Eigencircles and associated surfaces

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Introduction

Linear algebra has many fruitful connections with geometry. This article develops one such connection: the relationship between a $2 \times 2$ matrix and an associated circle which we call the eigencircle.

This connection was first investigated in a previous paper of ours [1], but the present paper is self-contained, and in fact introduces eigencircles in a different way. Here we discuss some surfaces containing the eigencircle which also have a number of interesting properties and connections with the associated matrix.

Our study may be regarded as a special case of multiparameter spectral theory, which has been studied extensively at a general level [2, 3, 4, 5, 6]. We concentrate on this special case, since it is elementary, accessible and, we believe, illuminating.

An eigenvalue of a square matrix $A$ is, as any linear algebra textbook explains, a number $\lambda$ such that $Ax = \lambda x$ for some $x \neq 0$. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a $2 \times 2$ matrix and $x = \begin{pmatrix} x \\ y \end{pmatrix}$, then we may write this as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

with $x, y$ not both zero. Note particularly the matrix $\lambda I_2$ on the right-hand side, and recall that the set of such matrices, under matrix addition and multiplication and with $\lambda \in \mathbb{R}$, is isomorphic (as a field) to the real numbers. This elementary fact can be used as a starting point for an extension of the concept of eigenvalues. Since the matrices $\lambda I_2$ represent real numbers (if $\lambda \in \mathbb{R}$), we ask what happens when we replace $\lambda I_2$ by real matrices that represent complex numbers.

It is well known that the set of matrices

$$\begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix},$$

$\lambda, \mu \in \mathbb{R}$, is isomorphic to the field $\mathbb{C}$ of complex numbers via the field isomorphism

$$\begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix} \rightarrow \lambda + i\mu.$$

Let us then extend the definition in (1) by replacing the matrix $\lambda I_2$ by
We thus ask for pairs \((\lambda, \mu) \in \mathbb{R}^2\) such that

\[
\begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix}
\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]

for some \(x, y\) not both zero. We call such a pair \((\lambda, \mu)\) an eigenpair of \(A\), and a corresponding \(x \neq 0\) is a \((\lambda, \mu)\)-eigenvector. The set of \(x\) satisfying (2) is the \((\lambda, \mu)\)-eigenspace associated with \((\lambda, \mu)\). The unadorned terms eigenvector and eigenspace have their usual meanings; if \((\lambda, 0)\) is an eigenpair, \(\lambda\) is an eigenvalue.

The set of eigenpairs will be seen to form a circle in the real \(\lambda, \mu\)-plane. We call this circle the eigencircle of \(A\). It turns out to provide useful geometric illustrations, and indeed proofs, of a number of properties of \(A\). These include:

- geometric pictures illustrating the determinant, eigenvectors (for real eigenvalues), and real or complex eigenvalues;
- a geometric derivation of the angle between the eigenvectors of \(A\);
- a 'proof without words' of the fact that a symmetric \(2 \times 2\) matrix has orthogonal eigenvectors, using the fact that the angle subtended by the diameter of a circle at any point on it is a right angle;
- a geometric proof that the product of real or complex eigenvalues is the determinant;
- visualisation of the family of \(2 \times 2\) matrices with given eigenvalues.

These and other aspects of the eigencircle are discussed in [1].

In the present paper, we use the eigencircle to give:

- a geometric interpretation of the discriminant of the characteristic equation of \(A\);
- geometric pictures of complex eigenvectors for complex eigenvalues, which completes the work begun in [1] of using the eigencircle to visualise real and complex eigenvalues and eigenvectors of \(A\);
- surfaces that contain the eigencircle and allow further geometric views of properties of \(2 \times 2\) matrices;
- an extension to quaternions, where the eigencircle becomes the equator of an eigensphere.

Throughout, if \(P\) and \(Q\) are points then \(\overrightarrow{PQ}\), \(\overrightarrow{Q}\) and \(PQ\) denote the vector from \(P\) to \(Q\), the line segment between \(P\) and \(Q\) and the length of this line segment, respectively.
Properties of the eigencircle

If (2) is to have a non-zero solution for \( x \), then (with obvious similarity to ordinary eigenvalues) we require

\[
\begin{vmatrix}
  a - \lambda & b - \mu \\
  c + \mu & d - \lambda \\
\end{vmatrix} = 0.
\]

(3)

The set of all eigenpairs of \( A \) in \( \mathbb{R}^2 \) is therefore described by the equation

\[
(a - \lambda)(d - \lambda) - (b - \mu)(c + \mu) = 0,
\]

(4)
or, equivalently,

\[
\lambda^2 + \mu^2 - (a + d)\lambda - (b - c)\mu + ad - bc = 0.
\]

(5)

Putting

\[
f = \frac{a + d}{2},
\]

(6)

\[
g = \frac{b - c}{2},
\]

(7)

\[
r^2 = f^2 + g^2,
\]

(8)

\[
\rho^2 = \left(\frac{a - d}{2}\right)^2 + \left(\frac{b - c}{2}\right)^2
\]

\[= r^2 - \det A,
\]

(9)

we can rewrite (5) as

\[
(\lambda - f)^2 + (\mu - g)^2 = \rho^2.
\]

(10)

The solution set of (5) can thus be seen to be a circle (the eigencircle of \( A \)) with centre \( C(f, g) \) (at distance \( r \) from the origin) and radius \( \rho \). It is illustrated in Figures 1 and 2(a). These figures assume that \( a < d \) and \( b < -c \), but the other possibilities are easily envisaged too (for example, by using the given figures with reversal of axes as appropriate). We find it convenient to refer at times to the projection \( Y = (f, 0) \) of the eigencircle's centre \( C \) onto the \( \lambda \)-axis.

\[\text{FIGURE 1: The eigencircle of the } 2 \times 2 \text{ matrix } A, \text{ showing } \rho, r \text{ and } \sqrt{\pm \det A} \text{ as the sides of a right triangle } OCD \text{ (see [9])}: (a) } \det A > 0, \text{ (b) } \det A < 0 \]
The eigencircle always contains the eigenpairs \((a, b), (d, b), (a, -c)\) and \((d, -c)\). (See Figure 2(a).) Generally these are four distinct points, defining a rectangle, but they may be just two (if \(a = d\) or \(b = c\)), in which case they mark a vertical or horizontal diameter of the circle, or one (if \(a = d\) and \(b = -c\)), in which case the eigencircle is just a single point.

To each eigenpair \((\lambda, \mu)\) there corresponds a set of \((\lambda, \mu)\)-eigenvectors which form a linear subspace of \(\mathbb{R}^2\). If \(\rho > 0\) then using (2), (3) and (4) this \((\lambda, \mu)\)-eigenspace is found to be the one-dimensional subspace generated by \(\begin{pmatrix} \lambda - d \\ \mu + c \end{pmatrix}\) or \(\begin{pmatrix} -\mu + b \\ \lambda - a \end{pmatrix}\). This can be visualised on the eigencircle as shown in Figure 2(a). The line from \((d, -c)\) to \((\lambda, \mu)\), which is a chord on the eigencircle (or a tangent if these two eigenpairs coincide), has direction \(\begin{pmatrix} \lambda - d \\ \mu + c \end{pmatrix}\) and so indicates the direction of the \((\lambda, \mu)\)-eigenspace. This line is perpendicular to the line from \((a, b)\) to \((\lambda, \mu)\), so the vector \(\begin{pmatrix} -\mu + b \\ \lambda - a \end{pmatrix}\) also belongs to that \((\lambda, \mu)\)-eigenspace.

If \((\lambda, \mu)\) is an eigenpair, then so is \((a + d - \lambda, b - c - \mu)\). The \((\lambda, \mu)\)- and \((a + d - \lambda, b - c - \mu)\)-eigenvectors are orthogonal (assuming \(\rho > 0\)). (See Figure 2(b).)

The four special eigenpairs mentioned above have corresponding \((\lambda, \mu)\)-eigenvectors

\[
\begin{pmatrix} a - d \\ b + c \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -b - c \\ a - c \end{pmatrix},
\]

respectively. These indicate the directions of: the diameter from \((a, b)\) to \((d, -c)\); the vertical and horizontal sides of the rectangle defined by the four
special points; and the tangents to the eigencircle where the aforementioned diameter meets it.

Note that any direction in the \((\lambda, \mu)\) plane gives a \((\lambda, \mu)\)-eigenvector. The eigencircle chord through \((d, -c)\) with this direction gives the corresponding eigenpair as the other intersection of this chord with the eigencircle.

**Power, determinant and discriminant**

In the previous paper [1], we made significant use of the following theorem from Euclid’s *Elements*, [7, III.35–36]: given a point \(O\) and a circle, and *any* line \(l\) through \(O\) that meets the circle, say in points \(P\) and \(Q\) (which may coincide, when \(l\) is tangent to the circle), then the product of lengths \(OP \cdot OQ\) is constant (i.e. independent of the choice of \(l\), provided it meets the circle). This is true regardless of whether the point \(O\) is inside, outside, or on the circle. The product \(OP \cdot OQ\) is known as the *power* of the point \(O\) with respect to the circle. The lengths \(OP\) and \(OQ\) are treated as signed, so that if \(O\) is inside the circle then the power of \(O\) is negative.

It follows from the work above (see (9) and Figure 1(a,b)) that the determinant of \(A\) is the power of the origin with respect to the eigencircle, as discussed in [1].

We now show that the discriminant of the characteristic equation of \(A\) is also determined by the power of an appropriate point with respect to the eigencircle. We use this fact later when we consider some surfaces containing the eigencircle.

The quadratic characteristic equation for ordinary eigenvalues \((\mu = 0)\) has discriminant

\[
\Delta = (a + d)^2 - 4(ad - bc)
\]

\[
= 4\left(p^2 - \det A\right)
\]

\[
= -4\left(g^2 - \rho^2\right)
\]

\[
= -4(g - \rho)(g + \rho),
\]

by (8) and (9). The eigenvalues may be written \(\lambda = f \pm \sqrt{\Delta/4}\).

Let \(P_0\) and \(Q_0\) be the lowest and highest points, respectively, on the eigencircle (i.e. where the eigencircle meets its vertical diameter), with \(\mu\)-coordinates \(g - \rho\) and \(g + \rho\) respectively (Figure 3). Then, as we saw above, the discriminant is just \(-4YP_0 \cdot YQ_0\), with lengths again signed by direction. Furthermore, if \(\overline{PYQ}\) is any line through \(Y\) and meeting the eigencircle in \(P\) and \(Q\), then we still have \(\Delta = -4YP \cdot YQ\), by the power property, and \(YP \cdot YQ\) is just the power of \(Y\) with respect to the eigencircle: see Figure 3. For any such \(P\) and \(Q\), then, the eigenvalues of \(A\) can be written as \(f \pm \sqrt{-YP \cdot YQ}\).

In the real case, the eigenvalues (being \(\lambda\)-coordinates of the points \(\Lambda_1, \Lambda_2\) in Figure 3) are \(f \pm \Lambda_1Y = f \pm Y\Lambda_2 = f \pm \sqrt{-Y\Lambda_1 \cdot Y\Lambda_2}\); we are, in effect, just taking \(P = \Lambda_1\) and \(Q = \Lambda_2\).
EIGENCIRCLES AND ASSOCIATED SURFACES

**Figure 3:** The power of $Y$ with respect to the eigencircle, which is related to the discriminant

**Complex eigenpairs**

So far, and in [1], we have required $\lambda$ and $\mu$ to be real. We now consider what happens when we allow them to be complex. We consider three- and four-dimensional surfaces associated with the matrix which contain the eigencircle and have a number of interesting properties.

Put $\lambda = \lambda_1 + i\lambda_2$ and $\mu = \mu_1 + i\mu_2$. The real and imaginary parts of (10) give

\[
(\lambda_1 - f)^2 + (\mu_1 - g)^2 - \lambda_2^2 - \mu_2^2 = \rho^2, \quad (11)
\]

\[
(\lambda_1 - f)\lambda_2 + (\mu_1 - g)\mu_2 = 0. \quad (12)
\]

The complex eigenpairs correspond to real eigentuples $(\lambda_1, \mu_1, \lambda_2, \mu_2)$ satisfying (11) and (12), which form a two-dimensional eigensurface $\mathcal{E}$ in four-dimensional real space. On the face of it, the four-dimensional nature of this space may make it difficult to visualise things such as complex eigenvalues and eigenvectors. However, we shall see how to do this with an appropriate three-dimensional picture.

The eigencircle equation (10) has the parametric form

\[
\lambda = f + \rho \cos \phi, \\
\mu = g + \rho \sin \phi.
\]

If the argument $\phi = \phi_1 + i\phi_2$ is complex, then we have a complex parametric form of the eigensurface $\mathcal{E}$, from which a description in terms of two real parameters can be derived using elementary properties of circular and hyperbolic functions:

\[
\lambda_1 = f + \rho \cos \phi_1 \cosh \phi_2, \quad (13)
\]

\[
\mu_1 = g + \rho \sin \phi_1 \cosh \phi_2, \quad (14)
\]
\[ \lambda_2 = -\rho \sin \phi_1 \sinh \phi_2, \]  
\[ \mu_2 = \rho \cos \phi_1 \sinh \phi_2, \]  
where \( \phi_1 \in [0, 2\pi) \), \( \phi_2 \in \mathbb{R} \). Put

\[ \sigma = \rho \cosh \phi_2, \]  
\[ \tau = \rho \sinh \phi_2, \]  
so \( \sigma \geq \rho, \tau \in \mathbb{R} \). Observe that

\[ \sigma^2 - \tau^2 = \rho^2, \]  
\[ (\lambda_1 - f)^2 + (\mu_1 - g)^2 = \sigma^2, \]  
\[ \lambda_2^2 + \mu_2^2 = \tau^2, \]
and that the equations for \( \xi \) may now be written

\[ \lambda_1 = f + \sigma \cos \phi_1, \]  
\[ \mu_1 = g + \sigma \sin \phi_1, \]  
\[ \lambda_2 = -\tau \sin \phi_1, \]  
\[ \mu_2 = \tau \cos \phi_1, \]  
with \( \sigma \) and \( \tau \) related by (19).

We now look at some views of the eigensurface which give insight into the shape of that surface and the properties of \((\lambda, \mu)\)-eigenvectors, with particular application to ordinary complex eigenvectors.

The real part of an eigentuple \( E = (\lambda_1, \mu_1, \lambda_2, \mu_2) \) is \( R = (\lambda_1, \mu_1) \), and these real parts may be illustrated by their projections onto the plane \( \lambda_2 = \mu_2 = 0 \), which may be called the real plane. Equation (11) shows that the eigensurface \( \xi \) meets the real plane in the eigencircle \( \xi \), and that the projection \( R \) cannot lie inside the eigencircle. However, any point in the real plane outside the eigencircle gives the real part of some eigentuple. The components of the vector \( \overrightarrow{CR} \) in the real plane are \( (\lambda_1 - f, \mu_1 - g) = (\sigma \cos \phi_1, \sigma \sin \phi_1) \) with \( \sigma \geq \rho \) (Figure 4(a)).

Similarly the imaginary part \( I = (\lambda_2, \mu_2) \) may be represented by a projection onto the imaginary plane \( \lambda_1 = \mu_1 = 0 \).

Now suppose the imaginary plane is superimposed on the real plane, with the \( \lambda_2 \)- and \( \mu_2 \)-axes parallel to the \( \lambda_1 \)- and \( \mu_1 \)-axes respectively, and the \( \lambda_2, \mu_2 \) origin superimposed on the centre \( C \) of the eigencircle. Then equation (12) requires that an imaginary part \( I \), corresponding to real part \( R \) in Figure 4(a), lies on a line \( \mathcal{L}_{\phi_1} \) in the imaginary plane through \( O \) which is perpendicular to \( CR \) (Figure 4(b)). In (24) and (25), \( \tau = OI \); from (19), \( \tau = \pm \sqrt{\sigma^2 - \rho^2} \) so we take \( \tau \) as a signed radial distance \( OI \) along \( \mathcal{L}_{\phi_1} \). Thus for each real part \( R \) outside the eigencircle there are two possible imaginary parts differing in sign; two such eigentuples are conjugate. Observe in
passing that if $R = O$ and $\det A > 0$ (so $O$ is outside the eigencircle) then the corresponding imaginary parts have $\tau = \pm \sqrt{\det A}$.

It will be convenient to refer also to the line through $C$ and parallel to $\mathcal{L}_{\phi}$. We call this $\mathcal{L}_{\phi}'$.

We now give a two-dimensional picture of the relationship between the lengths $\sigma$ and $\tau$ of the real and imaginary parts. Consider the eigentuples corresponding (via (13, 14, 15, 16)) to angles $\phi_1 = \nu$ and $\phi_1 = \nu + \pi$. Such angles determine a diameter $PCQ$ of the eigencircle, shown in Figure 4(a) (by the angles made with the positive $\lambda_1$-axis). These angles also determine the lines $\mathcal{L}_\nu$ (Figure 4(b)) and $\mathcal{L}_\nu'$ (by the angles made with the positive $\mu_2$-axis). It is helpful to picture the $\sigma, \tau$-plane $\mathcal{P}_\nu$ determined by the lines $\overline{PCQ}$ and $\mathcal{L}_\nu'$ and perpendicular to the real plane. The eigentuples with $\phi_1 \in \{\nu, \nu + \pi\}$ lie on the hyperbola (19) in this plane (see Figure 5). Call this hyperbola $h_\nu$. The eigensurface is thus the union of these $h_\nu$ over all $\nu$.

Note, however, that the eigensurface is not a hyperboloid, since as $\nu$ varies, so does the direction of the $\tau$-axis $\mathcal{L}_\nu'$ in $\mathbb{R}^4$. Rather, $\mathcal{E}$ may be obtained by rotating $h_\nu$ about $\mathcal{L}_\nu'$ while simultaneously rotating $\mathcal{L}_\nu$ about $O$ in the imaginary plane, and rotating $\mathcal{L}_\nu'$ about $C$ so that it remains parallel to $\mathcal{L}_\nu$. Nonetheless, certain hyperboloids will be useful in forming a three-dimensional view of some aspects of the eigensurface.

Equations (19), (20) describe a hyperboloid of one sheet, $\mathcal{H}_\nu$, in the subspace of $\mathbb{R}^4$ determined by the $\lambda_1$-axis, $\mu_1$-axis, and $\mathcal{L}_\nu'$ as the $\tau$-axis. The axis of symmetry of $\mathcal{H}_\nu$ is $\mathcal{L}_\nu'$. Observe that

$$\mathcal{H}_\nu \cap \mathcal{P}_\nu = h_\nu$$

and

$$\mathcal{H}_\nu \cap \mathcal{E} = h_\nu \cup \mathcal{E}.$$
FIGURE 5: (a) The hyperbola $h_v$ in the $\sigma, \tau$-plane $P_v$, showing lengths of the real and imaginary parts ($R$ and $I$) of an eigentuple. (b) Conjugate eigentuples $E_1, E_2$ and associated $E_i$-eigenvectors shown with $h_v$ and the eigencircle $C$. The hyperboloid $\mathcal{H}_v$ is obtained by revolving $h_v$ around $L_v$.

In particular, $\mathcal{H}_v \neq C$. The mapping $\eta_v : C \rightarrow \mathcal{H}_v$ given by

$$(\lambda_1, \mu_1, \lambda_2, \mu_2) \rightarrow (\lambda_1, \mu_1, -\tau \sin \nu, \tau \cos \nu)$$

using equations (13) to (18), is nonlinear. However, its restriction to $\mathcal{H}_v \cap C$ is the identity map on that set.

$(\lambda, \mu)$-eigenvectors corresponding to a complex eigenpair $(\lambda, \mu)$ may be obtained as in the real case (see the section 'Properties of the eigencircle' above) just by allowing $\lambda$ and $\mu$ to be complex. The $(\lambda, \mu)$-eigenvectors are thus complex multiples of $\begin{pmatrix} \lambda - d \\ \mu + c \end{pmatrix}$, with $\lambda, \mu \in \mathbb{C}$. It is straightforward to represent these complex vectors in $\mathbb{R}^4$ as vectors from $T(d, -c, 0, 0)$ to $E(\lambda_1, \mu_1, \lambda_2, \mu_2)$.

We can use $\mathcal{H}_v$ to picture $(\lambda, \mu)$-eigenvectors as follows. Suppose $(\lambda_1, \mu_1, \lambda_2, \mu_2) \in C$. Find $\nu \in [0, \pi)$ such that $(\lambda_1, \mu_1, \lambda_2, \mu_2) \in h_v$. Then the eigentuples $E_i = (\lambda_1, \mu_1, \pm \lambda_2, \pm \mu_2), (i = 1, 2)$, belong to $\mathcal{H}_v \cap C$. $E_1$ and $E_2$ are conjugate in the sense that their real parts are identical and their imaginary parts are oppositely signed. As seen above, the $E_i$-eigenvectors corresponding to the eigentuple $E_i$ are (complex) multiples of the vector from $T$ to $E_i, (i = 1, 2)$. Now $T$ also belongs to $\mathcal{H}_v \cap C$, since it lies on the eigencircle. Since all three points $T, E_1, E_2$ are in $\mathcal{H}_v \cap C$, and the latter set is fixed under $\eta_v$, we can picture these points—and hence those $(\lambda, \mu)$-eigenvectors described above—in a three-dimensional picture of $\mathcal{H}_v$, knowing that they bear the same relationship to each other in this picture as they do in the eigensurface. See Figure 5(b).

An important special case is when $\nu = -\pi/2$, when $\mathcal{H}_{-\pi/2}$ lies in a subset of $(\lambda_1, \mu_1, \lambda_2, \mu_2)$-space in which $\mu_2 = 0$. We find it convenient to write $\mathcal{H}_{\perp} = \mathcal{H}_{-\pi/2}$. If the matrix $A$ has complex eigenvalues, then from our
earlier discussion on the discriminant (with $\Delta < 0$) we see that these eigenvalues are $\lambda^{(i)} = f \pm i\sqrt{p_Y}$, $(i = 1, 2)$, where $p_Y = -\Delta / 4$ is the power of $Y$ with respect to the eigencircle. We can picture these eigenvalues on $\mathcal{H}_\perp \cap \mathcal{E}$ using the points $E_1 = (f, 0, \sqrt{p_Y}, 0)$ and $E_1 = (f, 0, -\sqrt{p_Y}, 0)$. The power $p_Y$ is the square of the length $YE_i$. We can therefore give a geometric view of certain complex eigenvectors corresponding to these complex eigenvalues, in $(\lambda_1, \mu_1, \tau)$-space, in the manner described in the previous paragraph and without resorting to a four-dimensional picture. The resulting diagram is similar to Figure 5(b), and has $R = Y$ as the midpoint of $E_1E_2$. The $\tau$-axis $\mathcal{L}_{-\pi/2}$ is now parallel to the $\lambda_2$-axis, and $\mathcal{L}_{-\pi/2}$ is the $\lambda_2$-axis itself.

**Lines in the hyperboloid**

Assume that $A$ has complex eigenvalues, as in the previous section. Let $P_1$ be a point on the eigencircle $\mathcal{C}$ such that the tangent to $\mathcal{C}$ at $P_1$ passes through $Y$. It is evident that the line $P_1\Lambda_1$ is tangent to the hyperboloid $\mathcal{H}_\perp$ at $P_1$, since it (like the line $YP_1$) belongs to the vertical tangent plane to $\mathcal{H}_\perp$ at $P_1$. Since $P_1\Lambda_1$ is tangent to $\mathcal{H}_\perp$ at $P_1$ and meets $\mathcal{H}_\perp$ again at $\Lambda_1$, it must in fact lie entirely in $\mathcal{H}_\perp$: see [8, §10.4]. The same property holds for $P_1\Lambda_2$, of course.

For another pair of lines related in some special manner to $A$ and contained in one of our hyperboloids, suppose $\det A > 0$ and allow $A$ to have real or complex eigenvalues. Let $I_1, I_2$ be those eigentuples with real part $(0, 0)$: $I_1, I_2 = \pm (0, 0, \sqrt{-\det A} \sin \alpha, -\sqrt{-\det A} \cos \alpha) \in \mathcal{H}_\alpha$, where $\alpha = \angle YOC$. Recall the tangent line $\overline{OD}$ to the eigencircle at $D$ (see Figure 1(a)). Observe that $\overline{OD}, \overline{OI_1}, \overline{OI_2}$ all belong to the vertical tangent plane to $\mathcal{H}_\alpha$ at $D$. Each of $\overline{OI_1}, \overline{OI_2}$ must lie entirely in $\mathcal{H}_\alpha$, since each is tangent to $\mathcal{H}_\alpha$ at one point and meets it again at another.

**Quaternions**

The algebra of quaternions consists of 4-tuples which may be written

$$\alpha + \beta i + \gamma j + \delta k,$$

where $i^2 = j^2 = k^2 = ijk = -1$. (See, for example, [9, §41].) These may be represented by certain $2 \times 2$ complex matrices:

$$\begin{pmatrix} u & v \\ -v & u \end{pmatrix},$$

where $u = \alpha + i\beta$ and $v = \gamma + i\delta$ are complex numbers. Mimicking our original line of enquiry, we take any real $2 \times 2$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and ask for pairs $(\lambda, \mu) \in \mathbb{C}^2$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda & \mu \\ -\bar{\mu} & \bar{\lambda} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$
for some $x, y$ not both 0. We obtain the following equations, where $\lambda = \lambda_1 + i\lambda_2, \mu = \mu_1 + i\mu_2$ and $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$:

\[
(\lambda_1 - f)^2 + (\mu_1 - g)^2 + \lambda_2^2 + \mu_2^2 = \rho^2, \quad (26)
\]

\[
\lambda_2 (a - d) + \mu_2 (b + c) = 0. \quad (27)
\]

Again, we have a two-dimensional surface $\mathcal{E}_Q$ in four-dimensional space. We call the points on $\mathcal{E}_Q$ quaternion eigentuples. Equation (27) restricts the imaginary part $J$ of a quaternion eigentuple to lie on a line $\mathcal{L}$ in the imaginary plane, where (after superimposition of real and imaginary planes as in the section on complex eigenpairs) $\mathcal{L}$ is perpendicular to the diagonal of the rectangle defined by $A$. In (26) put $\lambda_2^2 + \mu_2^2 = \tau^2$, where $|\tau| = OJ$ is the size of the imaginary part $J$. Then (26) shows that $\mathcal{E}_Q$ is a sphere, which (inevitably) we call the eigensphere. It sits in the three-dimensional subspace spanned by the $\lambda$-axis, $\mu$-axis and $\mathcal{L}$ (which are mutually orthogonal). Note that the direction of $\mathcal{L}$, and hence the orientation in $\mathbb{R}^4$ of the eigensphere, depends on the exact choice of $A$, in contrast to the complex eigenpair situation where the eigensurface is determined entirely by the eigencircle. On the other hand, the direction of $\mathcal{L}$ in the imaginary plane is fixed: it does not depend on the real part, as the analogous line $\mathcal{L}_v$ did earlier. Any point inside the eigencircle gives the real part of some quaternion eigentuple (and those outside do not), again in contrast to complex eigenpairs where the points inside the eigencircle were precisely those that were not the real part of any eigentuple.

\[\text{Figure 6: The eigensphere, with a quaternion } E \text{-eigenvector } \overrightarrow{TE} \]

Equation (26) is invariant under negation of the imaginary part (equivalently, $\tau \rightarrow -\tau$), so any quaternion eigentuple $E$ has a 'dual' eigentuple $\overline{E}$ obtained by reflecting $E$ in the real plane. The quaternion $E$-eigenvectors can be obtained in $\mathbb{C}^2$ or $\mathbb{R}^4$. The quaternion $E$-eigenspace is
spanned by the vector \( \overrightarrow{TE} \), where \( T = (d, -c, 0, 0) \). These vectors may be pictured using the eigensphere in \( \lambda_1, \mu_1, \tau \) coordinates as shown in Figure 6. (Compare Figures 2(a) and 5(b).)

**Invitation**

We have found the eigencircle to be a rich source of geometric insight into algebraic properties of \( 2 \times 2 \) matrices. We invite the reader to explore its properties further, and to consider what useful analogues it might have for higher-order matrices. One possibility is to start with a version of (2) for \( 4 \times 4 \) matrices, with the matrix \( \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix} \) replaced by a \( 4 \times 4 \) real matrix representing a general quaternion (see [9, §41]). Instead of an eigencircle, we would have a fourth-degree hypersurface in four-dimensional space.

We also invite the reader to play with the eigencircle applet [10], and send us comments on it.

**References**


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