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Weighted Model Sets and their Higher Point-Correlations

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Abstract. Examples of distinct weighted model sets with equal 2, 3, 4, 5-point correlations are given.

1 Introduction

This short paper can be thought of as an extension of [4] where we proved that a regular real model set with a real internal space is, up to translation and alterations of density zero, uniquely determined by its 2- and 3-point correlation measures. There we have also given an example in the case that the internal space is the product space of a real space and a finite group that shows that this result does not extend in general to more complicated internal spaces. More precisely, we have given an example of two distinct model sets having equal 2- and 3-point correlation measures, created using a single cut-and-project scheme, that are not translationally equivalent, even allowing for alterations of density zero.

In this paper, extending the setting to *weighted* model sets, we offer examples of pairs of weighted model sets in $\mathbb{R}^d$ that have equal $2, 3, 4, 5$-point correlation measures, created using a single "cut and project" scheme, that are not translationally equivalent. These pairs are created by imposing a 6-colouring on a previously constructed aperiodic model set and then further imposing on them two different weighting schemes. The resulting weighted model sets are then different, not even having any weights in common, but their correlations measures, up to and including the fifth one, are identical.

The relevance of this type of result to the theory of long-range aperiodic order and crystallography along with some comments on its history are discussed in [4]. For more on the early history of mathematical diffraction, see [6]. Suffice it to say that uniformly discrete ergodic point sets in $\mathbb{R}^d$ are determined, up to local indistinguishability, by knowledge of all their point correlation measures ([3]). It is then a relevant question to ask how many of these correlations one really needs to distinguish two such *locally distinguishable* sets from each other. This sort of question is context sensitive; do we know *a priori* that the sets are model sets, or Meyer sets, or are from some other nice class of sets? The class of weighted model sets is particularly interesting because these sets are so often used in modelling quasicrystals (with the colours and weights representing different types of atoms and their masses or scattering densities) and have played such an important part in the development of

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the field of aperiodic order. One might suppose from the results stated above that at least for them, very few correlations would be necessary to separate them from each other. Our result here shows that even knowledge of the $2, 3, 4, 5$-point correlation measures need not be enough. Neither the system of weights nor the internal space that we use in the example are anything particularly extraordinary in the study of model sets or in applications of them to modelling physical systems. Even the model set construction of the often cited Penrose tilings and their accompanying Penrose point sets uses gradings and internal spaces that are products of a real space and a finite cyclic group.

The key to this result (and also to the previous examples for unweighted model sets) is to use corresponding results, developed in [6], for one dimensional periodic sets. There is a simple way in which to intertwine this periodicity with the structure of an ordinary model set, and this leads to aperiodic weighted model sets with points of several types or colours which have the same coincidence of $2, 3, 4, 5$-point correlations.

In brief outline, we start with a weighted periodic model set on the real line whose internal space is $\mathbb{Z}/N\mathbb{Z}$ for some $N > 1$. We let $\mathcal{S} = (\mathbb{R}^d \times H, \mathcal{L})$ be an arbitrary cut-and-project scheme, where H is a locally compact Abelian group and $\mathcal{L}$ is a lattice in $\mathbb{R}^d \times H$. As usual, we denote by L the projection of $\mathcal{L}$ into $\mathbb{R}^d$ and denote by $(\,\cdot\,)^\star$ the star map of $\mathcal{S}$ from L to H. Now assume that there is given a *surjective* homomorphism $\alpha\colon L \to \mathbb{Z}/N\mathbb{Z}$. Then there is a combined cut-and-project scheme specified by $\mathcal{S}^e = (\mathbb{R}^d \times (H \times \mathbb{Z}/N\mathbb{Z}), \mathcal{L}^e)$, where $\mathcal{L}^e := \{(x, (x^\star, \alpha(x))) : x \in L\}$. We call $\alpha(x)$ the colour or type of $x \in L$. We use this new cut-and-project scheme to create coloured model sets with several windows, each corresponding to a different colour. In order to get a weighted model set we weight the points of this coloured model set according to their type.

We assume that the reader is familiar with the basic theory of model sets ([8]). In Section 2 we briefly recall the definitions that we need and the important concepts of uniform distribution and the point-correlation measures. Closely connected to correlation measures are the pattern frequencies, which are generally more convenient for our purposes. In Section 3 we consider discrete *periodic* point sets on the real line, regarding them as model sets, and present the general formula for their finite-point correlation measures. In Section 4 we elaborate the method of building an *extended* weighted model set from a weighted periodic system and an aperiodic model set, and then determine the precise form of the resulting pattern frequencies. In Section 5 we take an example from [6] of two weighted periodic systems (based on $\mathbb{Z}/6\mathbb{Z}$) that have equal $2, 3, 4, 5$-point correlation measures (we offer a short proof of this), and then use it to show that our extension construction will produce any number of aperiodic weighted real model sets with equal $2, 3, 4, 5$-point correlation measures. Finally, by way of illustration, we offer an example of this using a 6-colouring of the vertices of Franz Gähler's shield tiling.

2 Model Sets and Correlations

We work in $\mathbb{R}^d$. The usual Lebesgue measure will be denoted by ℓ. The open cube of side length R centred at 0 is denoted by C_R. We begin with the *cut-and-project scheme*

$\mathcal{S} = (\mathbb{R}^d, H, \mathcal{L})$ consisting of a compactly generated, locally compact Abelian group H and a lattice $\mathcal{L} \subset \mathbb{R}^d \times H$ for which the projection mappings π_1 and π_2 from $\mathbb{R}^d \times H$ onto $\mathbb{R}^d$ and H are injective and have dense image respectively:

$$\begin{array}{ccccc} \mathbb{R}^d & \overset{\pi_1}{\longleftarrow} & \mathbb{R}^d \times H & \overset{\pi_2}{\longrightarrow} & H \\ & & \cup & & \\ L & \overset{\simeq}{\longleftrightarrow} & \mathcal{L} & & \\ x & \leftrightarrow & \tilde{x} & \mapsto & x^\star \end{array} . \tag{2.1}$$

Then $L := \pi_1(\mathcal{L})$ is isomorphic as a group to $\mathcal{L}$ and we have the mapping $(\,\cdot\,)^\star \colon L \to H, x \mapsto \tilde{x} \mapsto x^\star$, with dense image, defined by $\pi_2 \circ (\pi_1|_{\mathcal{L}})^{-1}$.

The statement that $\mathcal{L}$ is a lattice is equivalent to saying that it is a discrete subgroup of $\mathbb{R}^d \times H$ and that the quotient group $\mathbb{T} := (\mathbb{R}^d \times H)/\mathcal{L}$ is compact. We let θ_H be a Haar measure on H, normalized so that the product measure $\ell \otimes \theta_H$ on $\mathbb{R}^d \times H$ gives total measure 1 to a fundamental region of the lattice $\mathcal{L}$.[1] We let $\theta_{\mathbb{T}}$ be the canonical Haar measure on $\mathbb{T}$ whose total measure is 1.

For $W \subset H$,

$$\mathit{\Lambda}(W) := \{u \in L : u^\star \in W\}.$$

A set $W \subset H$ is called a *window* if $\Sigma^\circ \subset W \subset \Sigma$ for some compact set $\Sigma \subset H$ that satisfies $\overline{\Sigma^\circ} = \Sigma$.

We shall assume throughout that all windows that we use have boundaries of θ_H-measure 0.

We will deal with multiple windows, and it is convenient to allow windows to be empty (which is allowed by the definition).

A (regular) *model set* or *cut-and-project set* is a set of the form $\Lambda = x + \mathit{\Lambda}(W)$, where W is a window and $x \in \mathbb{R}^d$. In the sequel we shall only have need of the simpler model sets of the form $\Lambda = \mathit{\Lambda}(W)$.

Model sets are uniformly distributed point sets:

Theorem 2.1 ([9]) *Let $W \subset H$ be a window. Then*

$$\lim_{R\to\infty} \frac{1}{\ell(C_R)} \operatorname{card}(\mathit{\Lambda}(W) \cap C_R) = \theta_H(W).$$

Let m be any positive integer and let $\mathbf{m} := \{1, \dots, m\}$. We assume that we are given *disjoint* windows $W_1, \dots, W_m$ and then the corresponding model sets

$$\Lambda_j = \mathit{\Lambda}((W_j) \subset \mathbb{R}^d.$$

Let $\Lambda := \bigcup_{i=1}^m \Lambda_i$ (a disjoint union). We assume given a set $\mathbf{w} = (w_1, \dots, w_m)$ of weights $w_j \in \mathbb{R}$. For $x \in \mathbb{R}^d$ we define

$$w(x) = \begin{cases} w_j & \text{if } x \in \Lambda_j, \\ 0 & \text{otherwise.} \end{cases}$$

[1]Other normalizations are possible. The one we have chosen leads to the formula of Theorem 2.1. Other normalizations produce multiplicative factors in this formula. None of this is relevant to what we need here.

In our examples below we shall only use non-negative weights.

We call $(\Lambda_1, \dots, \Lambda_m)$ with the weights $\mathbf{w}$ a *weighted model set*. More generally, one would allow arbitrary translations of the colour component sets Λ_i provided the translated point sets do not overlap, but we have no need of this here. For notational simplicity we shall use the symbol $\Lambda^{\mathbf{w}}$ (or often simply Λ if the context is clear) to denote the coloured/weighted model set that we have just described.

The $n+1$-point *correlation* $(n = 1, 2, \dots)$ of a model set $\Lambda^{\mathbf{w}}$ (or more generally any weighted locally finite subset of $\mathbb{R}^d$) is the measure on $(\mathbb{R}^d)^n$ defined by

$$\begin{aligned}\gamma_\Lambda^{(n+1)}(f) &= \lim_{R\to\infty} \frac{1}{\ell(C_R)} \sum_{y_1,\dots,y_n,x \in C_R\cap\Lambda} w(x)w(y_1)\dots w(y_n) f(-x+y_1,\dots,-x+y_n)\\ &= \lim_{R\to\infty} \frac{1}{\ell(C_R)} \sum_{\substack{x\in C_R\cap\Lambda\\ y_1,\dots,y_n\in\Lambda}} w(x)w(y_1)\dots w(y_n) f(-x+y_1,\dots,-x+y_n),\end{aligned}$$

for all continuous compactly supported functions f on $(\mathbb{R}^d)^n$ ([3]). The simpler second sum is a result of the compactness of the support K of f and the van Hove property of the averaging sequence $\{C_R\}$ (namely that the measures of the K-boundaries of the sets C_R have vanishing relevance relative to the total volume of C_R as $R \to \infty$). Of course, it is not immediately clear whether or not these limits exist, but they do for model sets, as we shall now see.

Because model sets are Meyer sets, the sets of elements $y_j - x$ that make up the values of the arguments of f occurring in the sums lie in the *uniformly discrete* set $\Lambda - \Lambda$. For any $\mathbf{z} = (z_1, \dots, z_n) \in (\Lambda - \Lambda)^n$ and any $\mathbf{k} = (k(0), \dots, k(n)) \in \mathbf{m}^{n+1}$ we can count the occurrences of $z_1, \dots, z_n$ in the form $-x + y_1, \dots, -x + y_n$, where $x \in \Lambda_{k(0)}$, $y_j \in \Lambda_{k(j)}$, $j = 1, \dots, m$, namely

$$\begin{aligned}&\mathrm{freq}_{\mathbf{k}}(z_1,\dots,z_n)\\ &\quad= \lim_{R\to\infty}\frac{1}{\ell(C_R)} \operatorname{card}\{x\in C_R : x\in\Lambda_{k(0)}, y_j\in\Lambda_{k(j)}, x = -z_j + y_j, j = 1,\dots,m\}\\ &\quad= \lim_{R\to\infty}\frac{1}{\ell(C_R)} \operatorname{card}\{x\in L\cap C_R : x^\star \in W_{k(0)} \cap \bigcap_{j=1}^m (-z_j^\star + W_{k(j)})\}\\ &\quad= \theta_H\Big(W_{k(0)} \cap \bigcap_{j=1}^m \big(-z_j^\star + W_{k(j)}\big)\Big),\end{aligned}$$

where we have used the uniform distribution theorem.

Hence for model sets we find that all these correlation measures exist and

$$\gamma_\Lambda^{(n+1)} = \sum_{z_1,\dots,z_n\in\Lambda-\Lambda} \Big(\sum_{\mathbf{k}\in\mathbf{m}^{n+1}} w_{k(0)}\dots w_{k(n)}\ \mathrm{freq}_{\mathbf{k}}(z_1,\dots,z_n)\Big)\,\delta_{(z_1,\dots,z_n)}.$$

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3 A Periodic Example

Let N be a positive integer and let $H = \mathbb{Z}/N\mathbb{Z}$. Then we have the cut-and-project scheme

$$\begin{array}{ccccc} \mathbb{R} & \xleftarrow{\pi_1} & \mathbb{R}\times\mathbb{Z}/N\mathbb{Z} & \xrightarrow{\pi_2} & \mathbb{Z}/N\mathbb{Z} \\ & & \cup & & \\ \mathbb{Z} & \xleftrightarrow{\simeq} & \widetilde{\mathbb{Z}} & & \\ x & \longleftrightarrow & (x, x_N) & \mapsto & x_N, \end{array} \tag{3.1}$$

where $x_N := x \mod N$ and $\widetilde{\mathbb{Z}} := \{(x, x_N) : x \in \mathbb{Z}\}$. Let $\mathbf{m}$ be as above. We assume that we are given disjoint subsets (windows) $A_1, \dots, A_m \subset \mathbb{Z}/N\mathbb{Z}$ and corresponding colour weights $w_1, \dots, w_m$. The corresponding model set is $\Lambda = \bigcup \Lambda_j$, where $\Lambda_j = \mathit{\Lambda}(A_j) = \{x \in \mathbb{Z} : x_N \in A_j\}$, $j = 1, \dots m$. Each colour set Λ_j is periodic, repeating modulo N, containing all those integers congruent mod N to an element of A_j.

For each $j = 0, \dots, N-1$ define

$$c_j = \begin{cases} w_k & \text{if } j \in A_k, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

A given pattern $(r_1, \dots r_n)$ of integers modulo N must occur in Λ in the form

$$s + r_1, \dots, s + r_n \mod N, \quad s = 0, 1, \dots, N-1\,.$$

If it occurs with colours $k(1), \dots, k(n)$ and $k(0)$ is the colour of s, then $r_j \in \Lambda_{k(j)} - \Lambda_{k(0)}$ for all j, and the weighting is

$$w_{k(0)} \dots w_{k(n)} = c_s c_{s+r_1} \dots c_{s+r_n}\,.$$

Thus, for this weighted model set we have

$$\gamma_\Lambda^{(n+1)} = \frac{1}{N} \sum_{(r_1,\dots,r_n)} M_n(r_1, \dots, r_n)\delta_{(r_1,\dots,r_n)},$$

where the sum runs over all possible patterns of length n and where

$$M_n(r_1, \dots, r_n) := \sum_{s=0}^{N-1} c_s c_{s+r_1} \dots c_{s+r_n}\,. \tag{3.3}$$

By the above arguments,

$$M_n(r_1, \dots, r_n) = \sum_{\mathbf{k}\in\mathbf{m}^{n+1}} w_{k(0)} \dots w_{k(n)}\, \theta_{\mathbb{Z}/N\mathbb{Z}}\Big(A_{k(0)} \cap \bigcap_{j=1}^{n} \big(-r_j^\star + A_{k(j)}\big)\Big)\,. \tag{3.4}$$

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4 Aperiodic Examples

Start with the cut-and-project scheme $\mathcal{S}$ of Section 2, along with the corresponding notation. Let $\alpha\colon L \to \mathbb{Z}/N\mathbb{Z}$ be any surjective homomorphism with the property that

$$\{(x^\star, \alpha(x)) : x \in L\} \text{ is dense in } H \times \mathbb{Z}/N\mathbb{Z}.$$

Form the new cut-and-project scheme $\mathcal{S}^e = (\mathbb{R}^d, H \times \mathbb{Z}/N\mathbb{Z}, \mathcal{L}^e)$, where $\mathcal{L}^e := \{(\tilde{x}, \alpha(x)) : x \in L\}$:

$$\begin{array}{ccccc} \mathbb{R}^d & \longleftarrow & \mathbb{R}^d \times H \times \mathbb{Z}/N\mathbb{Z} & \longrightarrow & H \times \mathbb{Z}/N\mathbb{Z} \\ & & \cup & & \\ L & \overset{\simeq}{\longleftrightarrow} & \mathcal{L}^e & & \\ x & \longleftrightarrow & (\tilde{x}, \alpha(x)) & \mapsto & (x^\star, \alpha(x)). \end{array} \tag{4.1}$$

$\mathcal{L}^e$ is clearly discrete, and since the group $\mathbb{Z}/N\mathbb{Z}$ is finite, the index

$$[\mathcal{L}^e : (\{\tilde{x} \in \mathcal{L} : \alpha(x) = 0\}, 0)]$$

is finite and the quotient of $\mathbb{R}^d \times H \times \mathbb{Z}/N\mathbb{Z}$ by $\mathcal{L}^e$ is compact. In short, $\mathcal{L}^e$ is a lattice in $\mathbb{R}^d \times H \times \mathbb{Z}/N\mathbb{Z}$.

Now we let $\mathbf{m}$ and the sets A_j be as in Section 3. Let W be a non-empty window in H and set $W_j := W \times A_j \subset H \times \mathbb{Z}/N\mathbb{Z}$. This produces from $\mathcal{S}^e$ coloured model sets

$$\Lambda_j^e = \mathit{\Lambda}^e(W_j). \tag{4.2}$$

Let $\Lambda^e := \bigcup_{j=1}^m \Lambda_j^e$. Notice that the actual points of the model sets involved here form a subset of the model set determined by the original cut-and-project scheme $\mathcal{S}$, whereas the colours are determined by the periodic scheme.

The $(n+1)$-point correlation for Λ^e is

$$\gamma_\Lambda^{(n+1)} = \sum_{z_1,\dots,z_n \in \Lambda^e - \Lambda^e} \Bigl(\sum_{\mathbf{k} \in \mathbf{m}^{n+1}} w_{k(0)} \dots w_{k(n)} \ \mathrm{freq}_{\mathbf{k}}(z_1, \dots, z_n) \Bigr) \delta_{(z_1,\dots,z_n)}, \tag{4.3}$$

where

$$\mathrm{freq}_{\mathbf{k}}(\{z_1, \dots, z_n\}) = \theta_{H\times\mathbb{Z}/N\mathbb{Z}} \Bigl(W_{k(0)} \cap \bigcap_{j=1}^n \bigl(-z_j^\star + W_{k(j)} \bigr) \Bigr).$$

Let $z_j^\star = (v_j^\star, r_j^\star) \in H \times \mathbb{Z}/N\mathbb{Z}$. Then for $(q, r) \in H \times \mathbb{Z}/N\mathbb{Z}$,

$$\begin{aligned} (q, r) \in W_{k(0)} \cap \bigcap_{j=1}^n (-z_j^\star + W_{k(j)}) & \\ \Leftrightarrow (q, r) \in W \times A_{k(0)} \text{ and } (q, r) \in (-v_j^\star, -r_j^\star) + W \times A_{k(j)} & \\ \Leftrightarrow q \in W \cap \bigcap (-v_j^\star + W) \text{ and } r \in A_{k(0)} \cap \bigcap (-r_j^\star + A_{k(j)}) . & \end{aligned}$$

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Thus the (relative) frequencies are given by

$$\mathrm{freq}_{\mathbf{k}}(z_1,\dots,z_n)=\theta_H\Big(W\cap\bigcap_{j=1}^{n}(-\nu_j^\star+W)\Big)\,\theta_{\mathbb{Z}/N\mathbb{Z}}\Big(A_{k(0)}\cap\bigcap_{j=1}^{n}(-r_j^\star+A_{k(j)})\Big).$$

The first term of this factorization is independent of $\mathbf{k}$, and as a consequence we can rewrite (4.3) as

$$\gamma_\Lambda^{(n+1)}=\sum_{z_1,\dots,z_n\in\Lambda^e-\Lambda^e}\theta_H\Big(W\cap\bigcap_{j=1}^{n}(-\nu_j^\star+W)\Big)M_n(r_1,\dots,r_n)\delta_{(z_1,\dots,z_n)}.$$

It is not particularly important for our purposes that the frequencies here be absolute. As we already pointed out, that depends on normalizing the Haar measures so that the corresponding Haar measure on $(\mathbb{R}^d\times H\times\mathbb{Z}/N\mathbb{Z})/\mathcal{L}^e$ has total measure equal to 1. What is important is to realize that if we colour and weight by a second set of weights $w_1',\dots,w_m'$ and a second set of windows $A_1',\dots,A_m'$ so that the expressions of (3.4) are equal for some n, then the $(n+1)$-point correlations of the two corresponding weighted model sets Λ^e arising from using one or the other of these two colour/weighting schemes will also be the same. This information is contained in (3.3), and it is this form that we shall see in the examples.

It is interesting to note here that the formula for the frequencies makes it look as if we are dealing with a simple product structure. However, the points $z_j=(\nu_j,r_j)$ are not truly from an unrestricted product. In fact, $r_j=\alpha(\nu_j)$. The reason for the frequencies to be given as they are is that the lattice $\mathcal{L}^e$ already has this special structure built into it. We have assumed that its image is dense in $H\times\mathbb{Z}/N\mathbb{Z}$, so we have a cut-and-project scheme, and this allows us to use the uniform distribution of model sets to derive the frequencies in terms of the measures of the windows.

5 Examples

With $N=6$, there are two sets of weights

(5.1) $\quad\mathbf{ws}_1:=[11,25,42,45,31,14]\qquad\mathbf{ws}_2:=[10,21,39,46,35,17]$

which, when used to weight the sets $A_j=A_j'=\{j\}\mod 6$, $j=0,\dots,5$, determine identical results on the left side of (3.3) for $n=1,2,3,4$; see [6, §5.3]. It follows from our discussions that the corresponding weighted model sets built in (4.2), though quite different, nonetheless have equal $2,3,4,5$-point correlations.

Although the information needed to show that the sums arising in (3.3) from these two sets of weights are the same is implicit in [6], and although it would be easy to check the result on a computer, it is interesting to see what lies behind this.

Given a collection of disjoint subsets $A_1,\dots,A_m$ of $\mathbb{Z}/N\mathbb{Z}$ and a set of weights w_k, $k=1,\dots m$, we define $\mathbf{c}=(c_0,\dots,c_{N-1})$, where the $c_j\in\mathbb{Z}$, using (3.2). The corresponding weighted Dirac comb is

$$D:=\sum_{j=0}^{N-1}c_j\delta_j,$$

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and its Fourier transform is $\widehat{D}$ given by

$$\widehat{D}(k) = \sum_{j=0}^{N-1} c_j \exp\Big(\frac{-2\pi i jk}{N}\Big) = P(w^k),$$

where $w := e^{-2\pi i/N}$ and $P = P^{\mathbf{c}}$ is defined by

$$P(x) = \sum_{j=0}^{N-1} c_j x^j.$$

In the weighted periodic point set determined by our choice of the cut-and-project scheme (3.1), the windows $A_1, \dots A_m$, and the weighting system $\mathbf{c}$, the pattern frequency of $(l_1, \dots, l_n)$ is, up to the appropriate normalization factor, given by (3.3):

$$M_n(l_1, \dots l_n) = \sum_{l=0}^{n} c_l c_{l+l_1} \cdots c_{l+l_n}.$$

In this way we have a function

$$M_n \colon (\mathbb{Z}/N\mathbb{Z})^n \longrightarrow \mathbb{R}.$$

A straightforward calculation of the Fourier transform of M leads to

$$\begin{aligned}\widehat{M_n}(k_1, \dots, k_n) &= \widehat{D}(k_1) \cdots \widehat{D}(k_n)\widehat{D}(-(k_1 + \cdots + k_n))\\ &= P(w^{k_1}) \cdots P(w^{k_n})P(w^{-(k_1+\cdots+k_n)}).\end{aligned}$$

Since M_n and $\widehat{M_n}$ determine each other, knowing one is the same as knowing the other. In particular, if two weighting systems determine $\mathbf{c}$ and $\mathbf{c}'$, and these determine the same functions $P(w^{k_1}) \cdots P(w^{k_n})P(w^{-(k_1+\cdots+k_n)})$ for some n, then they also produce the same $n+1$-point pattern frequencies.

Let us apply this analysis to the two weighting systems for $\mathbb{Z}/6\mathbb{Z}$ given in (5.1). Here the two corresponding polynomials are

$$P_1(x) := (x+1)(x^2+x+1)(2x^2+5)(3x+1) =: Q_1(x)(3x+1)$$

and

$$P_2(x) := (x+1)(x^2+x+1)(2x^2+x+4)(3x+1) =: Q_2(x)(3x+1).$$

Since we are only interested in the values of these at powers of $w = e^{-2\pi i/6}$, we may alter P_1 and P_2, reducing their exponents of x modulo 6 so that they all lie in the range $0, \dots, 5$. Having done this one checks that these indeed produce the coefficients $c_0, \dots, c_5$ given by $\mathbf{ws}_1, \mathbf{ws}_2$. We also observe that by their construction these polynomials vanish at w^2, w^3, w^4. Also the two polynomials are equal for $x = 1$,

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and furthermore, $w^4 Q_1(w^{-1}) = Q_2(w)$ and $w^4 Q_2(w^{-1}) = Q_1(w)$. This means that $Q_1(w)P_1(w^{-1}) = Q_2(w)P_2(w^{-1})$ and hence $P_1(w)P_1(w^{-1}) = P_2(w)P_2(w^{-1})$.

Now consider the situation of the 5-point correlations. We have to look at

$$P(w^j)P(w^k)P(w^l)P(w^r)P(w^{-(j+k+l+r)})$$

for the two polynomials and show that these values are the same for all possible values of j, k, l, r. Assuming that none of these indices is 0 mod 6 and taking into account the vanishing properties above and the symmetry of the four indices j, k, l, r, we have the following table of possibilities:

j	k	l	r	-(j+k+l+r)
1	1	1	1	-4
1	1	1	-1	-2
1	1	-1	-1	0
1	-1	-1	-1	2
-1	-1	-1	-1	4

Given the vanishing properties of P_1, P_2, only the middle case is non-trivial. However, there the facts that 1 and -1 occur equally often and that the polynomials are equal at $w = 1$ gives the result.

If one of the indices, say r, is zero, then we are in the situation of the 4-point correlation. There the only non-obvious case is when $j, k, l = \pm 1$ and all three are not equal. Then along with $-(j + k + l)$ we have 1 twice and -1 twice, and so again the products are equal.

The situation for the 2- and 3-point correlations is equally simple.

We thus have the following theorem.

Theorem 5.1 *Given the cut-and-project scheme* (2.1) *and a nonempty window* $W \subset H$*, then the two (different!) aperiodic weighted model sets arising from the cut-and-project scheme* (4.1) *and the two mod 6 weighted colourings given by* $\mathbf{ws}_1$ *and* $\mathbf{ws}_2$ (5.1) *have the same* 2, 3, 4, 5*-point correlations.*

5.1 A Two Dimensional Example

The STS tiling, or *shield tiling*, is an aperiodic substitution tiling discovered by F. Gähler [5]. It consists of three types of tiles: squares, triangles and asymmetric hexagons that look like shields, see Figure 5.1. The vertices of an STS tiling can be realized as a model set $\mathcal{S} = (\mathbb{R}^2 \times \mathbb{R}^2, \mathcal{L})$ whose lattice can be described as

$$\mathcal{L} = \{(k_1, k_2, k_3, k_4) : 2k_i \in \mathbb{Z} \quad \text{for all } i, \text{all of them even or all of them odd}\}.$$

This lattice has an automorphism C of order 12. Lie theorists will recognize the lattice as the root lattice of type F_4, so C can be chosen as any one of its (conjugate) Coxeter transformations ([2]). The four eigenvectors of C lead to two real C-invariant 2D spaces of the real span of the lattice, and it is the projections onto these that create the cut-and-project scheme. C appears in each of these as a rotation of

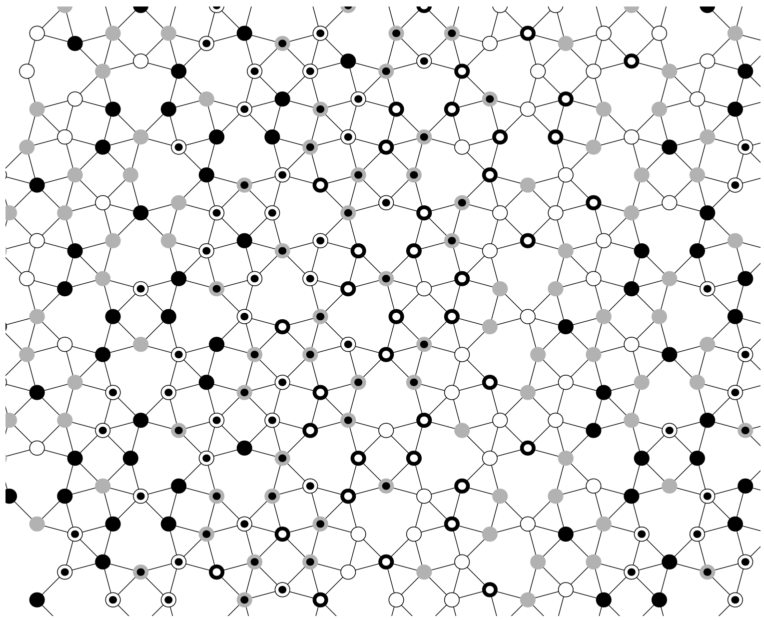

Figure 5.1: A fragment of the shield tiling with a 6-colouring

order 12. The window is a regular dodecagon W, displaced generically to avoid the projections of any of the lattice points of $\mathcal{L}$ falling on the boundary of W ([5]).

Inside $\mathcal{L}$ we have the sublattice of index 2 obtained by requiring the vectors to have integral components. Inside that, there is a sublattice $\mathcal{L}_0$ consisting of those vectors the sum of whose components is congruent to 0 modulo 3. Then $\mathcal{L}/\mathcal{L}_0 \simeq \mathbb{Z}/6\mathbb{Z}$ provides us with a homomorphism $\alpha\colon \mathcal{L} \to \mathbb{Z}/6\mathbb{Z}$ with which we can carry out the construction of Section 4.

The resulting colouring on the shield tiling is indicated in Figure 5.1 with the different colours indicated by different symbols. Symbols that differ only by the presence or absence of a centre dot correspond to colours differing by 3 modulo 6. There are no 6-colourings that respect the rotation C. Instead, what one can see here is the bands of like-colours moving in a roughly north-by-northeast direction (the shortest edge vector in this direction has degree 0) and then cycling through the six bands as one moves in the normal directions. According to the theory above, each of the two weighting systems of (5.1) can be applied to this model set and the resulting sets will be indistinguishable from the point of view of their 2, 3, 4, 5-point correlations. The results of [6] imply that they *are* distinguishable by their 6-point correlations.

The results of [6], on which the construction of this paper depends, seem not to have been generalized to dimensions greater than one. It would be interesting to do this, since it would probably give rise to other even more interesting examples of distinct model sets with many identical correlations. At the present time the whole question of classification of model sets by knowledge of a finite number of correlation measures seems wide open.

6 A Stochastic Interpretation

We finish by pointing out that it is possible to place the results and examples described in this paper into a stochastic setting. We use the same ingredients as before: an unweighted model set Λ, a homomorphism α, and a set of weights. We assume now that all the weights are non-negative and scaled so that their sum is 1. We imagine now that the points of the basic unweighted model set Λ are selected or not selected on the basis of independent random choices at each site, the probability of being selected being w_j if the point is of colour j. The resulting structure is a point process, each event being the outcome of independent Bernoulli trials made at every one of the sites according to the probabilities given by the weights. The moment measures that we have described then describe the expected values of the patterns that can occur in the point process. The consequences of this for the two-point correlation and the corresponding diffraction measure can be explicitly determined from [1, Thm. 2]. In particular the diffraction consists, almost surely, of a pure point part plus a continuous constant background.

More on pure point diffraction and higher moments will be found in the forthcoming [7].

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