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A SOLUTION OF A PROBLEM OF PLOTKIN AND VOVSI AND AN APPLICATION TO VARIETIES OF GROUPS

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Dedicated to Mike (M. F.) Newman on the occasion of his 65th birthday

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Abstract

Let K be an arbitrary field of characteristic 2, F a free group of countably infinite rank. We construct a finitely generated fully invariant subgroup U in F such that the relatively free group F/U satisfies the maximal condition on fully invariant subgroups but the group algebra $K(F/U)$ does not satisfy the maximal condition on fully invariant ideals. This solves a problem posed by Plotkin and Vovsi. Using the developed techniques we also construct the first example of a non-finitely based (nilpotent of class 2)-by-(nilpotent of class 2) variety whose Abelian-by-(nilpotent of class at most 2) groups form a hereditarily finitely based subvariety.

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1. Introduction

1. Let F be a free group. *A relatively free group* is a group of the form F/V, where V is a fully invariant subgroup (that is a subgroup closed under all endomorphisms of F). In particular, F itself is relatively free. A subgroup in a relatively free group F/V is *verbal* if and only if it is fully invariant (if G is not relatively free then it may contain fully invariant subgroups which are not verbal; see [6] for a definition of

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'verbal' in the general case). A relatively free group G is called *verbally Noetherian* if it satisfies the maximal condition on verbal subgroups (equivalently, if every verbal subgroup in G is finitely generated as a verbal subgroup).

Let K be an associative and commutative ring with an identity element, F/V a relatively free group, $K(F/V)$ the group algebra of F/V over K. An ideal in $K(F/V)$ is *verbal* if and only if it is fully invariant, that is closed under all endomorphisms of $K(F/V)$ induced by the endomorphisms of F/V (if G is not relatively free then verbal ideals are fully invariant in $K(G)$ but the converse, in general, does not hold). For terminology and basic facts related to identities and varieties of group representations we refer to Plotkin and Vovsi [8] and Vovsi [12]. The group algebra $K(G)$ of a relatively free group G is called *verbally Noetherian* if $K(G)$ satisfies the maximal condition on verbal ideals (equivalently, if every verbal ideal in $K(G)$ is finitely generated as a verbal ideal).

Clearly, if a relatively free group F/V is not verbally Noetherian then so is the group algebra $K(F/V)$ for every K (if N is a non-finitely generated verbal subgroup in F/V then the ideal generated by the set $(N-1)$ is a non-finitely generated verbal ideal in $K(F/V)$). No other ways to get examples of non-(verbally Noetherian) group algebras of relatively free groups of countably infinite rank over a Noetherian ring were known. The following problem is equivalent to the one posed by Plotkin and Vovsi (see [8, Problem 4.2.8]).

Does there exist a verbal subgroup U in a free group F of countably infinite rank such that U is finitely generated (as a verbal subgroup) and satisfies the following conditions:

(i) *F/U is verbally Noetherian;*
(ii) *over some field K the group algebra $K(F/U)$ is not verbally Noetherian?*

We resolve this by proving the following theorem. Let $(x, y) = xy - yx$, $(x, y, z) = ((x, y), z) = xyz - yxz - zxy + zyx$ and let $a^b = b^{-1}ab$. Define $a^{(x,y)} = a^{xy}a^{-yx}$ and $a^{(x,y,z)} = a^{xyz}a^{-yxz}a^{-zxy}a^{zyx}$. Let $a^{uv} = (a^u)^v$.

THEOREM 1. *Let K be an arbitrary field of characteristic 2, F the free group of countably infinite rank on free generators $x_1, x_2, \ldots$, U the verbal subgroup of F generated (as a verbal subgroup) by the elements*

$$(1) \qquad [[x_1, x_2, x_3], [x_4, x_5, x_6]],$$

$$(2) \qquad [x_1, x_2, x_3]^{(x_4,x_5,x_6)(x_7,x_8)}.$$

Then the group F/U is verbally Noetherian but the group algebra $K(F/U)$ is not verbally Noetherian.

REMARK. In fact, some extra calculations show that Theorem 1 remains valid if one replaces the element (2) with $[x_1, x_2, x_3]^{(x_4,x_5,x_6)}$.

2. The techniques developed in order to prove Theorem 1 can be applied to varieties of groups. Let $x_1, x_2, \dots$ be free generators of the free group F. For any $v = v(x_1, \dots, x_n) \in F$, $v \equiv 1$ is said to be an *identity* (or a *law* or an *identical relation*) of a group G if $v(g_1, \dots, g_n) = 1$ for all $g_1, \dots, g_n \in G$. The class of all groups satisfying a given set of identities is called a *variety* of groups. We refer to Neumann [6] for terminology and basic facts related to identities and varieties of groups. A variety of groups **V** is called *finitely based* if **V** can be defined by a finite set of identities. A group variety **V** is called *Specht* (or *hereditarily finitely based*) if all subvarieties of **V** including **V** itself are finitely based (equivalently: if each group in **V** has a finite basis for its identities).

Many varieties of groups are known to be Specht; in particular, this applies to the variety $\mathbf{N}_c\mathbf{A}$ of all (nilpotent of class at most c)-by-Abelian groups for each c (Cohen [2] for $c = 1$, Bryant and Newman [1] for $c = 2$, Krasil'nikov [5] for arbitrary c) and each variety var (G) generated by a finite group G (Oates and Powell [10]). On the other hand, the variety $\mathbf{N}_2\mathbf{N}_2$ of all (nilpotent of class at most 2)-by-(nilpotent of class at most 2) groups is known to be non-Specht (Vaughan-Lee [11]) as well as the variety $Z\mathbf{A}\mathbf{N}_2$ of all centre-by-Abelian-by-(nilpotent of class at most 2) groups (Gupta and Krasil'nikov [3]).

A variety **V** is called *just non-Specht* or *just non-finitely based* if **V** is non-Specht but all proper subvarieties of **V** are Specht (equivalently, if **V** is non-finitely based but all proper subvarieties of **V** are finitely based). It follows easily from Zorn's lemma that each non-Specht variety contains a just non-Specht subvariety so just non-Specht varieties of groups 'form the border' between Specht and non-Specht varieties. It is known that there are infinitely many just non-Specht varieties of groups (Newman [7]) but no examples of such varieties are as yet known. The following theorem gives the first example of a non-finitely based subvariety **V** of the variety $\mathbf{N}_2\mathbf{N}_2$ whose intersection with $\mathbf{A}\mathbf{N}_2$ is Specht. The variety **V** comes closest to being just non-Specht. We hope that it could give an approach to construct a just non-Specht variety of groups (a problem which remains open).

Recall that $a^{(x,y)} = a^{xy}a^{-yx}$ and $a^{(x,y,z)} = a^{xyz}a^{-yxz}a^{-zxy}a^{zyx}$. Let $a^{u_1 \cdots u_{k-1} u_k} = (a^{u_1 \cdots u_{k-1}})^{u_k}$ for all $k > 1$.

THEOREM 2. *Let* **V** *be the variety of groups defined by the identities*

$$(3) \qquad [[x_1, x_2, x_3], [x_4, x_5, x_6], [x_7, x_8]] \equiv 1$$

and

$$(4) \qquad [x_1, x_2, x_3]^{(x_4,x_5,x_6)(x_7,x_8,x_9)(x_{10},x_{11})(x_{12},x_{13})} \equiv 1.$$

Then the variety $\mathbf{V}$ *is not Specht but the intersection variety* $\mathbf{V} \cap \mathbf{AN}_2$ (*which is defined by the identities* $[[x_1, x_2, x_3], [x_4, x_5, x_6]] \equiv 1$ *and* (4)) *is Specht.*

REMARK. Theorem 2, in fact, remains valid if one replaces the identity (4) with the identity $[x_1, x_2, x_3]^{(x_4,x_5,x_6)(x_7,x_8,x_9)} \equiv 1$. The proof remains valid as well although some additional calculations are needed.

3. Let k be a positive integer, U_k the verbal subgroup of the free group F of countably infinite rank generated (as a verbal subgroup of F) by the elements

$$[[x_1, x_2, x_3], [x_4, x_5, x_6]], \qquad [x_1, x_2, x_3]^{(x_4,x_5,x_6)(x_7,x_8,x_9)\cdots(x_{3k-2},x_{3k-1},x_{3k})},$$

$\mathbf{U}_k$ the variety of groups corresponding to the verbal subgroup U_k so that $\mathbf{U}_k$ is defined by the identities

$$[[x_1, x_2, x_3], [x_4, x_5, x_6]] \equiv 1, \qquad [x_1, x_2, x_3]^{(x_4,x_5,x_6)(x_7,x_8,x_9)\cdots(x_{3k-2},x_{3k-1},x_{3k})} \equiv 1.$$

To prove that the variety $\mathbf{V} \cap \mathbf{AN}_2$ is Specht and the relatively free group F/U defined in Theorem 1 is verbally Noetherian we need the following.

PROPOSITION 1. *For every positive integer* k *the relatively free group* F/U_k *is verbally Noetherian.*

Since $\mathbf{V} \cap \mathbf{AN}_2 \subset \mathbf{U}_5$ and $U_k \subset U$ for all $k > 2$, Proposition 1 implies that the variety $\mathbf{V} \cap \mathbf{AN}_2$ is Specht and the relatively free group F/U is verbally Noetherian.

Let $\mathbb{N}$ be the set of all positive integers and let Φ be the set of all functions $\phi : \mathbb{N} \to \mathbb{N}$ such that $a\phi < b\phi$ when $a < b$. We also write Φ for the corresponding sets of endomorphisms of F (such that $x_i\phi = x_{i\phi}$ for all i) and of F/U_k. A subgroup L in F (in F/U_k) is called a Φ-*subgroup* if L is closed under all endomorphisms $\phi \in \Phi$.

In fact, rather than Proposition 1 we shall prove the following stronger assertion.

PROPOSITION 2. *For every positive integer* k *the relatively free group* F/U_k *satisfies the maximal condition on normal* Φ-*subgroups.*

2. Proof of Theorem 1

We write $\mathbb{Z}$ for the set of integers and $\mathbb{N}$ for the set of all positive integers. Since F/U is verbally Noetherian by Proposition 1, to prove Theorem 1 it suffices to check that $K(F/U)$ is not verbally Noetherian. Let $y_1, y_2, \ldots$ be free generators of the relatively free group F/U. For every $m \in \mathbb{N}$, define $v_m \in K(F/U)$ by

$$v_m = \big([y_1, y_2, y_3] - 1\big)\big([y_4, y_5] - 1\big)\cdots\big([y_{2m+2}, y_{2m+3}] - 1\big)\big([y_1, y_2, y_3] - 1\big).$$

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Let I be the verbal ideal in $K(F/U)$ generated by the elements v_m ($m \in \mathbb{N}$). Let, for each k, I_k denote the verbal ideal generated by all v_m ($m \neq k$). Using a construction from [3] we shall prove that, for each k, the element v_k is not contained in I_k and so I is not finitely generated as a verbal ideal.

In [3, Theorem 2′] for each $k \in \mathbb{N}$ there were constructed an algebra $\mathbf{R}_k$ over a field K of characteristic 2 and a subgroup $\mathbf{H}_k$ of the group of units $U(\mathbf{R}_k)$ which satisfy, in particular, the following conditions:

(i) $v_m(h_1, h_2, \ldots, h_{2m+3}) = 0$ for all $h_i \in \mathbf{H}_k$, $m \neq k$;

(ii) $v_k(h_1, h_2, \ldots h_{2k+3}) \neq 0$ for some $h_1, h_2, \ldots, h_{2k+3} \in \mathbf{H}_k$.

To check that $v_k \notin I_k$ it suffices to prove the following lemma.

LEMMA 1. *For each $k \in \mathbb{N}$ the group $\mathbf{H}_k$ satisfies the identities*

$$[[x_1, x_2, x_3], [x_4, x_5, x_6]] \equiv 1, \tag{5}$$

$$[x_1, x_2, x_3]^{(x_4, x_5, x_6)(x_7, x_8)} \equiv 1. \tag{6}$$

Indeed, let $h_1, \ldots, h_{2k+3}$ be elements of $\mathbf{H}_k$ such that $v_k(h_1, \ldots, h_{2k+3}) \neq 0$ and let χ be the map of the set $\{y_i \mid i \in \mathbb{N}\}$ into $\mathbf{H}_k$ such that $y_i\chi = h_i$ for $i = 1, \ldots, 2k+3$, $y_i\chi = 1$ for $i > 2k+3$. By Lemma 1, χ can be extended to a homomorphism of F/U into $\mathbf{H}_k$ which, in its turn, can be extended to a homomorphism (also denoted by χ) of the algebra $K(F/U)$ into $\mathbf{R}_k$. Then, by (i), $I_k\chi = 0$ but, by (ii), $v_k\chi \neq 0$ so $v_k \notin I_k$ as required.

PROOF (of Lemma 1). Let K be an arbitrary field of characteristic 2, k an arbitrary fixed positive integer. The algebra $\mathbf{R}_k$ and the group $\mathbf{H}_k$ were constructed in [3, Theorem 2′] in the following way.

Let G be a group given by the presentation

$$G = \langle x_1, x_2, \ldots \mid x_i^2,\ [x_{i_1}, x_{i_2}, x_{i_3}],\ i, i_1, i_2, i_3 \in \mathbb{N} \rangle \tag{7}$$

and let $\overline{G} = G/G'$. For each $g \in G$ put $\overline{g} = gG' \in \overline{G}$. Note that for each $c \in G'$ we have $c^2 = 1$ (because $[x_i, x_j]^2 = [x_i^2, x_j] = 1$ for every $i, j \in \mathbb{N}$).

Let T denote the ideal of the group algebra $K(G)$ generated by all elements

(8)
$$([g_1, g_2] + 1)([g_3, g_4] + 1) + ([g_1, g_4] + 1)([g_2, g_3] + 1), \quad g_1, g_2, g_3, g_4 \in G.$$

Denote $S = K(G)/T$. For each $f \in KG$ put $\hat{f} = (f + T) \in S$. Let M_k be the left $K(\overline{G})$-submodule of $K(\overline{G}) \otimes_K S$ generated by all elements

$$1 \otimes \hat{g} \quad (g \notin G'), \quad 1 \otimes 1 \quad \text{and} \quad 1 \otimes ([\hat{g}_1, \hat{g}_2] + 1) \cdots ([\hat{g}_{2m-1}, \hat{g}_{2m}] + 1)$$
$$(m \neq k,\ g_1, \ldots g_{2m} \in G).$$

The algebra $\mathbf{R}_k$ is the algebra of matrices

$$\mathbf{R}_k = \begin{pmatrix} K(\overline{G}) & K(\overline{G}) \otimes_K S & K(\overline{G}) \otimes_K S/M_k \\ 0 & S & S \\ 0 & 0 & K \end{pmatrix}$$

which is the quotient algebra of the algebra

$$\begin{pmatrix} K(\overline{G}) & K(\overline{G}) \otimes_K S & K(\overline{G}) \otimes_K S \\ 0 & S & S \\ 0 & 0 & K \end{pmatrix}$$

modulo the ideal

$$\begin{pmatrix} 0 & 0 & M_k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The group $\mathbf{H}_k$ is the subgroup of the group of units of $\mathbf{R}_k$ generated by the matrix C and all matrices $\mathbf{g}$ ($g \in G$), where

$$C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} \overline{g} & 0 & 0 \\ 0 & \hat{g} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so that $\mathbf{H}_k = \mathbf{BG}$ is the semidirect product of $\mathbf{B} = \operatorname{sgp}\{C^{\mathbf{g}} \mid \mathbf{g} \in \mathbf{G}\}$ with $\mathbf{G} = \operatorname{sgp}\{\mathbf{g} \mid g \in G\}$, where

$$C^{\mathbf{g}} = \begin{pmatrix} 1 & \overline{g}^{-1} \otimes \hat{g} & 0 \\ 0 & 1 & \hat{g}^{-1} \\ 0 & 0 & 1 \end{pmatrix}. \tag{9}$$

Now we are in position to prove that $\mathbf{H}_k$ satisfies the identity (5). Let $\mathbf{h}_1, \dots, \mathbf{h}_6 \in \mathbf{H}_k$, $C_1 = [\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3]$, $C_2 = [\mathbf{h}_4, \mathbf{h}_5, \mathbf{h}_6]$. Since $C_1, C_2 \in \mathbf{B}$, they are products of elements of the form (9) so for some $f_i, g_j \in G$

$$C_1 = \begin{pmatrix} 1 & \sum_i \overline{f}_i^{-1} \otimes \hat{f}_i & * \\ 0 & 1 & \sum_i \hat{f}_i^{-1} \\ 0 & 0 & 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & \sum_j \overline{g}_j^{-1} \otimes \hat{g}_j & * \\ 0 & 1 & \sum_j \hat{g}_j^{-1} \\ 0 & 0 & 1 \end{pmatrix}$$

(entries denoted by $*$ are not important for the argument). Therefore,

$$[C_1, C_2] = \begin{pmatrix} 1 & 0 & P \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$P = \left(\sum_i \overline{f}_i^{-1} \otimes \hat{f}_i\right)\left(\sum_j \hat{g}_j^{-1}\right) - \left(\sum_i \overline{g}_j^{-1} \otimes \hat{g}_j\right)\left(\sum_i \hat{f}_i^{-1}\right)$$
$$= \sum_{i,j}\left(\overline{f}_i^{-1} \otimes \hat{f}_i\hat{g}_j^{-1} - \overline{g}_j^{-1} \otimes \hat{g}_j\hat{f}_i^{-1}\right).$$

Note that if $f_i g_j^{-1} \notin G'$ for some i, j then $g_j f_i^{-1} \notin G'$ and

$$\overline{f}_i^{-1} \otimes \hat{f}_i\hat{g}_j^{-1}, \overline{g}_j^{-1} \otimes \hat{g}_j\hat{f}_i^{-1} \in M_k.$$

On the other hand, if $f_i g_j^{-1} = c \in G'$ for some i, j then $g_j f_i^{-1} = c^{-1} = c \in G'$ and $\overline{g}_j = \overline{f}_i$. So

$$\overline{f}_i^{-1} \otimes \hat{f}_i\hat{g}_j^{-1} - \overline{g}_j^{-1} \otimes \hat{g}_j\hat{f}_i^{-1} = \overline{f}_i^{-1} \otimes \hat{c} - \overline{f}_i^{-1} \otimes \hat{c} = 0.$$

Thus, $P = 0 \pmod{M_k}$ and $[[\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3], [\mathbf{h}_4, \mathbf{h}_5, \mathbf{h}_6]] = 1$ for all $\mathbf{h}_1, \dots, \mathbf{h}_6 \in \mathbf{H}_k$, that is $\mathbf{H}_k$ satisfies the identity (5).

Let us check that $\mathbf{H}_k$ satisfies the identity (6) as well. It was checked in [3, page 361] that for every $g_1, g_2, g_3 \in G$

$$(\hat{g}_1, \hat{g}_2, \hat{g}_3) = 0. \tag{10}$$

Let $\mathbf{h}_1, \dots \mathbf{h}_8 \in \mathbf{H}_k$. Let $D = [\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3]$, $\mathbf{h}_i = \mathbf{g}_i\mathbf{b}_i$, where $\mathbf{g}_i \in \mathbf{G}$, $\mathbf{b}_i \in \mathbf{B}$ $(i = 4, \dots, 8)$. Then

$$D = \begin{pmatrix} 1 & a & * \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$D^{(\mathbf{h}_4, \mathbf{h}_5, \mathbf{h}_6)} = D^{\mathbf{g}_4\mathbf{g}_5\mathbf{g}_6 - \mathbf{g}_5\mathbf{g}_4\mathbf{g}_6 - \mathbf{g}_6\mathbf{g}_4\mathbf{g}_5 + \mathbf{g}_6\mathbf{g}_5\mathbf{g}_4} = \begin{pmatrix} 1 & P_{12} & * \\ 0 & 1 & P_{23} \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$P_{12} = \overline{g_4g_5g_6}^{-1}a\hat{g}_4\hat{g}_5\hat{g}_6 - \overline{g_5g_4g_6}^{-1}a\hat{g}_5\hat{g}_4\hat{g}_6 - \overline{g_6g_4g_5}^{-1}a\hat{g}_6\hat{g}_4\hat{g}_5 + \overline{g_6g_5g_4}^{-1}a\hat{g}_6\hat{g}_5\hat{g}_4$$
$$= \overline{g}_4^{-1}\overline{g}_5^{-1}\overline{g}_6^{-1}a(\hat{g}_4\hat{g}_5\hat{g}_6 - \hat{g}_5\hat{g}_4\hat{g}_6 - \hat{g}_6\hat{g}_4\hat{g}_5 + \hat{g}_6\hat{g}_5\hat{g}_4)$$
$$= \overline{g}_4^{-1}\overline{g}_5^{-1}\overline{g}_6^{-1}a(\hat{g}_4, \hat{g}_5, \hat{g}_6)$$

and

$$\begin{aligned}P_{23} &= (\hat{g}_4\hat{g}_5\hat{g}_6)^{-1}b - (\hat{g}_5\hat{g}_4\hat{g}_6)^{-1}b - (\hat{g}_6\hat{g}_4\hat{g}_5)^{-1}b + (\hat{g}_6\hat{g}_5\hat{g}_4)^{-1}b \\ &= (\hat{g}_6^{-1}\hat{g}_5^{-1}\hat{g}_4^{-1} - \hat{g}_6^{-1}\hat{g}_4^{-1}\hat{g}_5^{-1} - \hat{g}_5^{-1}\hat{g}_4^{-1}\hat{g}_6^{-1} + \hat{g}_4^{-1}\hat{g}_5^{-1}\hat{g}_6^{-1})b \\ &= (\hat{g}_4^{-1}, \hat{g}_5^{-1}, \hat{g}_6^{-1})b.\end{aligned}$$

So, by (10), $P_{12} = P_{23} = 0$ and $[\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3]^{(\mathbf{h}_4, \mathbf{h}_5, \mathbf{h}_6)}$ is a matrix of the form

$$\begin{pmatrix} 1 & 0 & P \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $P \in K(\overline{G}) \otimes_K S/M_k$. It remains to note that

$$\begin{pmatrix} 1 & 0 & P \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\mathbf{h}_7\mathbf{h}_8} = \begin{pmatrix} 1 & 0 & \overline{g}_7\overline{g}_8 P \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & P \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\mathbf{h}_8\mathbf{h}_7},$$

that is $[\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3]^{(\mathbf{h}_4, \mathbf{h}_5, \mathbf{h}_6)\mathbf{h}_7\mathbf{h}_8} = [\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3]^{(\mathbf{h}_4, \mathbf{h}_5, \mathbf{h}_6)\mathbf{h}_8\mathbf{h}_7}$, so

$$[\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3]^{(\mathbf{h}_4, \mathbf{h}_5, \mathbf{h}_6)(\mathbf{h}_7, \mathbf{h}_8)} = \left([\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3]^{(\mathbf{h}_4, \mathbf{h}_5, \mathbf{h}_6)}\right)^{(\mathbf{h}_7\mathbf{h}_8 - \mathbf{h}_8\mathbf{h}_7)} = 1.$$

Thus, the identity (6) is satisfied in $\mathbf{H}_k$. This completes the proof of Lemma 1 and the proof of Theorem 1 (provided that Proposition 1 is proved). □

REMARK. It is possible to check that $\mathbf{H}_k$ satisfies the stronger identity

$$[x_1, x_2, x_3]^{(x_4, x_5, x_6)} \equiv 1$$

as well.

3. Proof of Theorem 2

Since the intersection variety $\mathbf{V} \cap \mathbf{AN}_2$ is Specht by Proposition 1, to prove Theorem 2 it suffices to prove that $\mathbf{V}$ is not Specht. We shall show this by proving the following.

THEOREM 3. *Let*

$$w_k = w_k(x_1, \dots, x_{2k+3}) = \left[[x_1, x_2, x_3]^{[x_4, x_5] \cdots [x_{2k+2}, x_{2k+3}]}, [x_1, x_2, x_3]\right],$$

$(k \in \mathbb{N})$, *$\mathbf{W}$ the subvariety of $\mathbf{V}$ defined by the system of identities $\{w_k \equiv 1 \mid k \in \mathbb{N}\}$. Then $\mathbf{W}$ is non-finitely based.*

In order to prove Theorem 3 we shall construct, for each $k \in \mathbb{N}$, a group $\mathscr{H}_k \in \mathbf{V}$ which satisfies the identity $w_k \equiv 1$ (so $\mathscr{H}_k$ satisfies all identities $w_l \equiv 1$ for $l \leq k$) but does not satisfy the identity $w_{k+1} \equiv 1$.

Let $\mathbf{F}_2 = \mathbb{Z}/2\mathbb{Z}$, k an arbitrary but fixed positive integer. Let G be a group given by the presentation (7), T the ideal of the group algebra $\mathbf{F}_2 G$ generated by all elements (8). Let $\tilde{T}$ denote the ideal of $\mathbf{F}_2 G$ generated by all elements

$$([g_1, g_2] + 1)([g_3, g_4] + 1) \quad (g_1, g_2, g_3, g_4 \in G)$$

so that $T \subseteq \tilde{T}$. Denote $S = \mathbf{F}_2 G/T$, $\tilde{S} = \mathbf{F}_2 G/\tilde{T}$. For each $f \in \mathbf{F}_2 G$ put $\hat{f} = (f + T) \in S$, $\tilde{f} = (f + \tilde{T}) \in \tilde{S}$. Let N_k be the left $\tilde{S}$-submodule of $\tilde{S} \otimes_{\mathbf{F}_2} S$ generated by all elements

$$1 \otimes \hat{g} \quad (g \notin G'), \qquad 1 \otimes 1 \quad \text{and} \quad 1 \otimes ([\hat{g}_1, \hat{g}_2] + 1) \cdots ([\hat{g}_{2m-1}, \hat{g}_{2m}] + 1)$$
$$(m \leq k, g_1, \dots, g_{2m} \in G).$$

Define $\mathscr{R}_k$ to be the algebra of matrices

$$\mathscr{R}_k = \begin{pmatrix} \tilde{S} & \tilde{S} \otimes_{\mathbf{F}_2} S & (\tilde{S} \otimes_{\mathbf{F}_2} S)/N_k \\ 0 & S & S \\ 0 & 0 & \mathbf{F}_2 \end{pmatrix}$$

which is the quotient algebra of the algebra

$$\mathscr{R} = \begin{pmatrix} \tilde{S} & \tilde{S} \otimes_{\mathbf{F}_2} S & \tilde{S} \otimes_{\mathbf{F}_2} S \\ 0 & S & S \\ 0 & 0 & \mathbf{F}_2 \end{pmatrix}$$

modulo the ideal

$$\begin{pmatrix} 0 & 0 & N_k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $\mathscr{H}_k$ be the subgroup of the group $U(\mathscr{R}_k)$ of all units of $\mathscr{R}_k$ generated by the matrix C and all matrices $\mathbf{g}$ ($g \in G$), where

$$C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} \tilde{g} & 0 & 0 \\ 0 & \hat{g} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\mathscr{H}_k = B\mathbf{G}$ is the semidirect product of $B = \operatorname{sgp}\{C^{\mathbf{g}} \mid \mathbf{g} \in \mathbf{G}\}$ with $\mathbf{G} = \operatorname{sgp}\{\mathbf{g} \mid g \in G\}$, where

$$C^{\mathbf{g}} = \begin{pmatrix} 1 & \tilde{g}^{-1} \otimes \hat{g} & 0 \\ 0 & 1 & \hat{g}^{-1} \\ 0 & 0 & 1 \end{pmatrix}. \tag{11}$$

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The following lemma can be proved similarly to corresponding assertions in the proof of Theorem 1.

LEMMA 2. *For every positive integer k we have* $\mathscr{H}_k \in \mathbf{V}$.

To complete the proof of Theorem 3 we need the following result.

LEMMA 3. *Let* $D \in B_k$,

$$D = \begin{pmatrix} 1 & \sum_i \tilde{g}_i^{-1} \otimes \hat{g}_i & * \\ 0 & 1 & \sum_i \hat{g}_i^{-1} \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} \tilde{c} & * & * \\ 0 & \hat{c} & * \\ 0 & 0 & 1 \end{pmatrix} \quad (c \in G').$$

Then

$$[D^{\mathbf{c}}, D] = \begin{pmatrix} 1 & 0 & P \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $P = (\tilde{c} + 1)(\sum_i \tilde{g}_i^{-1}) \otimes \hat{c} + N_k$.

PROOF. Since

$$D^{\mathbf{c}} = \begin{pmatrix} 1 & \tilde{c}\left(\sum_i \tilde{g}_i^{-1} \otimes \hat{g}_i\right)\hat{c} & * \\ 0 & 1 & \hat{c}\left(\sum_i \hat{g}_i^{-1}\right) \\ 0 & 0 & 1 \end{pmatrix},$$

we have

$$\begin{aligned} P &= \tilde{c}\left(\sum_i \tilde{g}_i^{-1} \otimes \hat{g}_i\right)\hat{c} \cdot \sum_i \hat{g}_i^{-1} + \left(\sum_i \tilde{g}_i^{-1} \otimes \hat{g}_i\right) \cdot \hat{c}\left(\sum_i \hat{g}_i^{-1}\right) + N_k \\ &= \sum_i \left(\tilde{c}\tilde{g}_i^{-1} \otimes \hat{c} + \tilde{g}_i^{-1} \otimes \hat{c}\right) \\ &\quad + \sum_{i<j} \left(\tilde{c}\tilde{g}_i^{-1} \otimes \hat{g}_i \hat{g}_j^{-1}\hat{c} + \tilde{c}\tilde{g}_j^{-1} \otimes \hat{g}_j \hat{g}_i^{-1}\hat{c} + \tilde{g}_i^{-1} \otimes \hat{g}_i \hat{g}_j^{-1}\hat{c} + \tilde{g}_j^{-1} \otimes \hat{g}_j \hat{g}_i^{-1}\hat{c}\right) + N_k \\ &= (\tilde{c} + 1)\left(\sum_i \tilde{g}_i^{-1}\right) \otimes \hat{c} + (\tilde{c} + 1)\sum_{i<j} f_{ij} + N_k, \end{aligned}$$

where $f_{ij} = \tilde{g}_i^{-1} \otimes \hat{g}_i \hat{g}_j^{-1}\hat{c} + \tilde{g}_j^{-1} \otimes \hat{g}_j \hat{g}_i^{-1}\hat{c}$. If $h_i h_j^{-1} c \notin G'$ then $h_j h_i^{-1} c \notin G'$ so $f_{ij} \in N_k$. If $h_i h_j^{-1} c = c' \in G'$ then $h_j h_i^{-1} c = c'$, $h_j = h_i c c'$ so

$$\begin{aligned} (\tilde{c} + 1) f_{ij} &= (\tilde{c} + 1)\tilde{h}_i^{-1} \otimes \hat{c}' + (\hat{c} + 1)\tilde{h}_i^{-1}\tilde{c}\tilde{c}' \otimes \hat{c}' \\ &= (\tilde{c} + 1)(\tilde{c}\tilde{c}' + 1)\tilde{h}_i^{-1} \otimes \hat{c}' = 0 \otimes \hat{c}' = 0. \end{aligned}$$

Thus, $(\tilde{c}+1)\sum_{i<j} f_{ij} + N_k = N_k$ and so $P = (\tilde{c}+1)(\sum_i \tilde{g}_i^{-1}) \otimes \hat{c} + N_k$ as required. This completes the proof of Lemma 3. □

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Lemma 3 implies that $\mathscr{H}_k$ satisfies the identity $w_k \equiv 1$. Indeed, for every $h_1, h_2, h_3 \in \mathscr{H}_k$, $[h_1, h_2, h_3] \in B_k$ so

$$[h_1, h_2, h_3] = \begin{pmatrix} 1 & \sum_i \tilde{g}_i^{-1} \otimes \hat{g}_i & * \\ 0 & 1 & \sum_i \hat{g}_i^{-1} \\ 0 & 0 & 1 \end{pmatrix}$$

for some $g_i \in G$. If $\mathbf{c} = [h_4, h_5] \cdots [h_{2k+2}, h_{2k+3}]$, $h_i \in \mathscr{H}_k$ for all i, then $\mathbf{c}$ is of the form

$$\begin{pmatrix} \tilde{c} & * & * \\ 0 & \hat{c} & * \\ 0 & 0 & 1 \end{pmatrix},$$

where $c \in G$, $c = [g_4, g_5] \cdots [g_{2k+2}, g_{2k+3}]$ for some $g_i \in G$ $(i = 4, \dots, 2k+3)$. Therefore, by Lemma 3,

$$\big[[h_1, h_2, h_3]^{\mathbf{c}}, [h_1, h_2, h_3]\big] = \begin{pmatrix} 1 & 0 & P \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $P = (\tilde{c} + 1)(\sum_i \tilde{g}_i^{-1}) \otimes \hat{c} + N_k$. Note that

$$c = \prod_{i=1}^{i=k} [g_{2i+2}, g_{2i+3}] = \prod_{i=1}^{i=k} (([g_{2i+2}, g_{2i+3}] + 1) + 1)$$

is a $\mathbf{F}_2$-linear combination (in the group algebra $\mathbf{F}_2 G$) of elements of the form $([f_1, f_2] + 1) \cdots ([f_{2l-1}, f_{2l}] + 1)$ $(l \le k)$, $f_i \in G$ $(i = 1, \dots, 2l)$, so $P = N_k$ and $[[h_1, h_2, h_3]^c, [h_1, h_2, h_3]] = 1$.

Thus, $w_k(h_1, \dots, h_{2k+3}) = 1$ for all $h_1, \dots, h_{2k+3} \in \mathscr{H}_k$, so the identity $w_k \equiv 1$ is satisfied in $\mathscr{H}_k$.

To prove that $\mathscr{H}_k$ does not satisfy the identity $w_{k+1} \equiv 1$ it suffices to check that $w_{k+1}(C, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_{2k+5}) \ne 1$, where

$$C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{x}_i = \begin{pmatrix} \tilde{x}_i & 0 & 0 \\ 0 & \hat{x}_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (i = 2, 3, \dots, 2k+5).$$

It is easy to check that

$$[C, \mathbf{x}_2, \mathbf{x}_3] = \begin{pmatrix} 1 & Q_{12} & * \\ 0 & 1 & Q_{23} \\ 0 & 0 & 1 \end{pmatrix},$$

where $Q_{12} = 1\otimes 1+\tilde{x}_2^{-1}\otimes\hat{x}_2+\tilde{x}_3^{-1}\otimes\hat{x}_3+\tilde{x}_3^{-1}\tilde{x}_2^{-1}\otimes\hat{x}_2\hat{x}_3$, $Q_{23} = 1+\hat{x}_2^{-1}+\hat{x}_3^{-1}+\hat{x}_3^{-1}\hat{x}_2^{-1}$. Let $c = [x_4, x_5]\cdots[x_{2k+4}, x_{2k+5}]$ and let

$$\mathbf{c} = \begin{pmatrix} \tilde{c} & 0 & 0 \\ 0 & \hat{c} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, by Lemma 3,

$$[[C, \mathbf{x}_2, \mathbf{x}_3]^{\mathbf{c}}, [C, \mathbf{x}_2, \mathbf{x}_3]] = \begin{pmatrix} 1 & 0 & P \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $P = (1 + \tilde{x}_2^{-1} + \tilde{x}_3^{-1} + \tilde{x}_3^{-1}\tilde{x}_2^{-1})(\tilde{c} + 1) \otimes \hat{c} + N_k$. Note that

$$(\tilde{c} + 1) = \sum_{i=2}^{i=k+1} \left(\left[\tilde{x}_{2i+2}, \tilde{x}_{2i+3}\right] + 1\right)$$

and

$$\begin{aligned}\hat{c} &= \prod_{i=2}^{i=k+1} \left[\hat{x}_{2i+2}, \hat{x}_{2i+3}\right] = \prod_{i=2}^{i=k+1} \left(\left(\left[\hat{x}_{2i+2}, \hat{x}_{2i+3}\right] + 1\right) + 1\right) \\ &= \prod_{i=2}^{i=k+1} \left(\left[\hat{x}_{2i+2}, \hat{x}_{2i+3}\right] + 1\right) + f,\end{aligned}$$

where $f \in N_k$. Therefore,

$$P = \sum_{i=2}^{i=k+1} f_i^{(1)} + \sum_{i=2}^{i=k+1} f_i^{(2)} + \sum_{i=2}^{i=k+1} f_i^{(3)} + \sum_{i=2}^{i=k+1} f_i^{(4)} + N_k,$$

where $f_i^{(1)} = f_i, f_i^{(2)} = \tilde{x}_2^{-1} f_i, f_i^{(3)} = \tilde{x}_3^{-1} f_i, f_i^{(4)} = \tilde{x}_3^{-1}\tilde{x}_2^{-1} f_i$,

$$f_i = \left(\left[\tilde{x}_{2i+2}, \tilde{x}_{2i+3}\right] + 1\right) \otimes \left(\prod_{i=2}^{i=k+1} \left(\left[\hat{x}_{2i+2}, \hat{x}_{2i+3}\right] + 1\right)\right).$$

The following lemmas can be deduced easily from the proof of Theorem 2 in [3].

LEMMA 4. *$\tilde{S} \otimes_{\mathbf{F}_2} S/N_k$ is a free left $\tilde{S}$-module freely generated by the set*

$$\{1 \otimes ([\hat{x}_{i_1}, \hat{x}_{i_2}] + 1)\cdots([\hat{x}_{i_{2l-1}}, \hat{x}_{i_{2l}}] + 1) + N_k \mid l > k,\ i_1 < i_2 < \cdots < i_{2l}\}.$$

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LEMMA 5. *Let*

$$U_1 = \{\tilde{x}_{i_1} \cdots \tilde{x}_{i_l} \mid l \geq 0, i_1 < \cdots < i_l\},$$
$$U_2 = \{\tilde{x}_{i_1} \cdots \tilde{x}_{i_l}([\tilde{x}_{j_1}, \tilde{x}_{j_2}] + 1) \mid l \geq 0, i_1 < \cdots < i_l, j_1 < j_2\}.$$

Let $U = U_1 \cup U_2$. Then U is a basis of $\tilde{S}$ over $\mathbf{F}_2$.

Lemmas 4 and 5 implies that the set

$$\{f_i^{(j)} + N_k \mid 2 \leq i \leq k+1, 1 \leq j \leq 4\}$$

is linearly independent over $\mathbf{F}_2$, so $P \not\equiv 0 \pmod{N_k}$ and

$$w_{k+1}(C, \mathbf{x}_2, \ldots, \mathbf{x}_{2k+5}) = \big[[C, \mathbf{x}_2, \mathbf{x}_3]^{\mathbf{c}}, [C, \mathbf{x}_2, \mathbf{x}_3]\big] \neq 1.$$

Thus, the group $\mathscr{H}_k$ does not satisfy the identity $w_{k+1} \equiv 1$.

This completes the proof of Theorem 3 as well as that of Theorem 2 (provided that Proposition 1 is proved).

4. Auxiliary results

Let $\mathscr{D}$ be an arbitrary associative ring generated by d_i ($i \in \mathbf{I}$) and let $\mathscr{T}$ be the two-sided ideal in $\mathscr{D}$ generated by all elements of the form

$$(u_1, u_2, u_3) \quad (u_i \in \mathscr{D}). \tag{12}$$

LEMMA 6 ([9]). *$\mathscr{T}$ is generated (as an ideal) by all elements of the forms*

$$(d_{i_1}, d_{i_2}, d_{i_3}) \quad (i_1, i_2, i_3 \in \mathbf{I}), \tag{13}$$
$$(d_{i_1}, d_{i_2})(d_{i_3}, d_{i_4}) + (d_{i_1}, d_{i_4})(d_{i_3}, d_{i_2}) \quad (i_1, i_2, i_3, i_4 \in \mathbf{I}). \tag{14}$$

PROOF. Assume that $\mathscr{D}$ is the free associative ring on a free generating set $\{d_i \mid i \in \mathbf{I}\}$ (it clearly suffices to prove Lemma under this assumption). Let $\mathscr{T}'$ be the two-sided ideal in $\mathscr{D}$ generated by all elements (13)–(14).

Using the identity $(uv, w) = u(v, w) + (u, w)v$, one can check easily that

$$(u_1u_3, u_2, u_4) = u_1(u_3, u_2, u_4) + (u_1, u_2, u_4)u_3 + (u_1, u_2)(u_3, u_4) + (u_1, u_4)(u_3, u_2)$$

so all elements of the form

$$(u_1, u_2)(u_3, u_4) + (u_1, u_4)(u_3, u_2) \quad (u_1, u_2, u_3, u_4 \in \mathscr{D}) \tag{15}$$

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are contained in $\mathscr{T}$. Therefore, $\mathscr{T}' \subseteq \mathscr{T}$.

Since elements (12) and (15) are multilinear with respect to u_i, one can assume that all u_i in (12) and (15) are monomials (on $\{d_i \mid i \in \mathbf{I}\}$). We shall prove that $\mathscr{T}'$ contains all elements of the forms (12) and (15) (and so $\mathscr{T} = \mathscr{T}'$) by induction on the degree of such an element.

Let $f \in \mathscr{D}$ be of the form (12) or (15). Note that every element f of the form (12) of degree 3 is contained in $\mathscr{T}'$ and so is every element f of the form (15) of degree 4 (because such f is of the form (13) or (14)). Suppose that f is of degree k ($k \geq 4$) and all elements of the forms (12) and (15) of degrees less than k are contained in $\mathscr{T}'$.

Consider $f = (u_1, u_2)(u_3, u_4) + (u_1, u_4)(u_3, u_2)$ of degree k. Suppose that $u_4 = u_4' u_4''$, where u_4', u_4'' are monomials of degree at least 1. Then

$$\begin{aligned} f &= (u_1, u_2)(u_3, u_4'u_4'') + (u_1, u_4'u_4'')(u_3, u_2) \\ &= (u_1, u_2)u_4'(u_3, u_4'') + (u_1, u_2)(u_3, u_4')u_4'' + u_4'(u_1, u_4'')(u_3, u_2) \\ &\quad + (u_1, u_4')u_4''(u_3, u_2). \end{aligned}$$

Since

$$\begin{aligned} &(u_1, u_2)u_4'(u_3, u_4'') + u_4'(u_1, u_4'')(u_3, u_2) \\ &\quad = u_4'[(u_1, u_2)(u_3, u_4'') + (u_1, u_4'')(u_3, u_2)] + (u_1, u_2, u_4')(u_3, u_4'') \end{aligned}$$

is contained in $\mathscr{T}'$ and so is (by the same argument)

$$(u_1, u_2)(u_3, u_4')u_4'' + (u_1, u_4')u_4''(u_3, u_2),$$

we have $f \in \mathscr{T}'$.

Note that if u_4 is of degree 1 but for some i ($1 \leq i \leq 3$) $u_i = u_i' u_i''$, where u_i', u_i'' are monomials of degree at least 1 then one can prove $f \in \mathscr{T}'$ in a quite similar way. Thus, under the inductive assumption each element f of the form (15) of degree k is contained in $\mathscr{T}'$.

Consider $f = (u_1, u_2, u_3)$ of degree k. If $u_3 = u_3' u_3''$, where u_3', u_3'' are monomials of degree at least 1 then

$$f = (u_1, u_2, u_3'u_3'') = u_3'(u_1, u_2, u_3'') + (u_1, u_2, u_3')u_3'' \in \mathscr{T}'.$$

If $u_2 = u_2' u_2''$ then

$$\begin{aligned} f &= (u_1, u_2'u_2'', u_3) = (u_2'(u_1, u_2'') + (u_1, u_2')u_2'', u_3) \\ &= u_2'(u_1, u_2'', u_3) + (u_2', u_3)(u_1, u_2'') + (u_1, u_2')(u_2'', u_3) + (u_1, u_2', u_3)u_2'' \in \mathscr{T}'. \end{aligned}$$

(If u_2, u_3 are of degree 1 but $u_1 = u_1' u_1''$ where u_1', u_1'' are monomials of degree at least 1 then the proof is quite similar). Thus, under the inductive assumption each element of the form (15) and of degree k is contained in $\mathscr{T}'$.

This completes the proof of Lemma 6. □

Let $A = F/\gamma_3(F)$, $\mathbb{Z}(A)$ the group ring of A. Suppose that $a_i = x_i\gamma_3(F)$ $(i \in \mathbb{N})$ so that $\{a_i \mid i \in \mathbb{N}\}$ is a free generating set of A. Let T be the two-sided ideal in $\mathbb{Z}(A)$ generated by all elements of the form (u_1, u_2, u_3) $(u_1, u_2, u_3 \in \mathbb{Z}(A))$.

LEMMA 7. *T is generated (as an ideal in $\mathbb{Z}(A)$) by all elements of the form*

$$(16)\qquad \big([a_{i_1}, a_{i_2}] - 1\big)\big([a_{i_3}, a_{i_4}] - 1\big) + \big([a_{i_1}, a_{i_4}] - 1\big)\big([a_{i_3}, a_{i_2}] - 1\big), \quad \textit{where}\quad i_1, i_2, i_3, i_4 \in \mathbb{N}.$$

PROOF. By Lemma 6, T is the two-sided ideal generated by all elements

$$(17)\qquad \big(a_{i_1}^{\varepsilon_1}, a_{i_2}^{\varepsilon_2}, a_{i_3}^{\varepsilon_3}\big) \quad (i_1, i_2, i_3 \in \mathbb{N}, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\})$$

and

$$(18)\qquad \big(a_{i_1}^{\varepsilon_1}, a_{i_2}^{\varepsilon_2}\big)\big(a_{i_3}^{\varepsilon_3}, a_{i_4}^{\varepsilon_4}\big) + \big(a_{i_1}^{\varepsilon_1}, a_{i_4}^{\varepsilon_4}\big)\big(a_{i_3}^{\varepsilon_3}, a_{i_2}^{\varepsilon_2}\big) \quad (i_1, i_2, i_3, i_4 \in \mathbb{N},\ \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \{-1, 1\}).$$

Let T_1 be the two-sided ideal in $\mathbb{Z}(A)$ generated by all elements

$$(19)\qquad \big(a_{i_1}^{\varepsilon_1}, a_{i_2}^{\varepsilon_2}\big)\big(a_{i_2}^{\varepsilon_2}, a_{i_3}^{\varepsilon_3}\big) \quad (i_1, i_2, i_3 \in \mathbb{N}, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\}).$$

Since

$$\big(a_{i_1}^{\varepsilon_1}, a_{i_2}^{\varepsilon_2}\big)\big(a_{i_2}^{\varepsilon_2}, a_{i_3}^{\varepsilon_3}\big) = \big(a_{i_1}^{\varepsilon_1}, a_{i_2}^{\varepsilon_2}\big)\big(a_{i_2}^{\varepsilon_2}, a_{i_3}^{\varepsilon_3}\big) + \big(a_{i_1}^{\varepsilon_1}, a_{i_3}^{\varepsilon_3}\big)\big(a_{i_2}^{\varepsilon_2}, a_{i_2}^{\varepsilon_2}\big) \in T$$

for all $i_1, i_2, i_3 \in \mathbb{N}$, $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\}$, $T_1 \subset T$.

Note that the ideal T_1 is generated by elements

$$(20)\qquad ([a_{i_1}, a_{i_2}] - 1)([a_{i_2}, a_{i_3}] - 1) \quad (i_1, i_2, i_3 \in \mathbb{N}).$$

Indeed,

$$(21)\qquad \big(a_i^{\varepsilon_1}, a_j^{\varepsilon_2}\big) = a_j^{\varepsilon_2} a_i^{\varepsilon_1}\big([a_i^{\varepsilon_1}, a_j^{\varepsilon_2}] - 1\big) = a_j^{\varepsilon_2} a_i^{\varepsilon_1}\big([a_i, a_j]^{\varepsilon_1\varepsilon_2} - 1\big)$$

and

$$(22)\qquad \big([a_i, a_j]^{-1} - 1\big) = -[a_i, a_j]^{-1}([a_i, a_j] - 1)$$

so

$$\big(a_{i_1}^{\varepsilon_1}, a_{i_2}^{\varepsilon_2}\big)\big(a_{i_2}^{\varepsilon_2}, a_{i_3}^{\varepsilon_3}\big) = \varepsilon g([a_{i_1}, a_{i_2}] - 1)([a_{i_2}, a_{i_3}] - 1)$$

for all $i_1, i_2, i_3 \in \mathbb{N}$, $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\}$ and some $g \in A$, $\varepsilon \in \{-1, 1\}$ which depend on i_1, i_2, i_3 and $\varepsilon_1, \varepsilon_2, \varepsilon_3$. Therefore, the elements (19) and (20) generate the same ideal (which is the ideal T_1).

Further, if $i_k = j_l = q$ for some k, l, $(1 \le k, l \le 2)$ then

$$\begin{aligned}\big([a_{i_1}^{\varepsilon_1}, a_{i_2}^{\varepsilon_2}] - 1\big)([a_{j_1}, a_{j_2}] - 1) &= \big([a_p, a_q]^{\delta_1} - 1\big)\big([a_q, a_r]^{\delta_2} - 1\big) \\ &= \varepsilon g([a_p, a_q] - 1)([a_q, a_r] - 1) \in T_1\end{aligned}$$

$$p \in \{i_1, i_2\},\ r \in \{j_1, j_2\},\ \delta_1, \delta_2, \varepsilon \in \{-1, 1\},\ g \in A$$

so

$$[a_{i_1}^{\varepsilon_1}, a_{i_2}^{\varepsilon_2}]([a_{j_1}, a_{j_2}] - 1) = ([a_{j_1}, a_{j_2}] - 1) \pmod{T_1} \tag{23}$$

if $\{i_1, i_2\} \cap \{j_1, j_2\} \neq \emptyset$. By (21),

$$\big(a_{i_1}^{\varepsilon_1}, a_{i_2}^{\varepsilon_2}\big)\big(a_{i_3}^{\varepsilon_3}, a_{i_4}^{\varepsilon_4}\big) = a_{i_2}^{\varepsilon_2} a_{i_1}^{\varepsilon_1} a_{i_4}^{\varepsilon_4} a_{i_3}^{\varepsilon_3}\big([a_{i_1}, a_{i_2}]^{\varepsilon_1\varepsilon_2} - 1\big)\big([a_{i_3}, a_{i_4}]^{\varepsilon_3\varepsilon_4} - 1\big)$$

so, by (23),

$$\big(a_{i_1}^{\varepsilon_1}, a_{i_2}^{\varepsilon_2}\big)\big(a_{i_3}^{\varepsilon_3}, a_{i_4}^{\varepsilon_4}\big) = a_{i_1}^{\varepsilon_1} a_{i_2}^{\varepsilon_2} a_{i_3}^{\varepsilon_3} a_{i_4}^{\varepsilon_4}\big([a_{i_1}, a_{i_2}]^{\varepsilon_1\varepsilon_2} - 1\big)\big([a_{i_3}, a_{i_4}]^{\varepsilon_3\varepsilon_4} - 1\big) \pmod{T_1}.$$

Similarly,

$$\big(a_{i_1}^{\varepsilon_1}, a_{i_4}^{\varepsilon_4}\big)\big(a_{i_3}^{\varepsilon_3}, a_{i_2}^{\varepsilon_2}\big) = a_{i_1}^{\varepsilon_1} a_{i_2}^{\varepsilon_2} a_{i_3}^{\varepsilon_3} a_{i_4}^{\varepsilon_4}\big([a_{i_1}, a_{i_4}]^{\varepsilon_1\varepsilon_4} - 1\big)\big([a_{i_3}, a_{i_2}]^{\varepsilon_2\varepsilon_3} - 1\big) \pmod{T_1},$$

so

$$\big(a_{i_1}^{\varepsilon_1}, a_{i_2}^{\varepsilon_2}\big)\big(a_{i_3}^{\varepsilon_3}, a_{i_4}^{\varepsilon_4}\big) + \big(a_{i_1}^{\varepsilon_1}, a_{i_4}^{\varepsilon_4}\big)\big(a_{i_3}^{\varepsilon_3}, a_{i_2}^{\varepsilon_2}\big) = a_{i_1}^{\varepsilon_1} a_{i_2}^{\varepsilon_2} a_{i_3}^{\varepsilon_3} a_{i_4}^{\varepsilon_4} f \pmod{T_1},$$

where

$$f = \big([a_{i_1}, a_{i_2}]^{\varepsilon_1\varepsilon_2} - 1\big)\big([a_{i_3}, a_{i_4}]^{\varepsilon_3\varepsilon_4} - 1\big) + \big([a_{i_1}, a_{i_4}]^{\varepsilon_1\varepsilon_4} - 1\big)\big([a_{i_3}, a_{i_2}]^{\varepsilon_2\varepsilon_3} - 1\big).$$

By (22) and (23),

$$\begin{aligned}&\big([a_{i_1}, a_{i_2}]^{\varepsilon_1\varepsilon_2} - 1\big)\big([a_{i_3}, a_{i_4}]^{\varepsilon_3\varepsilon_4} - 1\big) \\ &\qquad = \operatorname{sgn}(\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4)([a_{i_1}, a_{i_2}] - 1)([a_{i_3}, a_{i_4}] - 1) \pmod{T_1},\end{aligned}$$

while

$$\begin{aligned}&\big([a_{i_1}, a_{i_4}]^{\varepsilon_1\varepsilon_4} - 1\big)\big([a_{i_3}, a_{i_2}]^{\varepsilon_2\varepsilon_3} - 1\big) \\ &\qquad = \operatorname{sgn}(\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4)([a_{i_1}, a_{i_4}] - 1)([a_{i_3}, a_{i_2}] - 1) \pmod{T_1},\end{aligned}$$

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so for all $i_1, i_2, i_3, i_4 \in \mathbb{N}$, $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \{-1, 1\}$

$$\begin{aligned}(a_{i_1}^{\varepsilon_1}, a_{i_2}^{\varepsilon_2})(a_{i_3}^{\varepsilon_3}, a_{i_4}^{\varepsilon_4}) + (a_{i_1}^{\varepsilon_1}, a_{i_4}^{\varepsilon_4})(a_{i_3}^{\varepsilon_3}, a_{i_2}^{\varepsilon_2}) \\ = \operatorname{sgn}(\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4) a_{i_1}^{\varepsilon_1} a_{i_2}^{\varepsilon_2} a_{i_3}^{\varepsilon_3} a_{i_4}^{\varepsilon_4} \big(([a_{i_1}, a_{i_2}] - 1)([a_{i_3}, a_{i_4}] - 1) \\ + ([a_{i_1}, a_{i_4}] - 1)([a_{i_3}, a_{i_2}] - 1)\big) \pmod{T_1}.\end{aligned}$$

Therefore, the two-sided ideal T_2 in $\mathbb{Z}(A)$ generated by T_1 and the elements (18) coincides with the ideal generated by T_1 and the elements (16). Since T_1 can be generated by the elements (20) which are also of the form (16), it is clear that T_2 is generated by the elements (16).

Finally, it is easy to check using (21), (22) and (23) that every element

$$(a_{i_1}^{\varepsilon_1}, a_{i_2}^{\varepsilon_2}, a_{i_3}^{\varepsilon_3}) = (a_{i_1}^{\varepsilon_1}, a_{i_2}^{\varepsilon_2}) a_{i_3}^{\varepsilon_3} - a_{i_3}^{\varepsilon_3}(a_{i_1}^{\varepsilon_1}, a_{i_2}^{\varepsilon_2})$$

of the form (17) is contained in T_1. Thus, T_2 is generated by all elements (17) and (18) that is $T_2 = T$. This completes the proof of Lemma 7. □

Let $\mathscr{M}$ be the set of all elements of $\mathbb{Z}(A)/T$ of the form

(24) $$a_1^{n_1} a_2^{n_2} \cdots ([a_{i_1}, a_{i_2}] - 1) \cdots ([a_{i_{2l-1}}, a_{i_{2l}}] - 1) + T$$
$$(l \geq 0, \quad i_1 < i_2 < \cdots < i_{2l-1} < i_{2l}),$$

where $n_j \in \mathbb{Z}$ for all $j \in \mathbb{N}$ and $n_j = 0$ for almost all j.

LEMMA 8. *$\mathbb{Z}(A)/T$ is spanned by $\mathscr{M}$.*

PROOF. Since each element of A can be written in the form $a_1^{n_1} a_2^{n_2} \cdots c$, where $c \in A'$, it suffices to prove that $(c - 1) + T$ is a linear combination of elements

$$([a_{i_1}, a_{i_2}] - 1) \cdots ([a_{i_{2l-1}}, a_{i_{2l}}] - 1) + T \quad (i_1 < i_2 < \cdots < i_{2l-1} < i_{2l})$$

for every $c \in A'$. Note that, for each $c \in A'$, $(c-1)+T$ is clearly a linear combination of elements of the form

(25) $$\big([a_{i_1}, a_{i_2}]^{m_1} - 1\big) \cdots \big([a_{i_{2l-1}}, a_{i_{2l}}]^{m_l} - 1\big) + T.$$

Further, for each $m \in \mathbb{Z}$

(26) $$\big([a_i, a_j]^m - 1\big) + T = m([a_i, a_j] - 1) + T.$$

Indeed, if $m > 0$ then

$$\big([a_i, a_j]^m - 1\big) + T = ([a_i, a_j] - 1)\big([a_i, a_j]^{m-1} + \cdots + [a_i, a_j] + 1\big) + T$$

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which is equal, by (23), to $m([a_i, a_j] - 1) + T$. If $m < 0$ then

$$\begin{aligned}\big([a_i, a_j]^m - 1\big) + T &= -[a_i, a_j]^m([a_i, a_j]^{|m|} - 1) + T \\ &= -|m|[a_i, a_j]^m([a_i, a_j] - 1) + T\end{aligned}$$

which is equal, by (23), to $-|m|([a_i, a_j] - 1) + T = m([a_i, a_j] - 1) + T$.

By (26), each element of the form (25) is a linear combination of elements

$$([a_{i_1}, a_{i_2}] - 1) \cdots ([a_{i_{2l-1}}, a_{i_{2l}}] - 1) + T, \tag{27}$$

where, by (23), $i_p \neq i_q$ for all $p \neq q$ $(1 \leq p, q \leq 2l)$. Further,

$$([a_j, a_{j'}] - 1) = -([a_{j'}, a_j] - 1) \pmod T$$

by (22), (23) and

$$([a_{j_1}, a_{j_2}] - 1)([a_{j_3}, a_{j_4}] - 1) = -([a_{j_1}, a_{j_4}] - 1)([a_{j_3}, a_{j_2}] - 1) \pmod T$$

because every element (16) is contained in T. These equations imply that for every i_r, j_r $(1 \leq r \leq 4)$ such that $\{i_1, i_2, i_3, i_4\} = \{j_1, j_2, j_3, j_4\}$ we have

$$([a_{j_1}, a_{j_2}] - 1)([a_{j_3}, a_{j_4}] - 1) = \varepsilon([a_{i_1}, a_{i_2}] - 1)([a_{i_3}, a_{i_4}] - 1) \pmod T, \tag{28}$$

where $\varepsilon \in \{-1, 1\}$. Thus, for every $c \in A'$ the element $(c - 1) + T$ is a linear combination of elements of the form (27) with $i_1 < i_2 < \cdots < i_{2l}$. This completes the proof of Lemma 8. □

LEMMA 9. *$\langle \mathbb{Z}(A)/T, + \rangle$ is a free Abelian group with a basis $\mathscr{M}$.*

PROOF. By Lemma 8, it suffices to prove that the set $\mathscr{M}$ is linearly independent over $\mathbb{Z}$.

Let E be an associative algebra over $\mathbb{Q}$ with an identity element 1 defined by

$$E = \langle e_i \ (i \in \mathbb{N}) \mid e_i^2 = 0, e_i e_j = -e_j e_i \ (i, j \in \mathbb{N}) \rangle$$

Then E is (isomorphic to) the Grassmann (or exterior) algebra on a countably infinite-dimensional vector space over $\mathbb{Q}$ with a basis $\{e_i \mid i \in \mathbb{N}\}$. It is well-known (and easy to prove) that the set

$$\{e_{i_1} \cdots e_{i_k} \mid k \geq 0, i_1 < i_2 < \cdots < i_k\}$$

is a basis of E over $\mathbb{Q}$. Since $e_i^2 = 0$, elements $1 + e_i$ $(i \in \mathbb{N})$ are invertable in E and $(1 + e_i)^{-1} = 1 - e_i$. Note that

$$[1 + e_i, 1 + e_j] = (1 - e_i)(1 - e_j)(1 + e_i)(1 + e_j) = 1 + 2e_i e_j$$

for all $i, j \in \mathbb{N}$. Since elements $e_i e_j$ are central in E, the (multiplicative) group $\mathcal{G}$ generated by $\{1 + e_i \mid i \in \mathbb{N}\}$ is nilpotent of class 2. Therefore, the mapping $\xi : a_i \to 1 + e_i$ ($i \in \mathbb{N}$) can be extended to a homomorphism of A onto $\mathcal{G}$ which, in its turn, can be extended to a homomorphism of the group ring $\mathbb{Z}(A)$ into E. Since

$$(([a_{i_1}, a_{i_2}] - 1)([a_{i_3}, a_{i_4}] - 1) + ([a_{i_1}, a_{i_4}] - 1)([a_{i_3}, a_{i_2}] - 1))\xi$$
$$= 4e_{i_1}e_{i_2}e_{i_3}e_{i_4} + 4e_{i_1}e_{i_4}e_{i_3}e_{i_2} = 0$$

for all i_1, i_2, i_3, i_4, we have $T \subseteq \ker \xi$ so there is a homomorphism $\overline{\xi} : \mathbb{Z}(A)/T \to E$ such that $(a_i + T)\overline{\xi} = 1 + e_i$ ($i \in \mathbb{N}$). Since

$$(([a_{i_1}, a_{i_2}] - 1) \cdots ([a_{i_{2l-1}}, a_{i_{2l}}] - 1) + T)\overline{\xi} = 2^l e_{i_1}e_{i_2} \cdots e_{i_{2l}},$$

the set

$$\{(([a_{i_1}, a_{i_2}] - 1) \cdots ([a_{i_{2l-1}}, a_{i_{2l}}] - 1) + T \mid l \geq 0, i_1 < i_2 < \cdots < i_{2l}\}$$

is linearly independent, so it forms a $\mathbb{Z}$-basis for $\mathbb{Z}A' + T/T$.

Now to complete the proof of Lemma 9 it remains to note that if T is an ideal of $\mathbb{Z}(A)$ generated by elements of $\mathbb{Z}(A')$ such that $\langle \mathbb{Z}(A') + T/T, + \rangle$ is a free abelian group and $\{v_j + T \mid j \in J\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}(A') + T/T$ then the set of all elements

$$a_1^{n_1} a_2^{n_2} \cdots v_j + T \quad (j \in J)$$

with $n_l \in \mathbb{Z}$ for all $l \in \mathbb{N}$ and $n_l = 0$ for almost all l is a basis of $\mathbb{Z}(A)/T$ over $\mathbb{Z}$. □

Let q be a positive integer, $\mathbb{N}^q$ the set of ordered q-tuples of elements of $\mathbb{N}$. Suppose that M_q is the free right $\mathbb{Z}(A)$-module generated by all elements $(i_1, i_2, \ldots, i_q) \in \mathbb{N}^q$.

Recall that Φ is the set of all functions $\phi : \mathbb{N} \to \mathbb{N}$ such that $a\phi < b\phi$ when $a < b$. We also write Φ for the corresponding sets of endomorphisms of $\mathbb{Z}(A)$ (such that $a_i\phi = a_{i\phi}$ for all i) and of $\mathbb{Z}$-linear mappings of M_q into itself such that $((i_1, \ldots, i_q)f)\phi = (i_1\phi, \ldots, i_q\phi)(f\phi)$, where $f \in \mathbb{Z}(A)$. A $\mathbb{Z}(A)$-submodule L in M_q is called a *Φ-submodule* if L is closed under all mappings $\phi \in \Phi$.

The main result of the section is as follows.

PROPOSITION 3. *For every positive integer q the module M_q/M_qT satisfies the maximal condition on Φ-submodules.*

PROOF. Recall that $\mathcal{M}$ is the set of all elements in $\mathbb{Z}(A)/T$ which are of the form (24). Define on $\mathcal{M}$ a linear order denoted by $\leq$ and a partial order denoted by $\preceq$. Let $m, m' \in \mathcal{M}$, $m = m_1m_2$, $m' = m'_1m'_2$, where $m_i, m'_i \in \mathcal{M}$ ($i = 1, 2$),

(29) $$m_1 = a_1^{n_1}a_2^{n_2} \cdots + T, \quad m'_1 = a_1^{n'_1}a_2^{n'_2} \cdots + T,$$

(30) $$m_2 = ([a_{i_1}, a_{i_2}] - 1) \cdots ([a_{i_{2l-1}}, a_{i_{2l}}] - 1) + T \quad (i_1 < \cdots < i_{2l}),$$

(31) $$m'_2 = ([a_{i'_1}, a_{i'_2}] - 1) \cdots ([a_{i'_{2l'-1}}, a_{i'_{2l'}}] - 1) + T \quad (i'_1 < \cdots < i'_{2l'}).$$

Define

$$\operatorname{sgn}(n) = \begin{cases} 1, & \text{if } n > 0; \\ 0, & \text{if } n = 0; \\ -1, & \text{if } n < 0. \end{cases}$$

We write $m_1 < m'_1$ if and only if one of the following conditions (i)–(ii) holds:

(i) $|n_k| < |n'_k|$ for some k but $|n_j| = |n'_j|$ for all $j > k$;

(ii) $|n_j| = |n'_j|$ for all $j \in \mathbb{N}$, $\operatorname{sgn}(n_k) < \operatorname{sgn}(n'_k)$ for some k but $\operatorname{sgn}(n_j) = \operatorname{sgn}(n'_j)$ for all $j > k$.

Define $m_2 < m'_2$ if and only if $i_{2l-k} < i'_{2l'-k}$ for some k but $i_{2l-j} = i'_{2l'-j}$ for all j, $0 \le j < k$ or $i_{2l-j} = i'_{2l'-j}$ for all j, $0 \le j < 2l$, and $l < l'$. Put $m < m'$ if and only if one of the following conditions (i′)–(ii′) holds:

(i′) $m_1 < m'_1$;

(ii′) $m_1 = m'_1$, $m_2 < m'_2$.

It is easy to prove that $(\mathscr{M}, \le)$ is well-ordered.

We write $m_1 \preceq m'_1$ if and only if the following conditions (j)–(jj) hold:

(j) $|n_j| \le |n'_j|$ for all $j \in \mathbb{N}$;

(jj) $\operatorname{sgn}(n_j) = \operatorname{sgn}(n'_j)$ for all $j \in \mathbb{N}$ such that $n_j \ne 0$.

Put $m_2 \preceq m'_2$ if and only if $\{i_1, \dots, i_{2l}\} \subseteq \{i'_1, \dots, i'_{2l'}\}$. Define $m \preceq m'$ if $m_1 \preceq m'_1$ and $m_2 \preceq m'_2$.

LEMMA 10. *Let $m \preceq m'$ ($m, m' \in \mathscr{M}$). Then there exist $f \in \mathbb{Z}(A)$ such that the following conditions hold*:

(i) *$mf = m'$;*

(ii) *if $\overline{m} < m$ ($\overline{m} \in \mathscr{M}$) then $\overline{m}f = 0$ or $\overline{m}f = \sum \varepsilon_i \overline{m}_i$, where $\varepsilon_i \in \{-1, 1\}$ and $\overline{m}_i < m'$ for all i.*

PROOF. Let $m = m_1 m_2$, $m' = m'_1 m'_2$, where m_i, m'_i are as in (29)–(31). Suppose that $b = a_1^{(n'_1 - n_1)} a_2^{(n'_2 - n_2)} \cdots \in A$. Then $m_1 b = m'_1 c$ for some $c \in A'$. Let

$$\{i''_1, \dots, i''_{2l''}\} = \{i'_1, \dots, i'_{2l'}\} \backslash \{i_1, \dots, i_{2l}\}, \quad i''_1 < \cdots < i''_{2l''}$$

and let $f_2 = ([a_{i''_1}, a_{i''_2}] - 1) \cdots ([a_{i''_{2l''-1}}, a_{i''_{2l''}}] - 1)$. By (28), there is $\varepsilon \in \{-1, 1\}$ such that $\varepsilon m_2 f_2 = m'_2$. Take $f = \varepsilon b c^{-1} f_2$ then $mf = m'$.

Let $\overline{m} \in \mathscr{M}$, $\overline{m} = \overline{m}_1 \overline{m}_2$, where $\overline{m}_1 = a_1^{\overline{n}_1} a_2^{\overline{n}_2} \cdots + T$,

$$\overline{m}_2 = ([a_{j_1}, a_{j_2}] - 1) \cdots ([a_{j_{2k-1}}, a_{j_{2k}}] - 1) + T \quad (j_1 < \cdots < j_{2k})$$

and let $\overline{m} < m$. Consider $\overline{m}f$ and suppose first that $\overline{m}_1 = m_1$. Then $\overline{m}_2 < m_2$ and it is easy to check that $\varepsilon \overline{m}_2 f_2 = \overline{\varepsilon} \overline{m}'_2$, where $\overline{\varepsilon} \in \{-1, 1\}$, $\overline{m}'_2$ is of the form (30) and

$\overline{m}'_2 < \varepsilon m_2 f_2 = m'_2$ (or $\overline{m}_2 f_2 = 0$ if $\{j_1, \ldots, j_{2k}\} \cap \{i''_1, \ldots, i''_{2l''}\} \neq \emptyset$). Therefore, $\overline{m}f = \varepsilon m_1 bc^{-1}\overline{m}_2 f_2 = \overline{\varepsilon} m'_1 \overline{m}'_2 = \overline{\varepsilon}\,\overline{m}'$, where $\overline{m}' < m'$ or $\overline{m}' = 0$.

Further, suppose that $\overline{m}_1 < m_1$. Then $\overline{m}_1 bc^{-1} = \overline{m}'_1 \overline{c}$, where $\overline{c} \in A'$, $\overline{m}'_1 = a_1^{\overline{n}'_1} a_2^{\overline{n}'_2} \cdots$. It is easy to check that $\overline{m}'_1 < m'_1$. Therefore, $\overline{m}f = \varepsilon \overline{m}_1 bc^{-1}\overline{m}_2 f_2 = \varepsilon \overline{m}'_1 \overline{c} f_2 \overline{m}_2$, where $\varepsilon \overline{c} f_2 \overline{m}_2 = 0$ or $\varepsilon \overline{c} f_2 \overline{m}_2 = \sum \varepsilon_i \overline{m}_2^{(i)}$ with $\overline{m}_2^{(i)}$ of the form (30) and $\varepsilon_i \in \{-1, 1\}$ for all i. Since $\overline{m}'_1 < m'_1$, $\overline{m}_i = \overline{m}'_1 \overline{m}_2^{(i)} < m'_1 m'_2 = m'$ for all i as required. This completes the proof of Lemma 10. □

Let $\leq$ denote the lexicographic order on $\mathbb{N}^q$ (that is $(j_1, \ldots, j_q) < (j'_1, \ldots, j'_q)$ if and only if there exists k such that $j_k < j'_k$ but $j_l = j'_l$ for all $l < k$). Let $\mathscr{W} = \mathbb{N}^q \times \mathscr{M}$. Since the free $\mathbb{Z}(A)/T$-module freely generated by all elements $(i_1, \ldots, i_q) \in \mathbb{N}^q$ is naturally isomorphic to M_q/M_qT, we may assume that $\mathscr{W} \subset M_q/M_qT$ and M_q/M_qT is spanned by $\mathscr{W}$. Define on $\mathscr{W}$ a linear order denoted by $\leq$ and a partial order denoted by $\preceq_\Phi$. Let $w, w' \in \mathscr{W}$, $w = (j_1, \ldots, j_q)m$, $w' = (j'_1, \ldots, j'_q)m'$, where $j_l, j'_l \in \mathbb{N}$ for all l, $m, m' \in \mathscr{M}$.

We write $w < w'$ if and only if one of the following conditions holds:

(i) $(j_1, \ldots, j_q) < (j'_1, \ldots, j'_q)$;

(ii) $j_l = j'_l$ for all l, $1 \leq l \leq q$ and $m < m'$.

Note that $(\mathscr{W}, \leq)$ is well-ordered.

We write $w \preceq_\Phi w'$ if and only if there exists $\phi \in \Phi$ such that the following conditions hold:

(j) $j_l\phi = j'_l$ for all l, $1 \leq l \leq q$;

(jj) $m\phi \preceq m'$.

LEMMA 11. *Let $w \preceq_\Phi w'$ ($w, w' \in \mathscr{W}$). Then there exist $\phi \in \Phi$ and $f \in \mathbb{Z}(A)$ such that the following conditions hold*:

(j) $(w\phi)f = w'$;

(jj) *if $\overline{w} < w$ ($\overline{w} \in \mathscr{W}$) then $(\overline{w}\phi)f = 0$ or $(\overline{w}\phi)f = \sum \varepsilon_i \overline{w}^{(i)}$, where $\varepsilon_i \in \{-1, 1\}$ and $\overline{w}^{(i)} < w'$ for all i.*

PROOF. Let $w = (j_1, \ldots, j_q)m$, $w' = (j'_1, \ldots, j'_q)m'$, where $j_l, j'_l \in \mathbb{N}$ ($1 \leq l \leq q$), $m, m' \in \mathscr{M}$. Since $w \preceq_\Phi w'$, there exists $\phi \in \Phi$ such that $j_l\phi = j'_l$ for all l and $m\phi \preceq m'$. Since $m\phi \preceq m'$, by Lemma 10 there exists $f \in \mathbb{Z}(A)$ which satisfies the conditions (i)–(ii) of Lemma 10 (if one replace m with $m\phi$ in (i)–(ii)). By (i), $(m\phi)f = m'$ so the condition (j) of Lemma 11 holds.

Let $\overline{w} \in \mathscr{W}$, $\overline{w} = (\overline{j}_1, \ldots, \overline{j}_q)\overline{m}$, where $\overline{j}_l \in \mathbb{N}$ ($1 \leq l \leq q$), $\overline{m} \in \mathscr{M}$. Suppose that $\overline{w} < w$. Then $(\overline{j}_1, \ldots, \overline{j}_q) < (j_1, \ldots, j_q)$ or $(\overline{j}_1, \ldots, \overline{j}_q) = (j_1, \ldots, j_q)$, $\overline{m} < m$.

Suppose that $(\overline{j}_1, \ldots, \overline{j}_q) < (j_1, \ldots, j_q)$. Then

$$(\overline{j}_1\phi, \ldots, \overline{j}_q\phi) < (j_1\phi, \ldots, j_q\phi) = (j'_1, \ldots, j'_q) \tag{32}$$

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so, $(w\phi)f = 0$ or $(w\phi)f = \sum \varepsilon w^{(i)}$, where $w^{(i)} = (\bar{j}_1\phi, \dots, \bar{j}_q\phi)m^{(i)}$ for some $m^{(i)} \in \mathscr{M}$ and, by (32), $w^{(i)} < (j'_1, \dots, j'_q)m' = w'$ for all i.

Suppose that $(\bar{j}_1, \dots, \bar{j}_q) = (j_1, \dots, j_q)$, $\overline{m} < m$. Then it is easy to check that $\overline{m}\phi < m\phi$ so by Lemma 10 (replacing $\overline{m}$ with $\overline{m}\phi$) $(\overline{m}\phi)f = 0$ or $(\overline{m}\phi)f = \sum \varepsilon_i m^{(i)}$, where $m^{(i)} \in \mathscr{M}$, $m^{(i)} < m'$. Thus, $(\overline{w}\phi)f = 0$ or $(\overline{w}\phi)f = \sum \varepsilon_i w^{(i)}$, where $w^{(i)} = (j'_1, \dots, j'_q)m^{(i)} < (j'_1, \dots, j'_q)m' = w'$ and $\varepsilon_i \in \{-1, 1\}$ for all i.

Therefore, the condition (jj) of Lemma 11 holds. The proof of Lemma 11 is completed. □

Let $\mathbf{J}$ denote the set of non-negative integers. Let $S_2 = \{0, 1\}$, $S_3 = \{-1, 0, 1\}$. Let $S = \mathbf{J} \times S_3 \times S_2$, $0 = (0, 0, 0) \in S$. We shall write $V(S) = V(S, 0)$ for the set of all sequences $(s_i \mid i \in \mathbb{N})$ of elements of S in which the set $\{i \mid s_i \neq 0\}$ is finite. For $q \in \mathbb{N}$, we shall write $V_q(S) = V_q(S, 0) = \mathbb{N}^q \times V(S)$ for the set of pairs (u, v) $(u \in \mathbb{N}^q, v \in V(S))$.

Define the partial order $\preceq$ on S by putting

$$(n, s_1, s_2) \preceq (n', s'_1, s'_2) \quad (n, n' \in \mathbb{N}; s_1, s'_1 \in S_3, s_2, s'_2 \in S_2)$$

if and only if

$$n \leq n', \quad s_1 = s'_1, \quad s_2 = s'_2.$$

Then we can define a partial order $\preceq_\Phi$ on $V_q(S)$. We write

$$((n_1, \dots, n_q), (s_i \mid i \in \mathbb{N})) \preceq_\Phi ((n'_1, \dots, n'_q), (s'_i \mid i \in \mathbb{N}))$$

if and only if there is an element ϕ of Φ such that $n_k\phi = n'_k$ $(1 \leq k \leq q)$ and $s_i \preceq s'_{i\phi}$ for all $i \in \mathbb{N}$.

Let R be an arbitrary non-empty set, $\preceq$ a partial order on R. Recall that $(R, \preceq)$ is called *partially well-ordered* if and only if every infinite sequence $r_1, r_2, \dots$ of elements of R contains an infinite subsequence $r_{i_1}, r_{i_2}, \dots$ $(i_1 < i_2 < \cdots)$ such that

$$r_{i_1} \preceq r_{i_2} \preceq \cdots$$

(see [4] for equivalent definitions).

Note that $(S, \preceq)$ is clearly partially well-ordered so the following lemma can be deduced easily from [1, Lemma 3.2] which, in its turn, is deduced from [4, Theorem 4.3].

LEMMA 12. *$(V_q(S), \preceq_\Phi)$ is partially well-ordered.*

Define a mapping $\nu : \mathscr{W} \to V_q(S)$. Let $w = (j_1, \dots, j_q)m$, $m = m_1m_2$, where $m_1 = a_1^{n_1}a_2^{n_2}\cdots + T$, $m_2 = ([a_{i_1}, a_{i_2}] - 1)\cdots([a_{i_{2l-1}}, a_{i_{2l}}] - 1) + T$ $(i_1 < \cdots < i_{2l})$. Put

$$w\nu = ((j_1, \dots, j_q), \quad (s_i \mid i \in \mathbb{N})),$$

where $s_i = (|n_i|, \operatorname{sgn}(n_i), s_{3i})$,

$$s_{3i} = \begin{cases} 1, & \text{if } i_s = i \text{ for some } s,\ 1 \le s \le 2l; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, ν is injective.

LEMMA 13. *Let $w, w' \in \mathscr{W}$ and $w\nu \preceq_\Phi w'\nu$. Then $w \preceq_\Phi w'$.*

PROOF. Let $w = (j_1, \dots, j_q)m$, $w' = (j'_1, \dots, j'_q)m'$, $m = m_1 m_2$, $m' = m'_1 m'_2$, where m_i, m'_i are of the forms (29)–(31). Then $w\nu = ((j_1, \dots, j_q), (s_i \mid i \in \mathbb{N}))$, $w'\nu = ((j'_1, \dots, j'_q), (s'_i \mid i \in \mathbb{N}))$, where $s_i = (|n_i|, \operatorname{sgn}(n_i), s_{3i})$, $s'_i = (|n'_i|, \operatorname{sgn}(n'_i), s'_{3i})$ for all i.

Since $w\nu \preceq_\Phi w'\nu$, there exists $\phi \in \Phi$ such that $j_l\phi = j'_l$ $(1 \le l \le q)$ and $s_i \preceq s'_{i\phi}$ for all $i \in \mathbb{N}$ that is $|n_i| \le |n'_{i\phi}|$, $\operatorname{sgn}(n_i) = \operatorname{sgn}(n'_{i\phi})$, $s_{3i} = s'_{3(i\phi)}$ for all $i \in \mathbb{N}$. To prove $w \preceq_\Phi w'$ it suffices to check that $m\phi \preceq m'$.

Let $m'' = m\phi$. Then $m'' = m''_1 m''_2$, where

$$m''_2 = ([a_{i_1\phi}, a_{i_2\phi}] - 1) \cdots ([a_{i_{2l-1}\phi}, a_{i_{2l}\phi}] - 1) + T \quad (i_1 < \cdots < i_{2l}),$$

$m''_1 = a_1^{n''_1} a_2^{n''_2} \cdots + T$,

$$n''_j = \begin{cases} n_i, & \text{if } j = i\phi, \\ 0, & \text{if } j \notin \mathbb{N}\phi \end{cases}$$

for all $j \in \mathbb{N}$. To prove $m'' \preceq m'$ (equivalently, $m''_1 \preceq m'_1$ and $m''_2 \preceq m'_2$) we have to check that $|n''_j| \le |n'_j|$ for all j, $\operatorname{sgn}(n''_j) = \operatorname{sgn}(n'_j)$ for all j such that $n''_j \ne 0$ and $\{i_1\phi, \dots, i_{2l}\phi\} \subseteq \{i'_1, \dots, i'_{2l'}\}$.

Let $j \in \mathbb{N}\phi$, $j = i\phi$. Then $|n''_j| = |n_i| \le |n'_{i\phi}| = |n'_j|$ and $\operatorname{sgn}(n''_j) = \operatorname{sgn}(n_i) = \operatorname{sgn}(n'_{i\phi}) = \operatorname{sgn}(n'_j)$. Let now $j \notin \mathbb{N}\phi$. Then $n''_j = 0$ so $|n''_j| \le |n'_j|$. Therefore, $m''_1 \preceq m'_1$.

Consider an arbitrary s, $1 \le s \le 2l$. Then $s_{3i_s} = 1 = s'_{3(i_s\phi)}$ so $i_s\phi = i'_r$ for some r, that is $i_s\phi \in \{i'_1, \dots, i'_{2l'}\}$. Therefore, $\{i_1\phi, \dots, i_{2l}\phi\} \subseteq \{i'_1, \dots, i'_{2l'}\}$ and $m''_2 \preceq m'_2$. Thus, $m\phi = m'' \preceq m'$. This completes the proof of Lemma 13. □

Let $(w_i \mid i \in \mathbb{N})$ be an arbitrary sequence of elements of $\mathscr{W}$. Consider the sequence $(w_i\nu \mid i \in \mathbb{N})$. By Lemma 12, there exists a subsequence $(w_{i_l}\nu \mid l \in \mathbb{N})$ such that

$$w_{i_1}\nu \preceq_\Phi w_{i_2}\nu \preceq_\Phi \cdots \quad (i_1 < i_2 < \cdots).$$

Then, by Lemma 13,

$$w_{i_1} \preceq_\Phi w_{i_2} \preceq_\Phi \cdots \quad (i_1 < i_2 < \cdots).$$

Thus, we have the following.

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LEMMA 14. *($\mathscr{W}$, $\preceq_\Phi$) is partially well-ordered.*

Now we can complete the proof of Proposition 3 in a standard way (see [1,2]). Suppose, in order to get a contradiction, that

$$M^{(1)} \subset M^{(2)} \subset \cdots$$

is a strictly ascending chain of Φ-submodules in M_q/M_qT (that is $M^{(i)} \neq M^{(i+1)}$ for all i). For each $i \in \mathbb{N}$ let $\mathscr{W}_i$ be the set of all elements $w \in \mathscr{W}$ such that there exists $h \in M^{(i+1)} \backslash M^{(i)}$, $h = nw + \sum n_j w_j$, $n \neq 0$, $w_j < w$ for all j. Since $M^{(i+1)} \backslash M^{(i)} \neq \emptyset$, so is $\mathscr{W}_i$. Let $w^{(i)}$ be the smallest (in the well-order $\leq$) element of $\mathscr{W}_i$ and let $h^{(i)} = n^{(i)} w^{(i)} + \sum n_j^{(i)} w_j^{(i)}$, $h^{(i)} \in M^{(i+1)} \backslash M^{(i)}$, where $n^{(i)}, n_j^{(i)} \in \mathbb{Z}$, $n^{(i)} \neq 0$, $w_j^{(i)} < w^{(i)}$ for all j. By Lemma 14, ($\mathscr{W}$, $\preceq_\Phi$) is partially well-ordered. Therefore, by passing to an infinite subsequence we may assume that

$$w^{(1)} \preceq_\Phi w^{(2)} \preceq_\Phi \cdots .$$

Let $\mathscr{I} = \text{ideal}\,\{n^{(i)} \mid i \in \mathbb{N}\}$, $\mathscr{I} \subseteq \mathbb{Z}$. Then there is $m \in \mathbb{N}$ such that $\mathscr{I} = \text{ideal}\,\{n^{(i)} \mid i = 1, \ldots, m\}$ so $n^{(m+1)} = \sum_{i=1}^m n^{(i)} n'_i$ for some $n'_i \in \mathbb{Z}$ $(i = 1, \ldots, m)$. Consider $h^{(i)} = n^{(i)} w^{(i)} + \sum n_j^{(i)} w_j^{(i)} \in M^{(i+1)} \backslash M^{(i)}$, $i = 1, \ldots, m+1$. Since $w^{(i)} \preceq_\Phi w^{(m+1)}$ for $i = 1, \ldots, m$, there exist $\phi_i \in \Phi$ and $f_i \in \mathbb{Z}(A)$ $(i = 1, \ldots, m)$ such that $(w^{(i)}\phi_i) f_i = w^{(m+1)}$ but $(w_j^{(i)}\phi_i) f_i = \sum_{j,k} n_{jk}^{(i)} w_{jk}^{(i)}$, where $w_{jk}^{(i)} < w^{(m+1)}$ for all i, j, k. Therefore, $h^{(m+1)} - \sum_{i=1}^m n'_i (h^{(i)}\phi_i) f_i = \sum_j n_j^{(m+1)} w_j^{(m+1)} - \sum_{i,j,k} n'_i n_{jk}^{(i)} w_{jk}^{(i)}$, where $w_j^{(m+1)} < w^{(m+1)}$, $w_{jk}^{(i)} < w^{(m+1)}$ for all i, $1 \leq i \leq m$, and all j, k. This contradicts the choice of $h^{(m+1)}$ because $(h^{(m+1)} - \sum_{i=1}^m n'_i (h^{(i)}\phi_i) f_i) \in M^{(m+1)} \backslash M^{(m)}$. The proof of Proposition 3 is completed. □

COROLLARY 4. *For every positive integers q, l the module M_q/M_qT^l satisfies the maximal condition on Φ-submodules.*

PROOF. By an inductive argument it suffices to prove that M_qT^{l-1}/M_qT^l satisfies the maximal condition on Φ-submodules. It is easily deduced from Lemma 7 that T^{l-1} is generated (as $\mathbb{Z}(A)$-module) by the elements of the form

$$f_{i_1 i_2 i_3 i_4} f_{i_5 i_6 i_7 i_8} \cdots f_{i_{4l-7} i_{4l-6} i_{4l-5} i_{4l-4}},$$

where

$$f_{j_1 j_2 j_3 j_4} = ([a_{j_1}, a_{j_2}] - 1)([a_{j_3}, a_{j_4}] - 1) + ([a_{j_1}, a_{j_4}] - 1)([a_{j_3}, a_{j_2}] - 1)$$

for all j_1, j_2, j_3, j_4. Put $q' = q + 4(l-1)$. Define a $\mathbb{Z}(A)$-linear map χ of $M_{q'}$ onto M_qT^{l-1}/M_qT^l by

$$(j_1, \ldots j_{q'})\chi = (j_1, \ldots, j_q) f_{j_{q+1} j_{q+2} j_{q+3} j_{q+4}} \cdots f_{j_{q'-3} j_{q'-2} j_{q'-1} j_{q'}} + M_qT^l.$$

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Since $M_{q'}T \subseteq \ker\chi$, one can define a $\mathbb{Z}(A)$-linear map $\overline{\chi}$ from $M_{q'}/M_{q'}T$ onto M_qT^{l-1}/M_qT^l by

$$((j_1,\dots,j_{q'})f + M_{q'}T)\overline{\chi} = (j_1,\dots,j_{q'})\chi f + M_qT^l \quad (j_k \in \mathbb{N}, f \in \mathbb{Z}(A)).$$

It is clear that $\overline{\chi}\phi = \phi\overline{\chi}$ for all $\phi \in \Phi$.

Suppose that

$$M^{(1)} \subset M^{(2)} \subset \cdots$$

is an infinite strictly ascending chain of Φ-submodules in M_qT^{l-1}/M_qT^l. Then

$$M^{(1)}\overline{\chi}^{-1} \subset M^{(2)}\overline{\chi}^{-1} \subset \cdots$$

is an infinite strictly ascending chain of Φ-submodules in $M_{q'}/M_{q'}T$. This contradicts Proposition 3 and completes the proof of Corollary 4. □

5. Proof of Proposition 2

It is well known that $F/\gamma_3(F)$ satisfies the maximal condition on normal Φ-subgroups. Therefore, to prove Proposition 2 it suffices to show that the group $\gamma_3(F)/U_k$ satisfies the maximal condition on normal Φ-subgroups of F/U_k contained in $\gamma_3(F)/U_k$.

Recall that $A = F/\gamma_3(F)$. Let $V = [\gamma_3(F), \gamma_3(F)]$. Clearly, $\gamma_3(F)/V$ is an abelian subgroup in F/V generated by the elements

(33) $$[x_{j_1}, x_{j_2}, x_{j_3}]^g \cdot V \quad (j_1, j_2, j_3 \in \mathbb{N}, g \in F).$$

Then one can consider $\gamma_3(F)/V$ as a right multiplicative $\mathbb{Z}(A)$-module generated by elements

$$[x_{j_1}, x_{j_2}, x_{j_3}] \cdot V \qquad (j_1, j_2, j_3 \in \mathbb{N})$$

with elements of A acting by conjugation:

$$[x_{j_1}, x_{j_2}, x_{j_3}]^g \cdot V = g^{-1}[x_{j_1}, x_{j_2}, x_{j_3}]g \cdot V.$$

Note that U_k/V is generated (as a subgroup in F/V) by all elements of the form

$$[v_1, v_2, v_3]^{(u_1,u_2,u_3)\cdots(u_{3k-5},u_{3k-4},u_{3k-3})u} \cdot V \quad (v_i, u_j, u \in F).$$

Since $[v_1, v_2, v_3] \cdot V$ is a product of elements of the form (33) and their inverses, the group U_k/V is generated by the elements of the form

$$(34) \qquad [x_{j_1}, x_{j_2}, x_{j_3}]^f \cdot V \quad (f \in I),$$

where I is the two-sided ideal in $\mathbb{Z}(A)$ generated by the elements

$$(u_1, u_2, u_3) \cdots (u_{3k-5}, u_{3k-4}, u_{3k-3}) \quad (u_i \in A).$$

Note that $I = T^{k-1}$ (recall that T is the ideal in $\mathbb{Z}(A)$ generated by all elements (u_1, u_2, u_3), where $u_i \in \mathbb{Z}(A)$). Indeed, obviously $I \subseteq T^{k-1}$. On the other hand, since

$$v(u_1, u_2, u_3) = (u_1u_2, u_3, v) - (u_2u_1, u_3, v) + (u_1, u_2, u_3)v$$

for all $u_i, v \in A$, each element

$$v_1(u_1, u_2, u_3)v_2 \cdots v_{k-1}(u_{3k-5}, u_{3k-4}, u_{3k-3})v_k \quad (u_i, v_j \in A)$$

can be rewritten in the form

$$(u'_1, u'_2, u'_3) \cdots (u'_{3k-5}, u'_{3k-4}, u'_{3k-3})v' \quad (u'_i, v' \in A)$$

so $T^{k-1} \subseteq I$.

Define a $\mathbb{Z}(A)$-linear mapping α of M_3 onto $\gamma_3(F)/V$ by

$$(j_1, j_2, j_3)\alpha = [x_{j_1}, x_{j_2}, x_{j_3}] \cdot V.$$

Clearly, $\phi\alpha = \alpha\phi$ for every $\phi \in \Phi$. Let β be the natural homomorphism of $\gamma_3(F)/V$ onto $\gamma_3(F)/U_k$. Since U_k/V is a $\mathbb{Z}(A)$-submodule in $\gamma_3(F)/V$ closed under all mappings $\phi \in \Phi$, $\gamma_3(F)/U_k$ is a right $\mathbb{Z}(A)$-module with mappings $\phi \in \Phi$ acting on it in such a way that $\beta\phi = \phi\beta$.

Define $\mu = \alpha\beta$, $\mu : M_3 \to \gamma_3(F)/U_k$. Then $\mu\phi = \phi\mu$ for all $\phi \in \Phi$. Since $((j_1, j_2, j_3)f)\mu$ $(f \in I)$ is of the form (34), $M_3T^{k-1} \subseteq \ker(\mu)$ so one can define a $\mathbb{Z}(A)$-linear homomorphism $\overline{\mu}$ of M_3/M_3T^{k-1} onto $\gamma_3(F)/U_k$ by $(m + M_3T^{k-1})\overline{\mu} = m\mu$ for all $m \in M_3$. Clearly, $\overline{\mu}\phi = \phi\overline{\mu}$ for all $\phi \in \Phi$. Note that if N is a normal subgroup of F/U_k contained in $\gamma_3(F)/U_k$ then N is a $\mathbb{Z}(A)$-submodule in $\gamma_3(F)/U_k$ so $N\overline{\mu}^{-1}$ is a $\mathbb{Z}(A)$-submodule in M_3/M_3T^{-1} and if N is closed under all $\phi \in \Phi$ then so is $N\overline{\mu}^{-1}$.

Now Proposition 2 follows immediately from Corollary 4. Indeed, if

$$N_1 \subset N_2 \subset \cdots$$

is an infinite strictly ascending chain of normal Φ-subgroups of F/U_k contained in $\gamma_3(F)/U_k$ then

$$N_1\overline{\mu}^{-1} \subset N_2\overline{\mu}^{-1} \subset \cdots$$

is an infinite strictly ascending chain of Φ-submodules in M_3/M_3T^{k-1}. This contradicts Corollary 4 and completes the proof of Proposition 2.

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