may not then be able to get a Cartesian equation for the curve by then eliminating \( \psi \) using \( y' = \tan \psi \).

For the case raised in the introduction \( R = mg \cos^2 \psi \). Thus, from (7)
\[
\int \frac{dy}{y} = \int \frac{2 \sin \psi}{\cos \psi - \cos^2 \psi} d\psi = \int \frac{2 \sin \psi}{\cos \psi} + \frac{2 \sin \psi}{1 - \cos \psi} d\psi,
\]
so that \( \ln y = 2 \ln \left( 1 - \cos \psi \right) - 2 \ln \cos \psi - 2 \ln k \) (say) and
\[
y = \frac{1}{k^2} \left( \frac{1 - \cos \psi}{\cos \psi} \right)^2 = \frac{1}{k^2} (\sec \psi - 1)^2, \text{ where } k \text{ is a constant. Now}
\]
\( y' = \tan \psi \) gives
\[
x = \int \frac{dy}{\tan \psi} = \int \frac{dy}{\tan \psi} d\psi = \int \frac{1}{k^2} 2(\sec \psi - 1) \frac{\sec \psi \tan \psi}{\tan \psi} d\psi
\]
\[
= \frac{2}{k^2} \int \left( \sec^2 \psi - \sec \psi \right) d\psi
\]
or \( x = \frac{2}{k^2} \left[ \tan \psi - \ln(\sec \psi + \tan \psi) \right] \) (since \( x = 0 \) when \( \psi = 0 \)) whence, using \( \sec \psi = 1 + k \sqrt{y} \),
\[
x = \frac{2}{k^2} \left[ k \sqrt{y} + 2k \sqrt{y} \right].
\]

Certainly (Figure 2), this solution curve is 'between' the straight line and parabola of Figure 1, but who would have guessed at the complexity of its equation?
FIGURE 2: Solution curve for $R = mg \cos^2 \psi$ (with $k = 1$)

The $R = 0$ condition

We turn now to examples of curves where (6) has a solution. The familiar case of the circle (where $R = 0$ when $\cos \psi = \frac{1}{2}$) is subsumed by that of the ellipse $x = a \sin t$, $y = b(1 - \cos t)$ with eccentricity $e$. Substituting into (6) produces the equation

$$2 \sec^3 t - 3 \sec^2 t = \begin{cases} \frac{-e^2}{1 - e^2} & \text{if } b < a \\ -e^2 & \text{if } b > a \end{cases}$$

with a unique solution for $t$ (and hence $\psi$, since $\tan \psi = \frac{b}{a} \tan t$) because the left-hand side maps $[0, \frac{\pi}{2})$ monotonically onto $[-1, \infty)$. For the power curve $y = kx^a$ ($a > 0$), (6) yields the equation $k^2a(a - 2)x^{2a - 2} = 1$, which has no solution with $x > 0$ if $a < 2$ and a ‘loss of contact angle' $\psi = \tan^{-1}\sqrt{\frac{a - 2}{a}}$ if $a > 2$. The solution to (6) for some other curves is shown in Table 3.

<table>
<thead>
<tr>
<th>Equation of curve</th>
<th>Value of $\psi$ with $R = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = k \sinh x$</td>
<td>$\tan^{-1}\sqrt{2k^2 + 1}$</td>
</tr>
<tr>
<td>$y = k (\cosh x - 1)$</td>
<td>$\tan^{-1}\sqrt{2k + 1}$</td>
</tr>
<tr>
<td>$y = k \tan x$</td>
<td>$\tan^{-1}\left[\frac{1}{2} \left(2k + \sqrt{4k^2 + 3}\right)\right]$</td>
</tr>
<tr>
<td>$y = k (1 - \cos x)$</td>
<td>No solution for $0 &lt; x &lt; \frac{\pi}{2}$</td>
</tr>
</tbody>
</table>

**TABLE 3**

Finally, it may be worth remarking that, if we set $G(x) = 1 + y^2 - 2yy''$, then $G(0) > 0$ since $y(0) = 0$ and $G'(x) = -2yy'''$, so there are certainly no solutions of $G(x) = 0$ if $y''' < 0$: this covers the curves of Figure 1(i), (ii) and the last entry of Table 3.
We have assumed throughout that the particle starts from the origin at rest. If the particle has speed \( u \) at the origin, (3) becomes \( \frac{1}{2}mu^2 - \frac{1}{2}mu^2 = mg\ell \) and the form of the key equations remains unchanged if we replace \( y \) by \( Y = y + \frac{u^2}{2g} \). With this substitution, our analysis of the inverse problem and \( R = 0 \) problem proceed as before. The one new feature is that it is now possible for \( R = 0 \) to occur, since \( 2YY'' = 1 + Y'^2 \) solves to give the parabola \( Y = \frac{1}{4a} + a(x + b)^2 \) \((a > 0, b \text{ constants})\).

References

**Teaching Notes**

**Constant acceleration revisited**

Consider a particle moving in a straight line with constant acceleration \( f \) and initial (at \( t = 0 \)) speed \( u \). Then if \( v(u, t) \) and \( s(u, t) \) are respectively the speed and distance travelled after time \( t \), we have

\[
\begin{align*}
  v(u, t) &= u + ft, \\
  s(u, t) &= ut + \frac{1}{2}ft^2.
\end{align*}
\]

When initially presenting these formulae to students, I would like to suggest that it is worth pointing out that the above formulae satisfy an important necessary condition which is, of course, not restricted to constant acceleration. This derives from the fact that the distance travelled after time \( t + t' \) is equal to the distance travelled after time \( t \) plus the distance travelled after a further time \( t' \), with the initial speed during the second interval being suitably updated from its value \( u \) at \( t = 0 \). The condition may thus be expressed as

\[
  s(u, t + t') \equiv s(u, t) + s(v(u, t), t')
\]

and it is readily shown that equations (1) satisfy this requirement, since

\[
  u(t + t') + \frac{1}{2}f(t + t')^2 \equiv ut + \frac{1}{2}ft^2 + (u + ft)t' + \frac{1}{2}ft'^2.
\]

Finally, it is perhaps worth mentioning that if the first of equations (1) has been derived (based on the definition of acceleration), then the second equation (for \( s(u, t) \)) can be obtained by the following argument. When \( f = 0 \), it is clear that \( s = ut \) and it is therefore plausible to assume that for \( f \neq 0 \), \( s(u, t) = ut + \beta ft^2 \) for suitable (as yet undetermined) values of \( \beta \).