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Revisiting Bolgiano–Obukhov scaling for moderately stably stratified turbulence

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According to the celebrated Bolgiano–Obukhov (Bolgiano, *J. Geophys. Res.*, vol. 64 (12), 1959, pp. 2226–2229; Obukhov, *Dokl. Akad. Nauk SSSR*, vol. 125, 1959, p. 1246) phenomenology for moderately stably stratified turbulence, the energy spectrum in the inertial range shows a dual scaling: the kinetic energy follows (i) $\sim k^{-11/5}$ for $k < k_B$, and (ii) $\sim k^{-5/3}$ for $k > k_B$, where k_B is the Bolgiano wavenumber. The $k^{-5/3}$ scaling, akin to passive scalar turbulence, is a direct consequence of the assumption that buoyancy is insignificant for $k > k_B$. We revisit this assumption, and using the constancy of kinetic and potential energy fluxes and simple theoretical analysis, we find that the $k^{-5/3}$ spectrum is absent. This is because the velocity field at small scales is too weak to establish a constant kinetic energy flux as in passive scalar turbulence. A quantitative condition for the existence of the second regime is also derived in the paper.

Key words: stratified turbulence

1. Introduction

Stable density stratification is commonly observed in oceans and the nocturnal atmosphere (Sagaut & Cambon 2008; Turner 2009; Davidson 2013; Maffioli & Davidson 2016). Both atmospheric and oceanic flow can often be turbulent; such turbulence, commonly known as 'stably stratified turbulence' (SST), is different from the classical 'Kolmogorov turbulence', which is applicable to homogeneous and isotropic hydrodynamic turbulence.

Stably stratified turbulence is quite complex, and there are many unresolved issues in this field (Lindborg 2006, 2007; Brethouwer *et al.* 2007; Davidson 2013; Rosenberg *et al.* 2015; Verma 2018). One of the important parameters here is the Froude number

$$Fr \equiv \frac{U}{NL}, \tag{1.1}$$

where U and L are the large-scale velocity and length scale, respectively, and N is the Brunt–Väisälä frequency (defined in § 2) (Davidson 2013). A related parameter is

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the Richardson number,

$$Ri \equiv \frac{N^2}{(\partial u/\partial z)^2}, \tag{1.2}$$

which is the ratio of buoyancy and flow shear. The approximation $\partial u/\partial z \sim U/L$ yields $Ri \approx Fr^{-2}$ (Rosenberg *et al.* 2015; Maffioli, Brethouwer & Lindborg 2016). The degree of turbulence is quantified by the Reynolds number, $Re \equiv UL/\nu$, where ν is the kinematic viscosity of the fluid.

Based on the above parameters, stably stratified turbulent flows can be classified into three broad regimes:

(1) $Re \gg 1$ and $Fr \gg 1$ (turbulent SST with weak buoyancy): In this regime, strong nonlinearity ($\boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{u}$) in comparison to buoyancy yields scaling similar to passive scalar turbulence. Hence, both the kinetic energy spectrum, $E_u(k)$, and potential energy spectrum, $E_b(k)$, follow the Kolmogorov spectrum and scale as $\sim k^{-5/3}$, where k denotes wavenumber.

(2) $Re \gg 1$ and $Fr \ll 1$ (turbulent SST with strong buoyancy): Here, buoyancy is much stronger than the nonlinearity. The flow is strongly anisotropic, with strong horizontal velocity compared to the vertical velocity (Vallis, Shutts & Gray 1997; Lindborg 2006, 2007; Brethouwer *et al.* 2007; Davidson 2013). The flow of the terrestrial atmosphere is strongly stratified with typical $Fr \sim 0.01$ (Waite 2011). The physics of this regime is quite complex, and it is still being debated.

(3) $Re \gg 1$ and $Fr \approx 1$ (turbulent SST with moderate buoyancy): Here, buoyancy and nonlinearity are of comparable strength. Bolgiano (1959) and Obukhov (1959) constructed a model for this regime by arguing that the buoyancy force converts kinetic energy into potential energy. They argued for a dual scaling, with transition occurring at Bolgiano wavenumber k_B. For $k < k_B$, the kinetic energy flux $\Pi_u(k)$ decreases as $\sim k^{-4/5}$, but the potential energy flux $\Pi_b(k)$ is constant. Here, $E_u(k) \sim k^{-11/5}$ and $E_b(k) \sim k^{-7/5}$. For $k > k_B$, buoyancy is expected to be weak, and hence the scaling is similar to that for a passive scalar. We denote the above model as the Bolgiano–Obukhov (BO) phenomenology.

In this paper we focus on the third regime – moderately stratified turbulence. For this, computational studies by Waite & Bartello (2004) and Kumar, Chatterjee & Verma (2014) show that the flow remains approximately isotropic. Furthermore, the direct numerical simulation results of Kimura & Herring (1996) and Kumar *et al.* (2014), the shell-model results of Kumar & Verma (2015) and the global energy balance analysis of Bhattacharjee (2015) have unequivocally shown that the kinetic energy spectrum indeed scales as $\sim k^{-11/5}$ in a wavenumber band. However, we are not aware of any numerical or experimental work that convincingly demonstrates the dual scaling for such flows.

Several researchers have reported BO scaling for turbulent thermal convection, Rayleigh–Taylor turbulence and unstably stratified flows. Note, however, that Verma, Kumar & Pandey (2017) showed that BO scaling is not applicable to such flows in three dimensions; instead, they follow Kolmogorov-like turbulence phenomenology ($E_u(k) \sim k^{-5/3}$). Yet, in two dimensions, turbulent thermal convection and Rayleigh–Taylor turbulence exhibit BO scaling, as demonstrated by Boffetta *et al.* (2012) and Boffetta & Mazzino (2017). Verma *et al.* (2017) and Verma (2018) argued that the above phenomenon is due to the inverse cascade of kinetic energy.

The flow in the deep oceans is moderately stratified with $Fr \sim 1$ (Petrolo & Woods 2019). The atmosphere of some other planets could yield a wide range of Fr; hence,

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a clear understanding of SST with moderate buoyancy is essential. The goal of this paper is to revisit turbulent SST with moderate buoyancy and critically examine the validity of dual scaling of the BO phenomenology. We start with the constancy of the total energy flux (kinetic plus potential) and demonstrate that, for large wavenumbers, the velocity field becomes weak; hence, the assumption that buoyancy becomes weak at large wavenumbers leading to $k^{-5/3}$ spectra is improbable. We observe that $E_u(k) \sim k^{-11/5}$ for $k > 1/L$, where L is the system size, with no cross-over to $k^{-5/3}$ spectra. As an aside, we recover $E_u(k) = k^{-5/3}$ for $k < 1/L$, which may be possible in systems with large aspect ratio. Thus, we provide a revision of the celebrated BO phenomenology.

The outline of the paper is as follows. The equations governing SST are introduced in § 2. The BO phenomenology is described in § 3. In § 4.1 and § 4.2, respectively, numerical solution and asymptotic analysis of the equation for the total energy flux (a fifth-order equation) are presented. We conclude in § 5.

2. Governing equations

The governing Navier–Stokes equations for stably stratified flows (density stratification in the vertical (z) direction) under the Boussinesq approximation are (Davidson 2004, 2013; Lindborg 2006; Verma 2018)

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{1}{\rho_m}\nabla\sigma - Nb\hat{z} + \nu\nabla^2\boldsymbol{u} + \boldsymbol{F}_u, \tag{2.1a}$$

$$\frac{\partial b}{\partial t} + (\boldsymbol{u}\cdot\nabla)b = Nu_z + \kappa\nabla^2 b, \tag{2.1b}$$

$$\nabla\cdot\boldsymbol{u} = 0. \tag{2.1c}$$

Here $\boldsymbol{u} = (u_x, u_y, u_z)$ and σ are, respectively, the velocity and the pressure fields; ν and κ are respectively the kinematic viscosity and diffusivity of the density fluctuation; ρ_m is the mean density; $\boldsymbol{F}_u$ is the external force (in addition to the buoyancy); and b is the density fluctuation in velocity units, which is achieved by the following transformation (Lindborg 2006; Davidson 2013; Rosenberg *et al.* 2015):

$$b = \frac{g}{N}\frac{\rho}{\rho_m}, \tag{2.2}$$

where g is the acceleration due to gravity and ρ is the density fluctuation. The quantity

$$N = \sqrt{\frac{g}{\rho_m}\left|\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}z}\right|} \tag{2.3}$$

is the Brunt–Väisälä frequency. Note that $-Nb$ is buoyancy.

It is convenient to describe the flow behaviour in Fourier space since it captures the scale-by-scale energy transfer and interactions. The following one-dimensional kinetic spectrum, $E_u(k)$, and the potential energy spectrum, $E_b(k)$, which are the sums of the respective energy of all the modes of a shell of thickness dk, are introduced:

$$E_u(k,t)\,\mathrm{d}k = \sum_{k<|\boldsymbol{k}'|\leqslant k+\mathrm{d}k} \frac{1}{2}|\boldsymbol{u}(\boldsymbol{k}',t)|^2, \tag{2.4}$$

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$$E_b(k,t)\,\mathrm{d}k = \sum_{k<|\boldsymbol{k}'|\leqslant k+\mathrm{d}k} \frac{1}{2}|b(\boldsymbol{k}',t)|^2. \tag{2.5}$$

Note that $E_u(k)$ and $E_b(k)$ are averaged over polar angles; hence they do not capture the anisotropic effects. The ring spectrum proposed by Teaca *et al.* (2009) and Nath *et al.* (2016) captures the angular-dependent spectra.

Henceforth, the explicit time dependence in E_u and E_b is suppressed for brevity. The nonlinear energy transfers across modes are quantified using energy fluxes or energy cascade rates. The kinetic (potential) energy flux, $\Pi_{u(b)}(k_0)$, for a wavenumber sphere of radius k_0 is the total kinetic (potential) energy leaving the said sphere due to nonlinear interactions. These fluxes are computed using the following formulae (Dar, Verma & Eswaran 2001; Verma 2004, 2018):

$$\Pi_u(k_0) = \sum_{|\boldsymbol{k}|>k_0} \sum_{|\boldsymbol{p}|\leqslant k_0} \mathrm{Im}[\{\boldsymbol{k}\cdot\boldsymbol{u}(\boldsymbol{q})\}\{\boldsymbol{u}(\boldsymbol{p})\cdot\boldsymbol{u}^*(\boldsymbol{k})\}], \tag{2.6a}$$

$$\Pi_b(k_0) = \sum_{|\boldsymbol{k}|>k_0} \sum_{|\boldsymbol{p}|\leqslant k_0} \mathrm{Im}[\{\boldsymbol{k}\cdot\boldsymbol{u}(\boldsymbol{q})\}\{b(\boldsymbol{p})b^*(\boldsymbol{k})\}], \tag{2.6b}$$

where $\boldsymbol{k}=\boldsymbol{p}+\boldsymbol{q}$.

The dynamical equations for modal kinetic energy ($E_u(\boldsymbol{k}) = (1/2)|\boldsymbol{u}(\boldsymbol{k})|^2$) and potential energy ($E_b(\boldsymbol{k}) = (1/2)|b(\boldsymbol{k})|^2$), respectively, can be derived from (2.1*a*) and (2.1*b*), and are as follows (Davidson 2013; Verma 2018):

$$\frac{\mathrm{d}}{\mathrm{d}t}E_u(\boldsymbol{k}) = T_u(\boldsymbol{k}) + \mathcal{F}_B(\boldsymbol{k}) + \mathcal{F}_{ext}(\boldsymbol{k}) - D_u(\boldsymbol{k}), \tag{2.7a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}E_b(\boldsymbol{k}) = T_b(\boldsymbol{k}) - \mathcal{F}_B(\boldsymbol{k}) - D_b(\boldsymbol{k}). \tag{2.7b}$$

Here $T_{u(b)}(\boldsymbol{k})$ and $D_{u(b)}(\boldsymbol{k})$ are, respectively, the nonlinear kinetic (potential) energy transfer rate and dissipation rate, while $\mathcal{F}_B$ and $\mathcal{F}_{ext}$ denote the energy feed rate by the buoyancy and external force, respectively. These quantities are defined as follows (Verma *et al.* 2017; Verma 2018):

$$T_u(\boldsymbol{k}) = \sum_{\boldsymbol{p}} \mathrm{Im}[\{\boldsymbol{k}\cdot\boldsymbol{u}(\boldsymbol{q})\}\{\boldsymbol{u}(\boldsymbol{p})\cdot\boldsymbol{u}^*(\boldsymbol{k})\}], \tag{2.8a}$$

$$T_b(\boldsymbol{k}) = \sum_{\boldsymbol{p}} \mathrm{Im}[\{\boldsymbol{k}\cdot\boldsymbol{u}(\boldsymbol{q})\}\{b(\boldsymbol{p})b^*(\boldsymbol{k})\}], \tag{2.8b}$$

$$\mathcal{F}_B(\boldsymbol{k}) = -N\mathrm{Re}[b(\boldsymbol{k})u_z^*(\boldsymbol{k})], \tag{2.8c}$$

$$\mathcal{F}_{ext}(\boldsymbol{k}) = \mathrm{Re}[\boldsymbol{F}_u(\boldsymbol{k})\cdot\boldsymbol{u}^*(\boldsymbol{k})], \tag{2.8d}$$

$$D_u(\boldsymbol{k}) = 2\nu k^2 E_u(\boldsymbol{k}), \tag{2.8e}$$

$$D_b(\boldsymbol{k}) = 2\kappa k^2 E_b(\boldsymbol{k}), \tag{2.8f}$$

where $\boldsymbol{k}=\boldsymbol{p}+\boldsymbol{q}$. The kinetic and potential energy fluxes are related to nonlinear energy transfer terms as

$$\Pi_u(\boldsymbol{k_0}) = -\sum_{|\boldsymbol{k}|\leqslant k_0} T_u(\boldsymbol{k}), \quad \Pi_b(\boldsymbol{k_0}) = -\sum_{|\boldsymbol{k}|\leqslant k_0} T_b(\boldsymbol{k}). \tag{2.9a,b}$$

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We write (2.7*a*) and (2.7*b*) for the spheres of radii k and $k + \mathrm{d}k$ and take their difference, which yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k<|\boldsymbol{k}'|\leqslant k+\mathrm{d}k} E_u(\boldsymbol{k}') = \sum_{k<|\boldsymbol{k}'|\leqslant k+\mathrm{d}k} T_u(\boldsymbol{k}') + \mathcal{F}_B(\boldsymbol{k}') + \mathcal{F}_{ext}(\boldsymbol{k}') - D_u(\boldsymbol{k}'), \tag{2.10a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k<|\boldsymbol{k}'|\leqslant k+\mathrm{d}k} E_b(\boldsymbol{k}') = \sum_{k<|\boldsymbol{k}'|\leqslant k+\mathrm{d}k} T_b(\boldsymbol{k}') - \mathcal{F}_B(\boldsymbol{k}') - D_b(\boldsymbol{k}'), \tag{2.10b}$$

where

$$\sum_{k<|\boldsymbol{k}'|\leqslant k+\mathrm{d}k} T_u(\boldsymbol{k}') = -\Pi_u(k+\mathrm{d}k) + \Pi_u(k), \tag{2.11a}$$

$$\sum_{k<|\boldsymbol{k}'|\leqslant k+\mathrm{d}k} T_b(\boldsymbol{k}') = -\Pi_b(k+\mathrm{d}k) + \Pi_b(k). \tag{2.11b}$$

Now taking the limit $\mathrm{d}k \to 0$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}E_u(k) = -\frac{\mathrm{d}}{\mathrm{d}k}\Pi_u(k) + \mathcal{F}_B(k) + \mathcal{F}_{ext}(k) - D_u(k), \tag{2.12a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}E_b(k) = -\frac{\mathrm{d}}{\mathrm{d}k}\Pi_b(k) - \mathcal{F}_B(k) - D_b(k), \tag{2.12b}$$

where

$$\mathcal{F}_B(k)\,\mathrm{d}k = -\sum_{k<|\boldsymbol{k}'|\leqslant k+\mathrm{d}k} N\mathrm{Re}[b(\boldsymbol{k}')u_z^*(\boldsymbol{k}')], \tag{2.13a}$$

$$\mathcal{F}_{ext}(k)\,\mathrm{d}k = \sum_{k<|\boldsymbol{k}'|\leqslant k+\mathrm{d}k} \mathrm{Re}[\boldsymbol{F}_u(\boldsymbol{k}')\boldsymbol{\cdot}\boldsymbol{u}^*(\boldsymbol{k}')], \tag{2.13b}$$

$$D_u(k)\,\mathrm{d}k = 2\nu\sum_{k<|\boldsymbol{k}'|\leqslant k+\mathrm{d}k} k'^2 E_u(\boldsymbol{k}'), \tag{2.13c}$$

$$D_b(k)\,\mathrm{d}k = 2\kappa\sum_{k<|\boldsymbol{k}'|\leqslant k+\mathrm{d}k} k'^2 E_b(\boldsymbol{k}'). \tag{2.13d}$$

The above energetics is illustrated in figure 1.

Let us consider a statistically steady state ($\partial/\partial t \to 0$). In the inertial range, $\mathcal{F}_{ext} = 0$, and the dissipative effects are negligible, i.e. $D_u \to 0$ and $D_b \to 0$. Hence the equations for the kinetic and potential energies simplify to

$$\frac{\mathrm{d}}{\mathrm{d}k}\Pi_u(k) = \mathcal{F}_B(k), \tag{2.14a}$$

$$\frac{\mathrm{d}}{\mathrm{d}k}\Pi_b(k) = -\mathcal{F}_B(k). \tag{2.14b}$$

The sum of (2.14*a*) and (2.14*b*) yields

$$\Pi_u(k) + \Pi_b(k) = \Pi = \text{const.} \tag{2.15}$$

Hence the total energy flux is constant in the inertial range. We will employ (2.15) in the later part of the paper.

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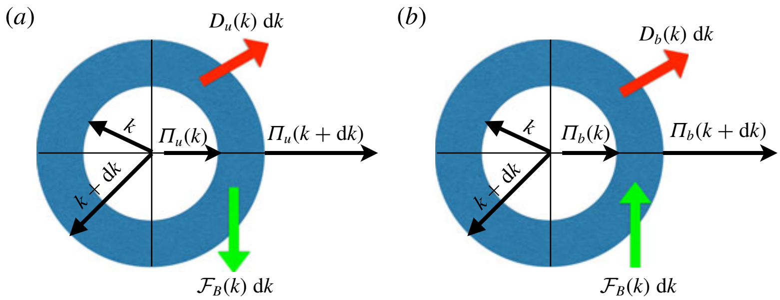

FIGURE 1. (Colour online) (*a*) The kinetic energy content of a wavenumber shell changes due to the kinetic energy flux difference $\Pi_u(k+\mathrm{d}k) - \Pi_u(k)$, energy removal rate by buoyancy $\mathcal{F}_B(k)\,\mathrm{d}k$, and viscous dissipation rate $D_u(k)\,\mathrm{d}k$. (*b*) The potential energy changes due to potential energy flux difference $\Pi_b(k+\mathrm{d}k) - \Pi_b(k)$, energy supply rate by buoyancy $\mathcal{F}_B(k)\,\mathrm{d}k$, and dissipation rate $D_b(k)\,\mathrm{d}k$.

Based on energetics arguments, it has been shown that the energy injection rate by buoyancy, $\mathcal{F}_B$, is negative. Hence $\Pi_u(k)$ decreases with k (Kumar *et al.* 2014; Verma 2018, 2019). Verma (2019) showed that, in the linear regime, gravity waves facilitate periodic exchange of kinetic and potential energies, hence $\mathcal{F}_B = 0$. Therefore, a non-dissipative gravity wave represents a neutral state. Since the system is stable, the nonlinearity makes $\mathcal{F}_B$ negative. If $\mathcal{F}_B > 0$, according to the integral form of (2.7*a*), the kinetic energy would grow in time, thus making the flow unstable. Hence, $\mathcal{F}_B < 0$. In addition, Kumar *et al.* (2014) and Verma (2019) go on to argue that $\mathcal{F}_B(k) < 0$. The above features have been verified numerically by Kumar *et al.* (2014) and Verma *et al.* (2017).

When we substitute negative $\mathcal{F}_B(k)$ in (2.14*a*) and (2.14*b*), we deduce that $\Pi_u(k)$ decreases with k, while $\Pi_b(k)$ increases with k. These features play an important role in the models of Bolgiano (1959) and Obukhov (1959).

The equations described in this section apply to all three regimes. In the following two sections we will focus on phenomenology of moderately stratified turbulence.

3. The Bolgiano–Obukhov phenomenology for moderately stably stratified turbulence

Bolgiano (1959) and Obukhov (1959) constructed a phenomenology for moderately stratified turbulence, which we refer to as BO phenomenology. In this regime, the flow is nearly isotropic. Kumar *et al.* (2014) showed that for $Fr \gtrsim 1$ the anisotropic ratio $E_\perp/2E_\parallel \approx 1$, where $E_\perp = (u_x^2 + u_y^2)/2$ and $E_\parallel = u_z^2/2$. Waite & Bartello (2004) also showed that the flow is approximately isotropic for $Fr = 1.3$, and anisotropy of stratification starts to become visible for $Fr \leqslant 0.21$. It has been conjectured that isotropy is also present in the inertial range of moderately SST.

According to the BO phenomenology, a force balance between the nonlinear term and buoyancy in (2.1*a*) yields

$$ku_k^2 = Nb_k, \tag{3.1}$$

where u_k and b_k are, respectively, the velocity and density fluctuations at wavenumber k. In addition, the BO phenomenology assumes that, in the inertial range, $\Pi_b(k) \approx$ const., and it equals the dissipation rate of the potential energy (ϵ_b):

$$\Pi_b(k) = kb_k^2 u_k = \epsilon_b. \tag{3.2}$$

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Equations (3.1) and (3.2) yield the following relations:

$$E_u(k) = \frac{u_k^2}{k} = c_1 \epsilon_b^{2/5} N^{4/5} k^{-11/5}, \tag{3.3a}$$
$$E_b(k) = \frac{b_k^2}{k} = c_2 \epsilon_b^{4/5} N^{-2/5} k^{-7/5}, \tag{3.3b}$$
$$\Pi_u(k) = k u_k^3 = c_3 \epsilon_b^{3/5} N^{6/5} k^{-4/5}, \tag{3.3c}$$
$$\Pi_b(k) = \epsilon_b. \tag{3.3d}$$

Bolgiano (1959) and Obukhov (1959) argued that the above-mentioned behaviour of the inertial range is true only for lower wavenumbers ($k < k_B$, where k_B will be defined below). For $k > k_B$ of the inertial range, the buoyancy effects are weak and hence cannot balance the inertial term (which is balanced by the pressure gradient). Hence in this region, the scaling of passive scalar (i.e. Kolmogorov) turbulence should be valid. The energy and flux relations obtained here are

$$E_u(k) = K_{Ko} \epsilon_u^{2/3} k^{-5/3}, \tag{3.4a}$$
$$E_b(k) = K_{OC} \epsilon_u^{-1/3} \epsilon_b k^{-5/3}, \tag{3.4b}$$
$$\Pi_u(k) = \epsilon_u, \tag{3.4c}$$
$$\Pi_b(k) = \epsilon_b, \tag{3.4d}$$

where ϵ_u is the viscous dissipation rate, and K_{Ko} and K_{OC} are the Kolmogorov and Obukhov–Corrsin constants. It is important to keep in mind that the viscous dissipation and thermal dissipation play a critical role in turbulence. They set up the fluxes, Π_u and Π_b, even though they are not very active in the inertial range.

The behavioural transition from one regime to another occurs near the Bolgiano wavenumber k_B, which is obtained by matching $\Pi_u(k)$ in the two regimes:

$$k_B \approx N^{3/2} \epsilon_u^{-5/4} \epsilon_b^{3/4}. \tag{3.5}$$

The nature of kinetic and potential energy fluxes, as well as dual scaling of moderately SST as predicted by Bolgiano (1959) and Obukhov (1959), are illustrated in figure 2. We also remark that $\Pi_u(k)$ decreases rapidly as $k^{-4/5}$ and then it tapers off to ϵ_u. However, $\Pi_b \approx \epsilon_b \approx \Pi$ (see (2.15)). Hence, $\epsilon_b \gg \epsilon_u$.

In addition to k_B, another important length referred to in SST is the 'Ozmidov length', which is defined as

$$L_O \equiv \sqrt{\frac{\epsilon_u}{N^3}}. \tag{3.6}$$

The corresponding wavenumber $k_O = 1/L_O$. At L_O, the time scales of gravity waves and local eddies match, i.e. $l/u_l \approx 1/N$. Using a numerical simulation, Waite & Bartello (2004, 2006) computed L_O for $Fr = 1.3$ and reported that L_O is approximately 1/31 of the system size.

For moderately stratified flows, $\Pi_u(k)$ varies with k; hence it is not obvious whether we should substitute $\epsilon_u = \Pi_u(k)$ of (3.3c), or ϵ_u of (3.4c). In any case, it is interesting to compare k_O with k_B. Using (3.5) and (3.6) we obtain

$$\frac{k_B}{k_O} = \epsilon_u^{-5/4+1/2} \epsilon_b^{3/4} \sim \left(\frac{\epsilon_b}{\epsilon_u}\right)^{3/4}. \tag{3.7}$$

Since $\epsilon_b \gg \epsilon_u$ for SST, we expect that $k_B \gg k_O$.

In the next section, we describe certain critical deficiencies of the BO phenomenology.

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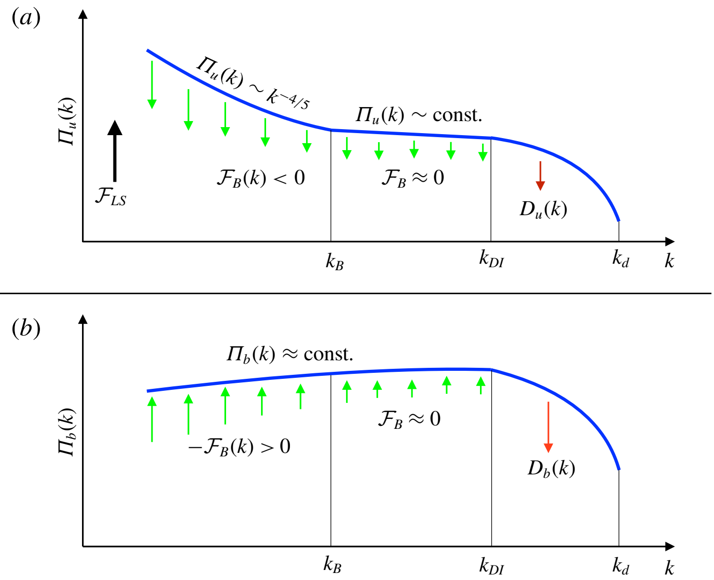

FIGURE 2. (Colour online) Schematic diagram of moderately SST according to the BO phenomenology. (*a*) Kinetic energy flux and (*b*) potential energy flux. The transition from the inertial regime to the dissipation regime occurs at wavenumber k_{DI}. Here k_d is the Kolmogorov wavenumber, and $k_d \gg k_{DI}$.

4. Revision of Bolgiano–Obukhov phenomenology for moderately stably stratified turbulence

A crucial assumption made in the BO phenomenology is that $\Pi_b(k) \approx$ const. in the inertial range (refer to (3.2)). This assumption needs a closer examination. A more rigorous approach would be to start with the constancy of total energy flux (2.15) that follows from the conservation of total energy (kinetic plus potential) in the inviscid limit.

We start with (2.15), and equate it to the total dissipation rate ϵ. That is,

$$\Pi_u(k) + \Pi_b(k) = ku_k^3 + kb_k^2 u_k = \epsilon. \tag{4.1}$$

In the above equation we eliminate b_k using (3.1), which yields the following fifth-order polynomial in u_k:

$$ku_k^3 + \frac{k^3 u_k^5}{N^2} = \epsilon. \tag{4.2}$$

There is no analytical solution for a fifth-order algebraic polynomial. Therefore, we employ numerical solution and asymptotic analysis to solve the above equation. These two results are consistent with each other.

4.1. *Numerical solution*

We numerically solve (4.2) using the ‘fsolve’ function of the SciPy library in Python, which uses Powell’s hybrid method to find zeros of nonlinear functions. We choose $N = 1.0$, and the total energy flux $\Pi = 1.0$, which is also equal to the total dissipation

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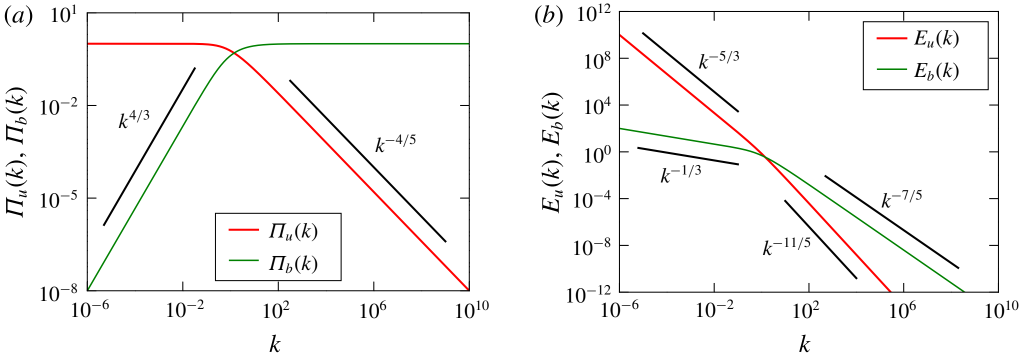

FIGURE 3. (Colour online) Fluxes and energy spectra for $N = 1.0$ and the total energy flux $\Pi = 1.0$. (*a*) Kinetic energy flux ($\Pi_u(k)$) is plotted in red and potential energy flux ($\Pi_b(k)$) is plotted in green. (*b*) Kinetic energy spectrum ($E_u(k)$) is plotted in red and potential energy spectrum ($E_b(k)$) is plotted in green. In both panels, black lines represent asymptotic behaviours in the extreme limits.

rate ϵ. We vary k from 10^{-6} to 10^{10} in logarithmic scale. These parameters can be treated as non-dimensional with the time period of a large-scale gravitational wave as the time scale, system size as the length scale, and large-scale velocity as the velocity scale. Using the numerically evaluated u_k and b_k, we evaluate $E_u(k) = u_k^2/k$, $E_b(k) = b_k^2/k$, $\Pi_u(k) = ku_k^3$ and $\Pi_b(k) = ku_k b_k^2$. The quantities are plotted in figure 3.

Figure 3 exhibits the fluxes and spectra of the kinetic and potential energies. For $1 < k < 10^{10}$, $\Pi_b \approx 1$, $\Pi_u(k) \sim k^{-4/5}$, $E_u \sim k^{-11/5}$ and $E_b \sim k^{-7/5}$, which are the predictions of the BO phenomenology for $k < k_B$. Surprisingly, there is no cross-over to $k^{-5/3}$ scaling of passive scalar turbulence. This is because $u_k \ll b_k$; hence u_k cannot induce a constant kinetic energy flux. We will show a more rigorous derivation in the next subsection.

Interestingly, for $k \ll 1$, we obtain $\Pi_u \approx 1$, $\Pi_b \sim k^{4/3}$, $E_u \sim k^{-5/3}$ and $E_b \sim k^{-1/3}$. That is, u_k dominates b_k at small k values, which leads to Kolmogorov's scaling for the velocity field. Note, however, that $k = 1$ corresponds to $1/L$. Hence, $k \ll 1$ is possible in SST when the transverse length scale is much larger than the vertical scale (L).

In the next two subsections we will perform asymptotic analysis of (4.1).

4.2. *Asymptotic analysis*

We examine the dominant balance for the two extreme limits of (4.2).

4.2.1. *Case 1: moderately stably stratified turbulence for $k \gg 1$*

In this situation, $\Pi_u \ll \Pi_b$, and hence the balance is between Π_b and ϵ:

$$\frac{k^3 u_k^5}{N^2} \approx \epsilon \quad \Longrightarrow \quad u_k \approx \epsilon^{1/5} N^{2/5} k^{-3/5}. \tag{4.3}$$

Using (3.1), b_k is found to be

$$b_k \approx \epsilon^{2/5} N^{-1/5} k^{-1/5}. \tag{4.4}$$

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Therefore, the kinetic and potential energy spectra and fluxes, as well as the energy feed by buoyancy, are given by

$$\Pi_u(k) \approx \epsilon^{3/5} N^{6/5} k^{-4/5}, \tag{4.5a}$$

$$\Pi_b(k) \approx \epsilon, \tag{4.5b}$$

$$\mathcal{F}_B(k) = \frac{\partial}{\partial k} \Pi_u(k) \approx -\frac{4}{5} \epsilon^{3/5} N^{6/5} k^{-9/5}, \tag{4.5c}$$

$$E_u(k) \approx \epsilon^{2/5} N^{4/5} k^{-11/5}, \tag{4.5d}$$

$$E_b(k) \approx \epsilon^{4/5} N^{-2/5} k^{-7/5}. \tag{4.5e}$$

Note that $u_k \sim k^{-3/5}$ decreases faster than $b_k \sim k^{-1/5}$. Therefore, buoyancy is strong enough so as to yield $E_u(k) \sim k^{-11/5}$ for the whole of the inertial range, without a transition to the $E_u(k) \sim k^{-5/3}$ regime. Note that the dissipation range starts after the inertial range.

A more quantitative condition for the absence of the second regime (k_B to k_{DI} of figure 2) is obtained as follows. Clearly, the Bolgiano wavenumber should be much smaller than the Kolmogorov wavenumber, k_d, which leads to

$$N^6 \epsilon_b^3 \epsilon_u^{-5} \ll \epsilon_u \nu^{-3}, \tag{4.6}$$

or

$$N^2 \nu \ll \epsilon_u^2 \epsilon_b^{-1}. \tag{4.7}$$

In the above equation, substitution of the following expressions for the Richardson number and thermal dissipation based on the root mean square quantities (Verma 2018),

$$Ri = \frac{N b_{rms} L}{U^2}, \tag{4.8}$$

$$\epsilon_b = \frac{U b_{rms}^2}{L} \tag{4.9}$$

yields

$$\epsilon_u \gg \frac{Ri}{\sqrt{Re}} \frac{U^3}{L}, \tag{4.10}$$

where L is the length scale of the system. Using $Ri \approx Fr^{-2}$, we obtain

$$Re_b = Re Fr^2 \gg \frac{U^3/L}{\epsilon_u}, \tag{4.11}$$

where Re_b is the buoyancy Reynolds number. As an example, for $Fr = 1.58$, Maffioli *et al.* (2016) obtained $Re_b = 10\,430$. Interestingly, (4.11) is similar to that obtained by Brethouwer *et al.* (2007) for strongly stratified turbulence.

Since $\epsilon_u \ll \epsilon_b$, the above condition may be very difficult to achieve in numerical simulations. If we assume that $\epsilon_u = 10^{-3} \epsilon_b \approx 10^{-3} U^3/L$, for $Fr = 1$, equation (4.11) predicts that $Re \gg 10^3$. Such a flow would be difficult to simulate. Therefore, we claim that the second regime of BO scaling is very difficult to find in numerical simulations. It would be interesting to attempt to find this regime in a shell model (Kumar & Verma 2015) or in some experiment.

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4.2.2. *Case 2: moderately stably stratified turbulence for lower wavenumbers* ($k \ll 1$)

Equations (4.3)–(4.4) indicate that $u_k \approx b_k$ near $k = 1$. For $k \ll 1$, $\Pi_u \gg \Pi_b$, implying that the dominant balance has to be between Π_u and ϵ:

$$ku_k^3 \approx \epsilon \quad \Longrightarrow \quad u_k \approx \epsilon^{1/3} k^{-1/3}. \tag{4.12}$$

Using (3.1), b_k is found to be

$$b_k \approx \epsilon^{2/3} N^{-1} k^{1/3}. \tag{4.13}$$

With the above u_k and b_k, the evaluated energy spectra and fluxes, as well as the energy feed by buoyancy in this situation, are given by

$$\Pi_u(k) \approx \epsilon, \tag{4.14a}$$
$$\Pi_b(k) \approx \epsilon^{5/3} N^{-2} k^{4/3}, \tag{4.14b}$$
$$\mathcal{F}_B(k) = -\frac{\partial}{\partial k}\Pi_b(k) \approx -\frac{4}{3}\epsilon^{5/3} N^{-2} k^{1/3} \tag{4.14c}$$
$$E_u(k) \approx \epsilon^{2/5} k^{-5/3}, \tag{4.14d}$$
$$E_b(k) \approx \epsilon^{4/3} N^{-2} k^{-1/3}. \tag{4.14e}$$

However, it is not certain whether the above scaling can be observed in realistic systems. The range $k \ll 1$ is possible in a large-aspect-ratio box, but such systems could exhibit two-dimensional or quasi-two-dimensional turbulence for which (4.1) is not valid. Hence this prediction needs to be tested thoroughly in future. Schematic diagrams exhibiting kinetic and potential energy fluxes based on the revised BO phenomenology are shown in figure 4.

5. Conclusions

In this paper, we revisit the celebrated Bolgiano–Obukhov (BO) phenomenology for SST under moderate stratification. The BO phenomenology predicts a dual scaling for the energy spectra: $E_u(k) \sim k^{-11/5}$ for $k < k_B$ and $E_u(k) \sim k^{-5/3}$ for $k > k_B$, where k_B is the Bolgiano wavenumber. The potential energy varies as $\sim k^{-7/5}$ and $\sim k^{-5/3}$, respectively, in the two regimes. The transition to $k^{-5/3}$ scaling is based on the argument that the energy supply rate from buoyancy becomes negligible when k is large, thus making density a passive scalar (such passive scalar behaviour of density is observed in weakly stratified turbulence).

In the present paper, we start with the constancy of the total energy flux, which yields a fifth-order algebraic equation for u_k. Numerical solution of the above equation yields $E_u(k) \sim k^{-11/5}$ and $\Pi_u(k) \sim k^{-4/5}$, with no transition to the Kolmogorov-like scaling for larger wavenumbers. The reason behind the absence of the second scaling is that u_k is too weak at large wavenumbers to be able to start a constant energy cascade. The above scaling is also substantiated using asymptotic analysis.

In addition, we also derive the quantitative condition for obtaining the Kolmogorov scaling; it is given by $k_B \ll k_d$, where k_d is Kolmogorov wavenumber. This condition yields $\epsilon_u \gg (Ri/\sqrt{Re})(U^3/d)$, which may be difficult to satisfy in numerical simulations considering the fact that $\epsilon_u \ll \epsilon_b$. However, it may be possible that such an extreme condition for observing the second regime of BO scaling could be satisfied in some shell models of SST.

In conclusion, we believe that our revised scaling of the BO formalism for moderately stable stratification will have important consequences in the modelling of buoyancy-driven flows.

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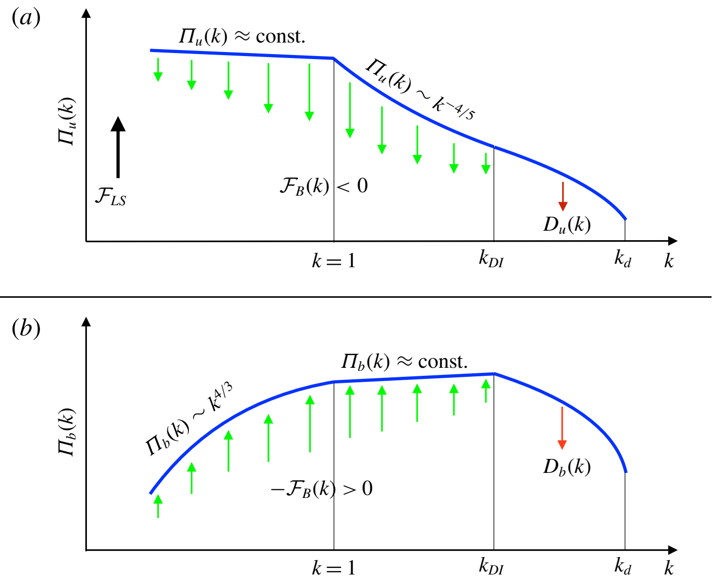

FIGURE 4. (Colour online) Schematic diagram of moderately SST according to the revised BO phenomenology, which is expected in numerical simulations. (*a*) Kinetic energy flux and (*b*) potential energy flux. Energy feed rate by buoyancy ($\mathcal{F}_B(k)$) is shown by green arrows. With $k \lesssim 1$ and $\Pi_u(k) \approx$ const., $\Pi_b(k)$ and $\mathcal{F}_B(k)$ increase with k as $\sim k^{4/3}$ and $\sim k^{1/3}$, respectively. With $k \gtrsim 1$ and $\Pi_b(k) \approx$ const., $\Pi_u(k)$ and $\mathcal{F}_B(k)$ decrease with k as $\sim k^{-4/5}$ and $\sim k^{-9/5}$, respectively.

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