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RESEARCH ARTICLE

Globally rigid graphs are fully reconstructible

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Abstract

A d-dimensional framework is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to $\mathbb{R}^d$. The length of an edge $uv \in E$ in (G, p) is the distance between $p(u)$ and $p(v)$. The framework is said to be globally rigid in $\mathbb{R}^d$ if the graph G and its edge lengths uniquely determine (G, p), up to congruence. A graph G is called globally rigid in $\mathbb{R}^d$ if every d-dimensional generic framework (G, p) is globally rigid.

In this paper, we consider the problem of reconstructing a graph from the set of edge lengths arising from a generic framework. Roughly speaking, a graph G is strongly reconstructible in $\mathbb{C}^d$ if the set of (unlabeled) edge lengths of any generic framework (G, p) in d-space, along with the number of vertices of G, uniquely determine both G and the association between the edges of G and the set of edge lengths. It is known that if G is globally rigid in $\mathbb{R}^d$ on at least $d + 2$ vertices, then it is strongly reconstructible in $\mathbb{C}^d$. We strengthen this result and show that, under the same conditions, G is in fact fully reconstructible in $\mathbb{C}^d$, which means that the set of edge lengths alone is sufficient to uniquely reconstruct G, without any constraint on the number of vertices (although still under the assumption that the edge lengths come from a generic realization).

As a key step in our proof, we also prove that if G is globally rigid in $\mathbb{R}^d$ on at least $d + 2$ vertices, then the d-dimensional generic rigidity matroid of G is connected. Finally, we provide new families of fully reconstructible graphs and use them to answer some questions regarding unlabeled reconstructibility posed in recent papers.

Contents

1. Introduction

A *(d-dimensional) framework* is a pair (G, p) where $G = (V, E)$ is a graph and $p : V \to \mathbb{R}^d$ is a map that assigns a point in $\mathbb{R}^d$ to each vertex of G. The *length* of an edge uv in (G, p) is the Euclidean distance between $p(u)$ and $p(v)$. We also call (G, p) a *realization* of G in $\mathbb{R}^d$. The framework is *generic* if the set of the coordinates of its points is algebraically independent over $\mathbb{Q}$. We say that a d-dimensional framework (G, p) is *globally rigid* if every other realization (G, q) of G in $\mathbb{R}^d$ in which corresponding edges have the same length is congruent to (G, p). That is, the graph G and its edge lengths in (G, p) uniquely determine the pairwise distances of all vertices in (G, p). It is known that for generic d-dimensional frameworks, global rigidity depends only on the graph G for all $d \geq 1$. We say that G is *(generically) globally rigid in* $\mathbb{R}^d$ if every (equivalently, if some) generic realization of G in $\mathbb{R}^d$ is globally rigid. In the rest of this section, we give a brief overview of our main results. Most of the definitions and more details are given in the next section.

In the context of rigidity theory, unlabeled reconstruction is the study of what combinatorial and geometric information is determined by the (multi)set of edge lengths arising from some d-dimensional framework (G, p). In [12], it was shown that if (G, p) is a generic globally rigid framework with n vertices in $\mathbb{R}^d$, where $n \geq d+2$, then there can be no distinct (not necessarily generic) realization (H, q) of any graph H with n vertices in $\mathbb{R}^d$ that produces the same edge lengths, up to trivialities. This result is essentially tight: If H is allowed to have more vertices or if G is not globally rigid, then p cannot be determined. To prove this result, it was sufficient to study the following, related graph reconstruction question.

We say that a graph G is *strongly reconstructible in* $\mathbb{C}^d$ if, whenever a pair of generic frameworks (G, p) and (H, q) in $\mathbb{C}^d$ have the same set of edge lengths and the same number of vertices, we have that G is isomorphic to H and the corresponding edges have the same length. In [12], it was shown that, for $d \geq 2$, globally rigid graphs in $\mathbb{R}^d$ on at least $d+2$ vertices are strongly reconstructible in $\mathbb{C}^d$. Although this result was sufficient to answer the original question of [12], as a pure reconstruction question, the dependence on n remains unsatisfying. To this end, we call a graph G *fully reconstructible in* $\mathbb{C}^d$ if, whenever a pair of generic frameworks (G, p) and (H, q) in $\mathbb{C}^d$ have the same set of edge lengths, but not necessarily the same number of vertices, we have that G is isomorphic to H and the corresponding edges have the same length. We stress that in the above definition we require both (G, p) and (H, q) to be generic. This, or some other nondegeneracy condition, is necessary since the edge lengths of (G, p) can be realized by an arbitrary forest H on $|E(G)|$ edges (but the realization (H, q) obtained in this way will usually be nongeneric).

As we shall see, both strong reconstructibility and full reconstructibility can be (perhaps more naturally) stated as a uniqueness condition not on the generic realizations of G but rather on a certain complex variety associated to G. This variety, which we call the d-dimensional measurement variety of G and denote by $M_{d,G}$, is the Zariski-closure of the set of all vectors arising as the squared edge lengths of d-dimensional realizations of G. Roughly speaking, G is strongly reconstructible in $\mathbb{C}^d$ if, whenever there is a bijection $\psi : E(G) \to E(H)$ that sends $M_{d,G}$ to $M_{d,H}$, where H has the same number of vertices as G, then G and H are isomorphic and ψ is consistent with an isomorphism between them (for detailed definitions see Subsection 2.6 and Theorem 2.19). Full reconstructibility has a similar characterization, with the only difference being that we do not require H to have the same number of vertices as G. Thus, strong reconstructibility in $\mathbb{C}^d$ asserts that G, in a sense, is uniquely

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determined by n, d and $M_{d,G}$, while full reconstructibility means that G is uniquely determined by d and $M_{d,G}$.

In [12], the question was posed whether global rigidity in $\mathbb{R}^d$ implies full reconstructibility in $\mathbb{C}^d$. For $d = 2$, the question was answered affirmatively in [8]. In this paper, as one of our main results, we substantially strengthen the previous results and show that, for all $d \geq 2$,

$$\textit{if } G \textit{ is globally rigid in } \mathbb{R}^d \textit{ on } n \geq d+2 \textit{ vertices, then } G \textit{ is fully reconstructible in } \mathbb{C}^d.$$

Results in [8] show that, for $d = 2$, this is tight: If a graph is not globally rigid in $\mathbb{R}^2$, then it is not even strongly reconstructible in $\mathbb{C}^2$. The $d = 1$ case is slightly different but also fully characterizable using 3-connectivity. For $d \geq 3$, a characterization of full reconstructibility seems more elusive. In particular, we show that global rigidity in $\mathbb{R}^d$ is not necessary for full reconstructibility in $\mathbb{C}^d$. We also prove some positive and negative results regarding possible sufficient and possible necessary conditions for strong and full reconstructibility. These answer a number of questions that were posed in [12] and [8].

To prove our result on full reconstructibility, we also prove a combinatorial theorem, interesting on its own right. We say that a graph is *$\mathcal{R}_d$-connected* if its d-dimensional (generic) rigidity matroid is connected (see the next section for detailed definitions). A combinatorial characterization of globally rigid graphs in $\mathbb{R}^d$ is known only for $d = 1, 2$ and is a major open problem for $d \geq 3$. In higher dimensions, Hendrickson's theorem (see Theorem 2.1) gives combinatorial necessary conditions that link global rigidity to connectivity and local rigidity properties of G. In this paper, as another main result, we strengthen one of Hendrickson's necessary conditions (redundant rigidity) by proving that, for all $d \geq 1$,

$$\textit{if } G \textit{ is globally rigid in } \mathbb{R}^d \textit{ on } n \geq d+2 \textit{ vertices, then it is } \mathcal{R}_d\textit{-connected}.$$

This result may lead to a better understanding of higher-dimensional global rigidity. In particular, we use it to find new examples of so-called *H-graphs*, graphs that satisfy Hendrickson's conditions but are not globally rigid.

The rest of the paper is laid out as follows. In Section 2, we recall the definitions and results from rigidity theory and algebraic geometry that we shall use throughout the paper. Section 3 contains our main results: After making some structural observations about the measurement variety of graphs, we show that globally rigid graphs in $\mathbb{R}^d$ on at least $d+2$ vertices are $\mathcal{R}_d$-connected (Theorem 3.5) and, for $d \geq 2$, fully reconstructible in $\mathbb{C}^d$ (Theorem 3.6). In Section 4, we illustrate the results of the previous section with several examples. In particular, we use Theorem 3.5 to give new examples of H-graphs. We also answer questions from [8] and [12] related to the unlabeled reconstructibility problem, as well as pose new open questions. Finally, in Section 5, we prove some new results regarding $\mathcal{R}_d$-connected graphs.

2. Preliminaries

We start by fixing some conventions. In the following, graphs will be understood to be simple, that is, without parallel edges and loops. To avoid some trivialities, we shall also implicitly assume that every graph considered has at least one edge (and thus at least two vertices). For a graph $G = (V, E)$, we shall use $\mathbb{R}^E$ and $\mathbb{C}^E$ to denote the $|E|$-dimensional real (complex, respectively) Euclidean space with axes labelled by the edges of G. We shall also often refer to the *configuration spaces* $\mathbb{R}^{nd}$ and $\mathbb{C}^{nd}$, where n denotes the number of vertices of G and $d \geq 1$ is some dimension. We shall really think of these spaces as $(\mathbb{R}^d)^V$ and $(\mathbb{C}^d)^V$, i.e., as n-tuples of d-dimensional vectors, indexed by the vertices of G. Nonetheless, as it is less cumbersome, we shall use the notation $\mathbb{R}^{nd}$ and $\mathbb{C}^{nd}$.

2.1. Real and complex frameworks

Let $G = (V, E)$ be a graph on n vertices and $d \geq 1$ some fixed integer. A *d-dimensional realization* of G is a pair (G, p), where $p : V \to \mathbb{R}^d$ maps the vertices of G into Euclidean space. We call such a map p (or equivalently, a point $p = (p_v)_{v \in V}$ in $\mathbb{R}^{nd}$) a *configuration*, and we say that the pair (G, p) is a *framework*. Two d-dimensional frameworks (G, p) and (G, q) are *equivalent* if $\|p(u) - p(v)\| = \|q(u) - q(v)\|$ for every edge $uv \in E$ and *congruent* if the same holds for every pair of vertices $u, v \in V$. Here, $\|\cdot\|$ denotes the Euclidean norm.

A framework is *(locally) rigid* if every continuous motion of the vertices which preserves the edge lengths takes it to a congruent framework and *globally rigid* if every equivalent framework is congruent to it.

We say that a configuration $p \in \mathbb{R}^{nd}$ is *generic* if its $n \cdot d$ coordinates are algebraically independent over $\mathbb{Q}$. It is known that, in any fixed dimension d, both local and global rigidity are generic properties of the underlying graph, in the sense that either every generic d-dimensional framework is locally/globally rigid or none of them are (see [5, 10]). Thus, we say that a graph is *rigid* (respectively, *globally rigid*) in d dimensions if every (or equivalently, if some) generic d-dimensional realization of the graph is rigid (respectively, globally rigid).

Let $G = (V, E)$ be a graph on n vertices and $d \geq 1$. The function $m_{d,G} : \mathbb{R}^{nd} \to \mathbb{R}^E$ mapping each realization of G to the sequence of its Euclidean squared edge lengths is called the *rigidity map* or *edge measurement map* of G in d dimensions. That is, for a d-dimensional realization (G, p) of G, the coordinate of $m_{d,G}(p)$ corresponding to the edge $uv \in E$ is $\|p(u) - p(v)\|^2$.

Analogously to the real case, we define a d-dimensional *complex framework* to be a pair (G, p), where $G = (V, E)$ is a graph and $p : V \to \mathbb{C}^d$ is a complex mapping. Given a framework (G, p) in $\mathbb{C}^d$ and a pair of vertices u, v in G, we define the *complex squared distance* of u and v in (G, p) by

$$m_{uv}(p) = (p(u) - p(v))^T \cdot (p(u) - p(v)) = \sum_{k=1}^{d} (p(u)_k - p(v)_k)^2,$$

where k indexes over the d dimension-coordinates. Note that in this definition we do not use conjugation, and thus, this is *not* the usual distance of $p(u)$ and $p(v)$. We also note that the mapping $m_{uv} : \mathbb{C}^{nd} \to \mathbb{C}$ depends on G and d, but since these will always be clear from the context, we shall omit them from our notation.

For an edge $e = uv$ of G, we say that $m_{uv}(p)$ is the *complex squared length* of the edge in (G, p). For real frameworks, this coincides with the usual (Euclidean) squared length, so we can extend $m_{d,G}$ to a $\mathbb{C}^{nd} \to \mathbb{C}^E$ function by letting

$$m_{d,G}(p) = \left(m_{uv}(p)\right)_{uv \in E}.$$

We say, as in the real case, that two frameworks (G, p) and (G, q) are *equivalent* if $m_{d,G}(p) = m_{d,G}(q)$, and they are *congruent* if $m_{d,K_V}(p) = m_{d,K_V}(q)$, where K_V is the complete graph on the vertex set V. A configuration $p \in \mathbb{C}^{nd}$ is, again, generic, if the coordinates of p are algebraically independent over $\mathbb{Q}$. A point $p \in \mathbb{R}^{nd}$ is generic as a real configuration precisely if it is generic as a complex one.

Although we will not explicitly need them, we note that one can define the analogues of rigidity and global rigidity for complex frameworks. It turns out that, as in the real case, the (global) rigidity of generic complex frameworks only depends on the underlying graph, and the corresponding graph property of being '(globally) rigid in $\mathbb{C}^d$' is equivalent to being (globally) rigid in $\mathbb{R}^d$; see [11, 12].

It follows from the definitions that globally rigid graphs are rigid. The following much stronger necessary conditions of global rigidity are due to Hendrickson [13]. We say that a graph is *redundantly rigid* in a given dimension if it remains rigid after deleting any edge. A graph is *k-connected* for some $k \geq 2$ if it has at least $k + 1$ vertices and it remains connected after deleting any set of less than k vertices.

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Theorem 2.1 [13]. *Let G be a graph on at least $d+2$ vertices for some $d \geq 1$. Suppose that G is globally rigid in $\mathbb{R}^d$. Then G is $(d+1)$-connected and redundantly rigid in $\mathbb{R}^d$.*

In $d = 1, 2$ dimensions, the conditions of Theorem 2.1 are, in fact, sufficient for global rigidity [14]. This fails in the $d \geq 3$ case, and a combinatorial characterization of globally rigid graphs in these dimensions is a major open question.

2.2. The rigidity matrix and the rigidity matroid

The rigidity matroid of a graph G is a matroid defined on the edge set of G which reflects the rigidity properties of all generic realizations of G. For a general introduction to matroid theory, we refer the reader to [18]. Let (G, p) be a realization of a graph $G = (V, E)$ in $\mathbb{R}^d$. The *rigidity matrix* of the framework (G, p) is the matrix $R(G, p)$ of size $|E| \times d|V|$, where, for each edge $v_i v_j \in E$, in the row corresponding to $v_i v_j$, the entries in the d columns corresponding to vertices v_i and v_j contain the d coordinates of $(p(v_i) - p(v_j))$ and $(p(v_j) - p(v_i))$, respectively, and the remaining entries are zeros. In other words, it is $1/2$ times the Jacobian of the rigidity map $m_{d,G}$. The rigidity matrix of (G, p) defines the *rigidity matroid* of (G, p) on the ground set E by linear independence of rows. It is known that any pair of generic frameworks (G, p) and (G, q) have the same rigidity matroid. We call this the d-dimensional *rigidity matroid* $\mathcal{R}_d(G) = (E, r_d)$ of the graph G. We can define the rigidity matrix $R(G, p)$ for complex frameworks in the same way as in the real case. This, again, allows us to define the rigidity matroid of the framework. It is not difficult to show that the rigidity matroid of a generic framework in $\mathbb{C}^d$ is, again, the d-dimensional rigidity matroid $\mathcal{R}_d(G)$.

We denote the rank of $\mathcal{R}_d(G)$ by $r_d(G)$. A graph $G = (V, E)$ is *$\mathcal{R}_d$-independent* if $r_d(G) = |E|$, and it is an *$\mathcal{R}_d$-circuit* if it is not $\mathcal{R}_d$-independent, but every proper subgraph G' of G is $\mathcal{R}_d$-independent. We note that in the literature such graphs are sometimes called M-independent in $\mathbb{R}^d$ and M-circuits in $\mathbb{R}^d$, respectively. An edge e of G is an *$\mathcal{R}_d$-bridge in G* if $r_d(G - e) = r_d(G) - 1$ holds. Equivalently, e is an $\mathcal{R}_d$-bridge in G if it is not contained in any subgraph of G that is an $\mathcal{R}_d$-circuit.

Gluck characterized rigid graphs in terms of their rank.

Theorem 2.2 [9]. *Let $G = (V, E)$ be a graph with $|V| \geq d + 1$. Then G is rigid in $\mathbb{R}^d$ if and only if $r_d(G) = d|V| - \binom{d+1}{2}$.*

Let M be a matroid on ground set E with rank function r. We can define a relation on the pairs of elements of E by saying that $e, f \in E$ are equivalent if $e = f$ or there is a circuit C of M with $\{e, f\} \subseteq C$. This defines an equivalence relation. The equivalence classes are the *connected components* of M. The matroid is said to be *connected* if there is only one equivalence class and *separable* otherwise. We shall use the fact that M is separable if and only if there is a partition $E = E_1 \cup E_2$ of E into two nonempty subsets for which

$$r(M) = r(M_1) + r(M_2),$$

holds, where M_i denotes the restriction of M to E_i, $i = 1, 2$.

Given a graph $G = (V, E)$, the subgraphs induced by the edge sets of the connected components of $\mathcal{R}_d(G)$ are the *$\mathcal{R}_d$-connected components* of G. The graph is said to be *$\mathcal{R}_d$-connected* if $\mathcal{R}_d(G)$ is connected, and *$\mathcal{R}_d$-separable* otherwise. See Figure 1 for an example of an $\mathcal{R}_3$-separable graph.

In one and two dimensions, $\mathcal{R}_d$-connectivity has close ties with (global) rigidity, as shown by the following result.

Theorem 2.3 [14]. *Let G be a graph without isolated vertices and $d \in \{1, 2\}$. Then*

1. *If G is globally rigid in $\mathbb{R}^d$ on at least $d + 2$ vertices, then it is $\mathcal{R}_d$-connected.*
2. *If G is $\mathcal{R}_d$-connected, then it is redundantly rigid in $\mathbb{R}^d$.*

It is known that part *(b)* of Theorem 2.3 is not true in $d \geq 3$ dimensions. On the other hand, we shall show that part *(a)* remains valid in d dimensions for all $d \geq 3$ (Theorem 3.5). We note that this is essentially a strengthening of the second part of Hendrickson's theorem (Theorem 2.1), which can

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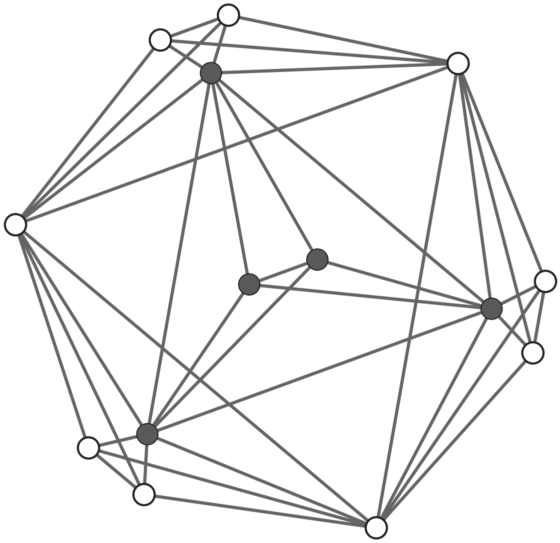

Figure 1. *A 3-connected, redundantly rigid and $\mathcal{R}_3$-separable graph. This graph satisfies $r_3(G) = 36 = 27 + 9 = r_3(G^o) + r_3(K_5)$, where G^o is the outer ring of K_5's and K_5 is the subgraph induced by the black (filled) vertices.*

be seen as follows. A graph is redundantly rigid in $\mathbb{R}^d$ if and only if it is rigid in $\mathbb{R}^d$ and contains no $\mathcal{R}_d$-bridges, and $\mathcal{R}_d$-connected graphs contain no $\mathcal{R}_d$-bridges (indeed, e is an $\mathcal{R}_d$-bridge in G if and only if $\{e\}$ is an $\mathcal{R}_d$-connected component of G). Since a globally rigid graph is clearly rigid, Theorem 3.5 implies that such a graph (on at least $d + 2$ vertices) is redundantly rigid as well.

Let $G = (V, E)$ be a graph on n vertices and (G, p) a framework in $\mathbb{C}^d$. The elements of $\ker(R(G, p)) \subseteq \mathbb{C}^{nd}$ are the *infinitesimal motions* of (G, p), while the elements of $\ker(R(G, p)^T) \subseteq \mathbb{C}^E$ are the *equilibrium stresses* (or *stresses*, for short) of (G, p). We shall also use the notation $S(G, p) = \ker(R(G, p)^T)$. Let us denote the column space of $R(G, p)$ by $\operatorname{span}(R(G, p))$. By basic linear algebra, $S(G, p)$ is the orthogonal complement[1] of $\operatorname{span}(R(G, p))$ in $\mathbb{C}^E$. Since the elements of $S(G, p)$ capture the row dependences of $R(G, p)$, we have that G is $\mathcal{R}_d$-independent if and only if for every generic realization (G, p) in $\mathbb{C}^d$, we have $S(G, p) = \{0\}$, and G is an $\mathcal{R}_d$-circuit if and only if every generic realization (G, p) in $\mathbb{C}^d$ has a unique (up to scalar multiple) nonzero equilibrium stress ω and ω is nonzero on every edge of G.

Suppose that G is $\mathcal{R}_d$-separable and let $E = E_1 \cup E_2$ be a partition of E into nonempty subsets such that for the graphs G_i induced by $E_i, i = 1, 2$, we have $r_d(G) = r_d(G_1) + r_d(G_2)$. It is not difficult to see that, in this case for any generic realization (G, p) in $\mathbb{C}^d$, we have $\operatorname{span}(R(G, p)) = \operatorname{span}(R(G_1, p)) \oplus \operatorname{span}(R(G_2, p))$ as linear subspaces of $\mathbb{C}^E$, under the identification $\mathbb{C}^E = \mathbb{C}^{E_1} \times \mathbb{C}^{E_2}$. This also implies $S(G, p) = S(G_1, p) \oplus S(G_2, p)$ under the same identification.

We close this section by recalling the well-known coning operation on a graph. Given a graph G, the *cone* of G is obtained by adding a new vertex v to G, along with new edges from v to every vertex of G. Coning provides a transfer between various rigidity properties in d and $d + 1$ dimensions, as shown by the following results.

Theorem 2.4. *Let $d \geq 1$, and let $G = (V, E)$ be a graph.*

1. *[22] G is rigid in $\mathbb{R}^d$ ($\mathcal{R}_d$-independent, respectively) if and only if the cone graph G^v is rigid in $\mathbb{R}^{d+1}$ ($\mathcal{R}_{d+1}$-independent, respectively).*
2. *[6] G is globally rigid in $\mathbb{R}^d$ if and only if the cone graph G^v is globally rigid in $\mathbb{R}^{d+1}$.*

[1]Note that here, as well as later on, we shall say that two vectors $x, y \in \mathbb{C}^E$ are orthogonal if $x^T y = 0$; that is, we work with the symmetric bilinear form $\langle x, y \rangle = x^T y$, and *not* the Hermitian bilinear form $\langle x, y \rangle = x^* y$.

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2.3. *Affine maps and conics at infinity*

We say that a framework (G, p) in $\mathbb{C}^d$ has *full affine span* if the affine span of the image of the vertices under p is all of $\mathbb{C}^d$. A configuration $q \in \mathbb{C}^{nd}$, viewed as a point $q = (q_v)_{v \in V}$, is an *affine image* of p if $q_v = Ap_v + b, v \in V$ for some matrix $A \in \mathbb{C}^{d \times d}$ and vector $b \in \mathbb{C}^d$. We say that p and q are *strongly congruent* if q can be obtained as the affine image of p under a rigid motion, i.e., an affine map $x \mapsto Ax + b$ such that $A^T A$ is the identity matrix.

Two frameworks (G, p) and (G, q) in $\mathbb{R}^d$ are strongly congruent if and only if they are congruent. This is not always the case for frameworks in $\mathbb{C}^d$. However, congruent frameworks that have full affine span are strongly congruent; see [11, Corollary 8].

Let (G, p) be a framework in $\mathbb{C}^d$. We say that the *edge directions of* (G, p) *lie on a conic at infinity* if there is a nonzero symmetric matrix Q such that, for every edge uv of G, $(p(u) - p(v))^T Q(p(u) - p(v)) = 0$. The following lemma is implied by results of Connelly (see, e.g., [5, Proposition 4.2]). For completeness, we give a proof.

Lemma 2.5. *Let $G = (V, E)$ be a graph and (G, p) a framework in $\mathbb{C}^d$ such that its edge directions do not lie on a conic at infinity. Let (G, q) be a framework such that q is an affine image of p. Then $m_{d,G}(q) = m_{d,G}(p)$ if and only if q and p are congruent.*

Proof. The 'if' direction is immediate. In the other direction, suppose that $m_{d,G}(q) = m_{d,G}(p)$. Let $x \mapsto Ax + b$ be an affine transformation that sends p to q. It follows from the definitions that, for any pair of vertices $u, v \in V$,

$$m_{uv}(q) = (p(u) - p(v))^T A^T A(p(u) - p(v)).$$

Therefore we have

$$m_{uv}(q) - m_{uv}(p) = (p(u) - p(v))^T (A^T A - I)(p(u) - p(v)). \tag{2.1}$$

By assumption, for every edge $uv \in E$, the left-hand side of equation (2.1) is zero. Since the edge directions of (G, p) do not lie on a conic at infinity, this implies $A^T A - I = 0$ so that the left-hand side is zero for every pair of vertices $u, v \in V$, which is what we wanted to show. □

The following lemma is stated in [5] for frameworks in $\mathbb{R}^d$, but the same proof works for frameworks in $\mathbb{C}^d$.

Lemma 2.6 [5, Proposition 4.3]. *Let G be a graph in which each vertex has degree at least d. Then for every generic realization (G, p) in $\mathbb{C}^d$, the edge directions of (G, p) do not lie on a conic at infinity.*

The following lemma is folklore.

Lemma 2.7. *Let $(G, p), (G, q)$ be frameworks in $\mathbb{C}^d$ and suppose that q is an affine image of p. Then $S(G, p) \subseteq S(G, q)$. If both (G, p) and (G, q) have full affine span, then $S(G, p) = S(G, q)$.*

Proof. Let $x \mapsto Ax + b$ be the affine transformation that maps p to q, and let A' be the $nd \times nd$ block matrix with n copies of A in its diagonal and zeroes elsewhere, where n denotes the number of vertices of G. In other words, A' is the Kronecker product $I_n \otimes A$ of the $n \times n$ identity matrix and A.

Direct calculation shows that $R(G, q) = R(G, p)A'$, which immediately implies $S(G, p) = \ker(R(G, p)^T) \subseteq \ker(R(G, q)^T) = S(G, q)$. If (G, p) and (G, q) have full affine span, then the affine map sending p to q must necessarily be invertible so that p is an affine image of q as well, implying $S(G, q) \subseteq S(G, p)$. □

Finally, we shall use the following property of globally rigid graphs which is easy to deduce from previous results on global rigidity and maximum rank stress matrices. We sketch the proof and refer the reader to [5, 10] for the definitions and key theorems.

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Theorem 2.8. *Let G be a globally rigid graph on $n \geq d+2$ vertices in $\mathbb{R}^d$, for some $d \geq 1$ and (G, p) a generic realization of G in $\mathbb{C}^d$. For every realization (G, q) in $\mathbb{C}^d$ with $S(G, p) = S(G, q)$, we must have that q is an affine image of p.*

Proof. Let (G, p_0) be a generic realization of G in $\mathbb{R}^d$. It was shown in [10] that there exists an equilibrium stress ω_0 for (G, p_0) for which the associated stress matrix has rank $n-d-1$. The complex version of [10, Lemma 5.8] then implies that (G, p) has an equilibrium stress ω such that the associated stress matrix has rank $n-d-1$.

If $S(G, p) = S(G, q)$ for some realization (G, q), then ω is a stress for (G, q) as well. Then (the complex version of) [5, Proposition 1.2] implies that q is an affine image of p. □

2.4. Algebraic geometry background

We briefly recall the notions from algebraic geometry that we shall use. For a more detailed exposition, see [12, Appendix A] or [8, Section 2.2]. We say that a subset $X \subseteq \mathbb{C}^m$ is a *variety* if it is the set of simultaneous vanishing points of some polynomials $f_1, \ldots, f_k \in \mathbb{C}[x_1, \ldots, x_m]$. The varieties in $\mathbb{C}^m$ form the closed sets of the so-called *Zariski topology*. For an arbitrary set $X \subseteq \mathbb{C}^m$, we shall use $\overline{X}$ to denote its closure in the Zariski topology on $\mathbb{C}^m$.

Let $X \subseteq \mathbb{C}^m$ be a variety. We denote by $I(X) \subseteq \mathbb{C}[x_1, \ldots, x_m]$ the set of polynomials that vanish on X. We say that X is *irreducible* if it cannot be written as the proper union of a finite number of varieties, and it is *defined over* $\mathbb{Q}$ if $I(X)$ has a generating set consisting of polynomials with rational coefficients. The *dimension* of an irreducible variety X is the largest number k such that there exists a chain $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_k = X$ of irreducible varieties. From this definition, the following useful fact is immediate: If $X, Y \subseteq \mathbb{C}^m$ are irreducible varieties of the same dimension with $X \subseteq Y$, then $X = Y$. We shall also use the following result.

Theorem 2.9 [21, Chapter 3, Theorem 1.6 and Chapter 6, Example 1.33]. *Let $X \subseteq \mathbb{C}^{m_1}, Y \subseteq \mathbb{C}^{m_2}$ irreducible varieties. Then the Cartesian product $X \times Y \subseteq \mathbb{C}^{m_1+m_2}$ is an irreducible variety of dimension* $\dim(X) + \dim(Y)$.

Let $X \subseteq \mathbb{C}^m$ be an irreducible variety. At each point $x \in X$, we define the *Zariski tangent space* of X at x, denoted by $T_x X$, to be the kernel of the Jacobian matrix of a set of generating polynomials of $I(X)$, evaluated at x. Thus, $T_x X$ is a linear subspace of $\mathbb{C}^m$. We say that x is *smooth* if $\dim(T_x X) = \dim(X)$. If X is *homogeneous* (i.e., it can be defined by homogeneous polynomials, or equivalently, $tx \in X$ for every $x \in X$ and $t \in \mathbb{C}$) and $x \in X$ is a smooth point, we define the *Gauss fiber corresponding to* x to be the set $\{y \in X : y \text{ is smooth and } T_y X = T_x X\}$.[2]

Let $X \subseteq \mathbb{C}^m$ be a variety defined over $\mathbb{Q}$. We say that a point $x \in X$ is *generic in* X if the only polynomials with rational coefficients satisfied by x are those in $I(X)$. Note that a framework (G, p) in $\mathbb{C}^d$ is generic if and only if p is generic as a point of the variety $\mathbb{C}^{nd}$. If X is an irreducible variety defined over $\mathbb{Q}$, then every generic point of X is smooth. We shall also need the following result.

Lemma 2.10 [12, Lemma A.6]. *Let $X \subseteq Y$ be irreducible varieties, with Y defined over $\mathbb{Q}$. Suppose that X has at least one point which is generic in Y. Then the points in X which are generic in Y are Zariski-dense in X.*

2.5. The measurement variety

Recall that for a graph $G = (V, E)$, we denote its d-dimensional edge measurement map by $m_{d,G} : \mathbb{C}^{nd} \to \mathbb{C}^E$.

[2]In other words, the Gauss fiber corresponding to x is the fiber over $T_x X$ of the rational map $X \dashrightarrow Gr(\dim(X), \mathbb{C}^m)$, defined by the mapping $x \mapsto T_x X$ on the smooth locus of X. Here, $Gr(\dim(X), \mathbb{C}^E)$ denotes the Grassmannian variety of $\dim(X)$-dimensional linear subspaces of $\mathbb{C}^n$.

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Definition 2.11. The *d-dimensional measurement variety* of a graph G (on n vertices), denoted by $M_{d,G}$, is the Zariski-closure of $m_{d,G}(\mathbb{C}^{nd})$.

We shall frequently use the following lemma on generic points. It follows by applying [12, Lemmas 4.4, A.7, A.8] to the varieties $\mathbb{C}^{nd}$, $M_{d,G}$ and the map $m_{d,G}$.

Lemma 2.12. *Let $x \in M_{d,G}$ be a point in the measurement variety of G. Then x is generic in $M_{d,G}$ if and only if there is a generic point $p \in \mathbb{C}^{nd}$ for which $x = m_{d,G}(p)$.*

It is known that the measurement variety, being the Zariski-closure of the image of an irreducible variety defined over $\mathbb{Q}$, is also an irreducible variety defined over $\mathbb{Q}$. It follows from the definition and basic topological considerations that if $E' \subseteq E$ is a subset of edges inducing a subgraph G' of G, then $M_{d,G'} = \overline{\pi_{E'}(M_{d,G})}$, where $\pi_{E'} : \mathbb{C}^E \to \mathbb{C}^{E'}$ is the projection onto the coordinate axes corresponding to E'; see [8, Lemma 3.8].

In what follows, we shall frequently compare the measurement varieties of different graphs, say $G = (V, E)$ and $H = (V', E')$, that have the same number of edges. Since $M_{d,G}$ and $M_{d,H}$ lie in different ambient spaces, to compare them we must specify an identification between $\mathbb{C}^E$ and $\mathbb{C}^{E'}$. To this end, we introduce the following notation. Let $\psi : E \to E'$ be a bijection between the edge sets of G and H. We write that $M_{d,G} =_\psi M_{d,H}$ if $\Psi(M_{d,G}) = M_{d,H}$, where $\Psi : \mathbb{C}^E \to \mathbb{C}^{E'}$ is the mapping induced by ψ in the natural way. Similarly, we write $M_{d,G} \subseteq_\psi M_{d,H}$ if $\Psi(M_{d,G}) \subseteq M_{d,H}$.

The following results show that $\mathcal{R}_d(G)$ is 'encoded' in the measurement variety in some sense. This has been observed before; see, e.g., [8, 12, 19]. Using the terminology of the latter paper, the situation can be summarized by saying that the algebraic matroid corresponding to the variety $M_{d,G}$ is $\mathcal{R}_d(G)$.

Lemma 2.13. *Let G be a graph on n vertices. Then*

$$\dim(M_{d,G}) = r_d(G).$$

In particular, for $n \geq d+1$ we have $\dim(M_{d,G}) \leq nd - \binom{d+1}{2}$ and equality holds if and only if G is rigid in $\mathbb{R}^d$. Moreover, G is $\mathcal{R}_d$-independent if and only if $M_{d,G} = \mathbb{C}^E$.

Theorem 2.14. *Let G and H be graphs with the same number of edges, and suppose that $M_{d,G} =_\psi M_{d,H}$ under some edge bijection $\psi : E(G) \to E(H)$. Then ψ defines an isomorphism between $\mathcal{R}_d(G)$ and $\mathcal{R}_d(H)$.*

The following result shows that the measurement variety also encodes the space of stresses of generic frameworks. This follows from the fact that $S(G, p)$ is the orthogonal complement of $\mathrm{span}(R(G, p))$ in $\mathbb{C}^E$ using standard results in differential geometry; see [10, Lemma 2.21] or [12, Lemma 4.10].

Lemma 2.15. *Let G be a graph and (G, p) a generic realization in $\mathbb{C}^d$ for some $d \geq 1$. Let $x = m_{d,G}(p) \in M_{d,G}$. Then the space of stresses $S(G, p)$ is the orthogonal complement of the tangent space $T_x(M_{d,G})$ in $\mathbb{C}^E$.*

The lemma implies that if (G, p) and (G, q) are generic frameworks in $\mathbb{C}^d$, then $S(G, p) \neq S(G, q)$ if and only if the Gauss fibers corresponding to $m_{d,G}(p)$ and $m_{d,G}(q)$ are different. We shall use this corollary later.

2.6. *Unlabeled reconstruction*

In what follows, it will be convenient to use the following notions. We say that two frameworks (G, p) and (H, q) in $\mathbb{C}^d$ are *length-equivalent* (*under the bijection* ψ) if there is a bijection ψ between the edge sets of G and H such that, for every edge e of G, the complex squared length of e in (G, p) is equal to the complex squared length of $\psi(e)$ in (H, q). For an edge bijection $\psi : E(G) \to E(H)$ and a graph isomorphism $\varphi : V(G) \to V(H)$, we say that ψ *is induced by* φ if for every edge $e = uv \in E(G)$ we have $\psi(e) = \varphi(u)\varphi(v)$.

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Definition 2.16. Let (G, p) be a generic realization of the graph G in $\mathbb{C}^d$. We say that (G, p) is *strongly reconstructible* if for every generic framework (H, q) in $\mathbb{C}^d$ that is length-equivalent to (G, p) under some edge bijection $\psi : E(G) \to E(H)$, where H has the same number of vertices as G, ψ is induced by a graph isomorphism $\varphi : V(G) \to V(H)$.

In this paper, we shall mainly focus on the following stronger property, where the condition on the number of vertices of H is omitted.

Definition 2.17. Let G be a graph without isolated vertices, and let (G, p) be a generic realization of G in $\mathbb{C}^d$. We say that (G, p) is *fully reconstructible* if, for every generic framework (H, q) in $\mathbb{C}^d$ that is length-equivalent to (G, p) under some edge bijection $\psi : E(G) \to E(H)$, where H has no isolated vertices, ψ is induced by a graph isomorphism $\varphi : V(G) \to V(H)$.

Note that, since we assume (G, p) to be generic, its edge lengths are pairwise distinct, and hence, the bijection ψ is unique in the above definitions. We also point out that, when considering full reconstructibility, it is natural to only consider graphs without isolated vertices since from any framework we can create other length-equivalent frameworks by adding isolated vertices. On the other hand, in the case of strong reconstructibility it is sensible to consider graphs with isolated vertices. In fact, it follows immediately from the definitions that G is fully reconstructible in $\mathbb{C}^d$ if and only if every graph obtained from G by adding zero or more isolated vertices is strongly reconstructible in $\mathbb{C}^d$.

As Theorem 2.19 below shows, both strong and full reconstructibility of a generic framework can be characterized in terms of a certain uniqueness condition on the measurement variety $M_{d,G}$ of the underlying graph, and in fact it is this formulation of reconstructibility that we shall most commonly use throughout the paper. This also implies that these reconstructibility notions are generic properties of a graph in the sense that if there is a generic framework (G, p) in $\mathbb{C}^d$ which is strongly (respectively, fully) reconstructible, then every generic realization of G in $\mathbb{C}^d$ is strongly (respectively, fully) reconstructible. This motivates the following definition.

Definition 2.18. A graph G is said to be *(generically) strongly reconstructible* (respectively, *(generically) fully reconstructible*) in $\mathbb{C}^d$ if every generic realization (G, p) of G in $\mathbb{C}^d$ is strongly (respectively, fully) reconstructible.

Theorem 2.19. *Let G be a graph and $d \geq 1$ be fixed. The following are equivalent.*

1. *G is generically strongly reconstructible (generically fully reconstructible, respectively) in $\mathbb{C}^d$.*
2. *There exists some generic framework (G, p) in $\mathbb{C}^d$ that is strongly reconstructible (fully reconstructible, respectively).*
3. *Whenever $M_{d,G} =_\psi M_{d,H}$ under an edge bijection $\psi : E(G) \to E(H)$ for some graph H, where H has the same number of vertices as G (where H has an arbitrary number of vertices, respectively), ψ is induced by a graph isomorphism.*

The 'strongly reconstructible' part of Theorem 2.19 is [8, Theorem 3.4]. The same proof works for the 'fully reconstructible' version after omitting the condition on the number of vertices of H.

We close this section by recalling the main result of [12].

Theorem 2.20 [12, Theorem 3.4]. *Let G be a graph on at least $d+2$ vertices, where $d \geq 1$. Suppose that*

- *$d = 1$ and G is 3-connected, or*
- *$d \geq 2$ and G is globally rigid in $\mathbb{R}^d$.*

Then G is strongly reconstructible in $\mathbb{C}^d$.

In the next section, we shall strengthen this result by proving that globally rigid graphs on at least $d + 2$ vertices are, in fact, fully reconstructible in $\mathbb{C}^d$ for $d \geq 2$. The cases $d = 1, 2$ were already settled in [8] by verifying the following equivalence.

Theorem 2.21 [8, Theorem 5.19, Corollary 5.22, Theorem 5.1]. *Let G be a graph on at least $d + 2$ vertices and without isolated vertices, where $d \in \{1, 2\}$. Then the following are equivalent.*

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- *$d = 1$ and G is 3-connected or $d = 2$ and G is globally rigid in $\mathbb{R}^2$.*
- *G is strongly reconstructible in $\mathbb{C}^d$.*
- *G is fully reconstructible in $\mathbb{C}^d$.*

In Section 4, we shall give examples showing that, for $d \geq 3$, there are fully reconstructible graphs in $\mathbb{C}^d$ (on at least $d + 2$ vertices) that are not globally rigid in $\mathbb{R}^d$. On the other hand, we do not know any example of a strongly reconstructible graph in $\mathbb{C}^d$ on at least $d+2$ vertices that is not fully reconstructible in $\mathbb{C}^d$, although it seems likely that such a graph exists.

3. Necessary conditions for global rigidity

In this section, we prove our main results: Globally rigid graphs in $\mathbb{R}^d$ on at least $d + 2$ vertices are $\mathcal{R}_d$-connected (Theorem 3.5) and fully reconstructible in $\mathbb{C}^d$ (Theorem 3.6). We start with some technical results about the structure of the measurement variety that we shall use in these proofs. Apart from Lemma 3.1, the lemmas in the next subsection are implicit in [12].

3.1. The structure of the measurement variety

The next lemma implies that the measurement variety of an $\mathcal{R}_d$-separable graph G is the product of the measurement varieties of its $\mathcal{R}_d$-connected components. A special case of this statement when G contains an $\mathcal{R}_d$-bridge was proved in [8, Theorem 3.13].

Lemma 3.1. *Let $d \geq 1$ and let $G = (V, E)$ be a graph. Suppose that there is a partition $E = E_1 \cup E_2$ of E into nonempty subsets such that $r_d(E) = r_d(E_1) + r_d(E_2)$. Let G_1 and G_2 be the subgraphs induced by E_1 and E_2, respectively. Then $M_{d,G} = M_{d,G_1} \times M_{d,G_2}$ (under the identification $\mathbb{C}^E = \mathbb{C}^{E_1} \times \mathbb{C}^{E_2}$).*

Proof. For $i = 1, 2$, M_{d,G_i} arises as the Zariski-closure of the projection of $M_{d,G}$ onto the coordinate axes corresponding to E_i, so we have that $M_{d,G} \subseteq M_{d,G_1} \times M_{d,G_2}$. By Theorem 2.9 and Lemma 2.13, $M_{d,G_1} \times M_{d,G_2}$ is an irreducible variety of dimension $r_d(E_1) + r_d(E_2)$. Since this dimension equals the dimension $r_d(E)$ of the irreducible variety $M_{d,G}$, the two varieties must be equal. □

Lemma 3.2. *Let G be a graph and (G, p) a generic framework in $\mathbb{C}^d$ with full affine span. Let $\mathcal{A} \subseteq \mathbb{C}^{nd}$ denote the set of affine images of p, and let $F \subseteq M_{d,G}$ be the Gauss fiber corresponding to $m_{d,G}(p)$. Then $\overline{m_{d,G}(\mathcal{A})} \subseteq \overline{F}$.*

Proof. For clarity, we shall write m and M instead of $m_{d,G}$ and $M_{d,G}$ in the following. For any $q \in \mathcal{A}$, if q has full affine span, then by Lemma 2.7, we have $S(G, p) = S(G, q)$.

Let $\mathcal{A}^g$ denote the set of frameworks in $\mathcal{A}$ that are generic. Since (G, p) is generic, this set is nonempty, and since $\mathcal{A}$ is irreducible (being a linear space), Lemma 2.10 implies $\overline{\mathcal{A}^g} = \mathcal{A}$. Now for any $q \in \mathcal{A}^g$, we have $S(G, q) = S(G, p)$ by Lemma 2.7, and it follows by Lemma 2.15 that $T_{m(q)}M = T_{m(p)}M$, or in other words, $m(q) \in F$.

This shows that $m(\mathcal{A}^g) \subseteq F$. Taking Zariski-closures and using the continuity of m with respect to the Zariski topology, we have

$$\overline{F} \supseteq \overline{m(\mathcal{A}^g)} = \overline{m(\overline{\mathcal{A}^g})} = \overline{m(\mathcal{A})},$$

as desired. □

Although we shall not use this fact, we note that, by [12, Lemma 4.6], $m_{d,G}(\mathcal{A})$ is a linear space, and in particular it is closed.

Let G be a graph and $d \geq 1$. We say that a Gauss fiber F of $M_{d,G}$ is *generic* if it contains a point that is generic in $M_{d,G}$. It will also be convenient to use the following notion in the next proof. Let $d \geq 2$, and let n denote the number of vertices of G. We say that a framework (G, p) in $\mathbb{C}^d$ is a *lifting* of the framework (G, q) in $\mathbb{C}^{d-1}$ if (G, q) is obtained by projecting the image of each vertex in (G, p) onto the

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first $d-1$ coordinate axes. If (G,q) is generic, then the generic liftings of (G,q) form a dense subset of the space of liftings of (G,q). This follows from the basic fact that for any finite set $S \subseteq \mathbb{C}$ that is algebraically independent over $\mathbb{Q}$, the numbers $x \in \mathbb{C}$ for which $S \cup \{x\}$ is also algebraically independent form a dense subset of $\mathbb{C}$.

Lemma 3.3. *Let $G=(V,E)$ be a graph and $d \geq 2$.*

1. *If G is not $\mathcal{R}_d$-independent, then for every point $x \in M_{d-1,G} \subseteq M_{d,G}$ that is generic in $M_{d-1,G}$ there are an infinite number of generic Gauss fibers F of $M_{d,G}$ with $x \in \overline{F}$.*
2. *If G is globally rigid in $\mathbb{R}^d$ on at least $d+2$ vertices and $x \in M_{d,G} \setminus M_{d-1,G}$, then there are at most a finite number of generic Gauss fibers F of $M_{d,G}$ with $x \in \overline{F}$.*

Proof. a) [Following [12, Proposition 4.21]] For clarity, we shall write m instead of $m_{d,G}$ in the following. We note first that, since G is not $\mathcal{R}_d$-independent, it has at least $d+2$ vertices and consequently any generic realization of G in $\mathbb{C}^d$ has full affine span.

Let $x \in M_{d-1,G}$ be a generic point. By Lemma 2.12, there is a generic framework (G,q) in $\mathbb{C}^{d-1}$ with $m_{d-1,G}(q)=x$. It is enough to find an infinite sequence of generic frameworks $(G,p_i), i \in \mathbb{N}$ in $\mathbb{C}^d$, with corresponding (generic) Gauss fibers $F_i, i \in \mathbb{N}$ such that $F_i \neq F_j$ for $i \neq j$ and such that q is an affine image of p_i since by Lemma 3.2 this implies $x \in \overline{F}_i$.

We shall find such frameworks (G,p_i) inductively. In fact, each framework will be a lifting of (G,q). For the base case, let p_1 be an arbitrary generic lifting of (G,q). Now suppose that for some $i>1$, we have already found suitable frameworks $(G,p_j), j<i$. Since G is not $\mathcal{R}_d$-independent, each of these frameworks has a nonzero stress ω_j. For a given lifting (G,p) of (G,q) to have ω_j as an equlibrium stress, the last coordinates $p(v)_d, v \in V$ must satisfy $|V|$ linear equations determined by ω_j, and since ω_j is nonzero, some of these equations are nontrivial. It follows that, for each $j<i$, the liftings of (G,q) that do not have ω_j as an equilibrium stress form a dense open subset of the space of liftings of (G,q). This implies that we can find a generic lifting (G,p_i) that does not satisfy any of the stresses $\omega_j, j<i$, and so in particular $S(G,p_i) \neq S(G,p_j)$ for $j<i$. Let F_i denote the Gauss fiber corresponding to $m(p_i)$. By Lemma 2.15, we must have $T_{m(p_i)}M_{d,G} \neq T_{m(p_j)}M_{d,G}$, and hence $F_i \neq F_j$ for $j<i$.

b) This is an immediate consequence of Proposition 4.20 and Remark 4.8 of [12]. □

Corollary 3.4. *Let G be a globally rigid graph in $\mathbb{R}^d$ on at least $d+2$ vertices for some $d \geq 2$, and suppose that $M_{d,G} =_\psi M_{d,H}$ under some edge bijection ψ for some graph H not necessarily on the same number of vertices as G. Then $M_{d-1,H} \subseteq_\psi M_{d-1,G}$.*

Proof. For clarity, we shall suppress the role of ψ and assume that $M_{d,G}$ and $M_{d,H}$ are in the same ambient space $\mathbb{C}^E$ so that $M_{d,G}=M_{d,H}$. Let R be the ring of polynomials over $\mathbb{C}$ with variables indexed by E and let $I(M_{d-1,G}), I(M_{d-1,H}) \subseteq R$ be the set of polynomials vanishing on $M_{d-1,G}$ and $M_{d-1,H}$, respectively. Moreover, let $R_\mathbb{Q}, I_\mathbb{Q}(M_{d-1,G})$ and $I_\mathbb{Q}(M_{d-1,H})$ denote the subset of $R, I(M_{d-1,G})$ and $I(M_{d-1,H})$, respectively, consisting of polynomials with rational coefficients.

Theorem 2.1 implies that G has no $\mathcal{R}_d$-bridges, so in particular it cannot be $\mathcal{R}_d$-independent. By Theorem 2.14, it follows that H is not $\mathcal{R}_d$-independent either, and thus, part *a)* of Lemma 3.3 implies that there is a generic point $x \in M_{d-1,H} \subseteq M_{d,H}$ such that there is an infinite number of generic Gauss fibers $F \subseteq M_{d,H}$ with $x \in \overline{F}$. Part *b)* of the same lemma then implies that x is in $M_{d-1,G}$. From this and the fact that x is generic in $M_{d-1,H}$, we have the following chain of containments:

$$I_\mathbb{Q}(M_{d-1,G}) \subseteq \{f \in R_\mathbb{Q} : f(x)=0\} = I_\mathbb{Q}(M_{d-1,H}). \tag{3.1}$$

Since both $M_{d-1,G}$ and $M_{d-1,H}$ are defined over $\mathbb{Q}$, $I_\mathbb{Q}(M_{d-1,G})$ and $I_\mathbb{Q}(M_{d-1,H})$ generate $I(M_{d-1,G})$ and $I(M_{d-1,H})$, respectively. Thus, equation (3.1) also implies $I(M_{d-1,G}) \subseteq I(M_{d-1,H})$, which is equivalent to $M_{d-1,H} \subseteq M_{d-1,G}$. □

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3.2. Globally rigid graphs are $\mathcal{R}_d$-connected

We are ready to prove the first main result of this section.

Theorem 3.5. *Let $G = (V, E)$ be a globally rigid graph in $\mathbb{R}^d$ on $n \geq d + 2$ vertices. Then G is $\mathcal{R}_d$-connected.*

Proof. Let (G, p) be a generic framework in $\mathbb{C}^d$. We shall show that if G is not $\mathcal{R}_d$-connected, then there is a framework (G, q) in $\mathbb{C}^d$ such that $S(G, p) = S(G, q)$ but where q is not an affine image of p. Then, from Theorem 2.8, G cannot be globally rigid in $\mathbb{R}^d$, and we are done.

If G is not $\mathcal{R}_d$-connected, there must be a partition $E = E_1 \cup E_2$ of E into nonempty subsets with $r_d(E) = r_d(E_1) + r_d(E_2)$. Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, and let $(G_1, p), (G_2, p)$ denote the respective subframeworks of (G, p).

By Theorem 2.1, G contains no $\mathcal{R}_d$-bridges, that is, every edge is contained in an $\mathcal{R}_d$-circuit. Since any $\mathcal{R}_d$-circuit that contains edges from E_1 must be contained in G_1, we have that G_1 contains an $\mathcal{R}_d$-circuit, so in particular it has a subgraph of minimum degree at least $d+1$. Thus, by Lemma 2.6, applied to this subgraph, the edge directions of (G_1, p) do not lie on a conic at infinity.

Now we find our promised (G, q). By Lemma 3.1, $M_{d,G} = M_{d,G_1} \times M_{d,G_2}$. Let $m_{d,G}(p) = (x_1, x_2) \in M_{d,G}$. Now $x_2 \in M_{d,G_2}$ implies $4x_2 \in M_{d,G_2}$ and consequently $(x_1, 4x_2) \in M_{d,G}$. Since (G, p) was generic, (x_1, x_2) is generic in $M_{d,G}$ by Lemma 2.12, and this implies that $(x_1, 4x_2)$ is generic in $M_{d,G}$ as well. Using Lemma 2.12, it follows that there is a generic framework (G, q) in $\mathbb{C}^d$ with $m_{d,G}(q) = (x_1, 4x_2)$.

Let us consider the subframeworks (G_1, q) and (G_2, q). Both of these frameworks are generic. Also note that (G_1, q) is equivalent to (G_1, p) and (G_2, q) is equivalent to $(G_2, 2p)$. Since $S(G, p) = S(G, 2p)$, this implies $S(G, p) = S(G_1, p) \oplus S(G_2, p) = S(G_1, q) \oplus S(G_2, q) = S(G, q)$, as desired.

We have that (G_1, q) is equivalent to (G_1, p). As established above, the edge directions of (G_1, p) do not lie on a conic at infinity. Thus, it follows from Lemma 2.5 that if q is an affine image of p, then q must be congruent to p. But q is clearly not congruent to p as (G_2, p) is not equivalent to (G_2, q). Thus, p is not an affine image of q, as desired. □

Theorem 3.5 was known to hold in $\mathbb{R}^1$ (where global rigidity, 2-connectivity and $\mathcal{R}_1$-connectivity are equivalent) and in $\mathbb{R}^2$, see [14] (c.f. Theorem 2.3 above). We note that in [17] it is conjectured that globally rigid graphs are 'nondegenerate' in $\mathbb{R}^d$, a condition that is stronger than $\mathcal{R}_d$-connectivity.

Underlying the proof of Theorem 3.5 is the following structural observation on the measurement variety of G, which we give without details. Since G is globally rigid, [10, Theorem 4.4] and [12, Lemma 4.24, Remark 4.25] imply that for any generic Gauss fiber F of $M_{d,G}$ we have $\dim(\overline{F}) = \binom{d+1}{2}$. On the other hand, it follows from the definitions that if $M_{d,G} = M_{d,G_1} \times M_{d,G_2}$, then $F = F_1 \times F_2$, where F_1 and F_2 are some generic Gauss fibers of M_{d,G_1} and M_{d,G_2}, respectively. Using Lemma 3.2, it can be shown that, under the assumptions on G_1 made in the proof of Theorem 3.5, we have $\dim(\overline{F_1}) \geq \binom{d+1}{2}$. Since $\dim(\overline{F_2}) \geq 1$, this gives

$$\binom{d+1}{2} = \dim(\overline{F}) = \dim(\overline{F_1}) + \dim(\overline{F_2}) > \binom{d+1}{2},$$

a contradiction.

3.3. Globally rigid graphs are fully reconstructible

Our goal in this subsection is to prove the following result, which gives an affirmative answer to [12, Question 7.5].

Theorem 3.6. *Let $d \geq 2$, and let G be a graph on $n \geq d + 2$ vertices that is globally rigid in $\mathbb{R}^d$. Then G is fully reconstructible in $\mathbb{C}^d$.*

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Theorem 4.7, below, will lead to examples that show that global rigidity is not necessary for full reconstructibility.

Our proof of Theorem 3.6 uses Theorem 3.5. In fact, as the following theorem shows, the former result is a strengthening of the latter.

Theorem 3.7. *Let $G = (V, E)$ be a graph without isolated vertices, and suppose that G is fully reconstructible in $\mathbb{C}^d$. Then G is $\mathcal{R}_d$-connected.*

Proof. Suppose, for a contradiction, that there is a partition $E = E_1 \cup E_2$ of E into nonempty subsets with $r_d(E) = r_d(E_1) + r_d(E_2)$, and let G_1 and G_2 be the subgraphs induced by E_1 and E_2, respectively. By Lemma 3.1, this implies $M_{d,G} = M_{d,G_1} \times M_{d,G_2}$. If G_1 and G_2 have at least one vertex in common, then let H be the graph consisting of disjoint copies of G_1 and G_2. Otherwise, let H be the graph obtained from disjoint copies of G_1 and G_2 by identifying some vertex of G_1 and some vertex of G_2. In both cases, we have $m_{d,H}(\mathbb{C}^{n'd}) = m_{d,G_1}(\mathbb{C}^{n_1 d}) \times m_{d,G_2}(\mathbb{C}^{n_2 d})$, where n', n_1 and n_2 denote the number of vertices of H, G_1 and G_2, respectively. It follows that $M_{d,H} = M_{d,G_1} \times M_{d,G_2}$.[3] Since by construction G and H are not isomorphic, Theorem 2.19 implies that G is not fully reconstructible, as desired. □

This theorem is similar in spirit to [8, Theorem 5.21], which states that if G is strongly reconstructible in $\mathbb{C}^d$ on at least $d+2$ vertices and without isolated vertices, then it cannot contain an $\mathcal{R}_d$-bridge. Example 4.6 in the next section shows that the converse of Theorem 3.7 is not true: The 6-ring depicted in Figure 4 is not even strongly reconstructible in $\mathbb{C}^3$, despite being $\mathcal{R}_3$-connected, 4-connected and redundantly rigid in $\mathbb{R}^3$.

We now turn our attention to Theorem 3.6. As our argument is quite involved, we start by giving a brief outline of the proof. By Theorem 2.19, to show that the globally rigid graph G is fully reconstructible, we need to show that, whenever $M_{d,G} =_\psi M_{d,H}$ for some graph H without isolated vertices and some edge bijection $\psi : E(G) \to E(H)$, we have that H is isomorphic to G (and the isomorphism induces the appropriate edge bijection). Let n and n' denote the number of vertices of G and H, respectively. If $n = n'$, then we are done by the strong reconstructibility of G (Theorem 2.20). If $n' < n$, then $M_{d,H}$ must necessarily be of lower dimension than $M_{d,G}$, which is impossible. The only remaining possibility to be ruled out is that $n' > n$; note that in this case $M_{d,G} =_\psi M_{d,H}$ (and in particular the equality of dimensions) implies that H is locally flexible in $\mathbb{R}^d$.

We rule this out as follows. From $M_{d,G} =_\psi M_{d,H}$ and $M_{d-1,H} \subseteq_\psi M_{d-1,G}$ (which follows from Corollary 3.4), we get bounds on k_d and k_{d-1}, the generic dimension of the space of infinitesimal motions of H in d and $d-1$ dimensions, respectively. Using Theorem 3.5, we also get that H is $\mathcal{R}_d$-connected and in particular connected. The crux of our argument is Corollary 3.12, which states that for a connected graph, $k_d - \binom{d+1}{2}$ (the 'd-dimensional degrees of freedom' of the graph) is larger by a multiplicative factor than $k_{d-1} - \binom{d}{2}$, the $(d-1)$-dimensional degrees of freedom of the graph. Applying this to H, we shall get a contradiction with the bounds on k_d and k_{d-1} obtained previously.

Before proving Corollary 3.12, we need a number of technical results. The first is about the structure of a certain variety, while the second gives a concrete example of an infinitesimal motion in d dimensions that, generically, cannot be decomposed into two $(d-1)$-dimensional infinitesimal motions. The subsequent lemmas use this result to obtain bounds on the generic dimension of infinitesimal motions in $\mathbb{C}^i$ for all $1 \le i \le d$.

Lemma 3.8. *Let $G = (V, E)$ be a graph on n vertices.*

1. *The variety $M_{1,G} \subseteq \mathbb{C}^E$ is not contained in any (linear) hyperplane in $\mathbb{C}^E$.*
2. *Consider the mapping $f : \mathbb{C}^{2n} \to \mathbb{C}^E$ defined by*

$$p = \left(p_v^{(1)}, p_v^{(2)}\right)_{v \in V} \longmapsto \left((p_v^{(1)} - p_u^{(1)})(p_v^{(2)} - p_u^{(2)})\right)_{uv \in E}.$$

[3] This follows from the basic fact that, for any $U \subseteq \mathbb{C}^{E_1}$, $V \subseteq \mathbb{C}^{E_2}$ we have $\overline{U \times V} = \overline{U} \times \overline{V}$, where closures are meant in the respective Zariski topologies see, e.g., [8, Lemma 2.4].

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Then $\overline{f(\mathbb{C}^{2n})}$ *(and consequently* $f(\mathbb{C}^{2n})$ *itself) is not contained in any (linear) hyperplane in* $\mathbb{C}^E$*.*

Proof. *a)* Let H be an arbitrary hyperplane in $\mathbb{C}^E$ whose orthogonal complement is generated by some nonzero $\omega \in \mathbb{C}^E$. The configurations $p \in \mathbb{C}^n$ for which ω is an equilibrium stress of (G, p) form a proper linear subspace of $\mathbb{C}^n$. In particular, we can find a generic framework (G, p) in $\mathbb{C}^1$ which does not have ω as an equilibrium stress. By Lemma 2.15, ω is not in the orthogonal complement of the tangent space of $M_{1,G}$ at $m_{1,G}(p)$. It follows that this tangent space is not contained in H, which implies that $M_{1,G}$ is not contained in H either, as desired.

b) By direct calculation[4] we have that $f = m_{2,G} \circ \alpha$ where $\alpha : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is defined by

$$p = \left(p_v^{(1)}, p_v^{(2)}\right)_{v\in V} \longmapsto \left(\frac{p_v^{(1)} + p_v^{(2)}}{2}, \frac{p_v^{(1)} - p_v^{(2)}}{2\sqrt{-1}}\right)_{v\in V}.$$

Since α is a linear automorphism of $\mathbb{C}^{2n}$, this implies that $f(\mathbb{C}^{2n}) = m_{2,G}(\mathbb{C}^{2n})$, and thus $\overline{f(\mathbb{C}^{2n})} = M_{2,G}$. Now by part *a)*, $M_{1,G}$ is not contained in any linear hyperplane in $\mathbb{C}^E$. Since $M_{1,G} \subseteq M_{2,G}$, the same holds for $M_{2,G}$. □

For the next two lemmas, we introduce the following notation. Let $d \geq 3$ and let $G = (V, E)$ be a graph on n vertices. For a framework (G, p) in $\mathbb{C}^d$, let $W_1(G, p) \leq \mathbb{C}^{nd}$ denote the set of infinitesimal motions of (G, p) that are supported on the first $d - 1$ coordinates, that is, infinitesimal motions of the form $(q_v, 0)_{v\in V}$, where $q_v \in \mathbb{C}^{d-1}$ for each vertex v. Similarly, let $W_2(G, p)$ denote the set of infinitesimal motions that are supported on the last $d - 1$ coordinates, and let $W(G, p)$ be the subspace of $\mathbb{C}^{nd}$ spanned by $W_1(G, p) \cup W_2(G, p)$.

Lemma 3.9. *Let* $d \geq 3$ *and let* $G = (V, E)$ *be a graph on n vertices that is not* $\mathcal{R}_{d-2}$*-independent. Then for any generic configuration* $p = \left(p_v^{(1)}, \dots, p_v^{(d)}\right)_{v\in V} \in \mathbb{C}^{nd}$*, the infinitesimal rotation* $\varphi = \varphi(p)$ *of* (G, p) *defined by*

$$\varphi_v = (-p_v^{(d)}, 0, \dots, 0, p_v^{(1)}) \qquad \forall v \in V$$

is not in the subspace $W(G, p)$*. In particular,* $W(G, p)$ *is a proper subspace of the space of infinitesimal motions of* (G, p)*.*

Proof. Consider an arbitrary realization (G, p) in $\mathbb{C}^d$, and let $\widetilde{p} \in \mathbb{C}^{n(d-2)}$ denote the projection of p onto the middle $d-2$ coordinate axes. Suppose that $\varphi(p)$ can be written as $\varphi(p) = q+r$, where $q \in W_1(G, p)$ and $r \in W_2(G, p)$. Then $q_v = (-p_v^{(d)}, \widetilde{q}_v, 0)$ for every vertex $v \in V$, where $\widetilde{q} = (\widetilde{q}_v)_{v\in V} \in \mathbb{C}^{n(d-2)}$. Since q is an infinitesimal motion, we have that

$$(p_v - p_u) \cdot (q_v - q_u) = 0, \qquad \forall uv \in E.$$

Using the above description of q and rearranging gives

$$(\widetilde{p}_v - \widetilde{p}_u) \cdot (\widetilde{q}_v - \widetilde{q}_u) = (p_v^{(1)} - p_u^{(1)})(p_v^{(d)} - p_u^{(d)}), \qquad \forall uv \in E. \tag{3.2}$$

We shall show that, for a generic realization (G, p) of G in $\mathbb{C}^d$, equation (3.2) cannot hold for any vector $(\widetilde{q}_v)_{v\in V}$. We start by observing that $(G, \widetilde{p})$ is a generic framework in $\mathbb{C}^{d-2}$, so $\mathrm{rank}(R(G, \widetilde{p}))$ is equal to the rank $r = r_{d-2}(G)$ of G, which is the maximal rank of the rigidity matrix of any framework in $\mathbb{C}^{d-2}$. Note that

$$\left((\widetilde{p}_v - \widetilde{p}_u) \cdot (\widetilde{q}_v - \widetilde{q}_u)\right)_{uv\in E} = R(G, \widetilde{p})\widetilde{q},$$

[4]This was already observed in [3].

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so it is sufficient to show that the vector $z = \left((p_v^{(1)} - p_u^{(1)})(p_v^{(d)} - p_u^{(d)})\right)_{uv \in E}$ is not contained in $\operatorname{span}(R(G, \widetilde{p}))$. This is the same as saying $\operatorname{rk}(M(p)) = r + 1$, where $M(p)$ is obtained by appending the column vector z to $R(G, \widetilde{p})$, which is further equivalent to the nonvanishing of some $(r+1) \times (r+1)$ subdeterminant of $M(p)$. The entries of $M(p)$ are polynomial functions in the coordinates of (G, p) with rational coefficients and consequently so are these $(r+1) \times (r+1)$ subdeterminants.

Since (G, p) is generic, it is sufficient to show that at least one of the polynomial functions describing these subdeterminants is not identically zero; in other words, that there is some framework (G, p') for which the corresponding matrix $M(p')$ has rank $r + 1$. In fact, we can find such a framework by only modifying the first and last coordinates of p_v for each $v \in V$. Since G is not $\mathcal{R}_{d-2}$-independent, $r < |E|$, and thus, $\operatorname{span}(R(G, \widetilde{p}))$ is contained in some linear hyperplane $H \subseteq \mathbb{C}^E$. By part *b)* of Lemma 3.8, we can find some vectors $(x_v)_{v \in V}$ and $(y_v)_{v \in V}$ such that the vector $((x_v - x_u)(y_v - y_u))_{uv \in E}$ is not contained in H. It follows that the framework (G, p') defined by $p'_v = (x_v, \widetilde{p}_v, q_v), v \in V$ satisfies $\operatorname{rk}(M(p')) = r + 1$, as desired. □

Lemma 3.10. *Let $d \geq 3$, and let G be a graph on n vertices that is not $\mathcal{R}_{d-2}$-independent. For $i = 1, \ldots, d$, let k_i denote the dimension of the space of infinitesimal motions of a generic realization of G in $\mathbb{C}^i$. Then for $i = 1, \ldots, d-2$, we have*

$$k_{i+2} - k_{i+1} \geq k_{i+1} - k_i + 1.$$

Proof. The assumption that G is not $\mathcal{R}_{d-2}$-independent implies that it is not $\mathcal{R}_i$-independent for all $1 \leq i \leq d-2$. Thus, it is sufficient to prove for $i = d-2$. By Lemma 3.9, there is a generic realization (G, p) in $\mathbb{C}^d$ such that $W(G, p)$ is a proper subset of the space of infinitesimal motions of (G, p). Note that $\dim(W_1(G, p)) = \dim(W_2(G, p)) = k_{d-1}$, and by the choice of (G, p), we have $\dim(W(G, p)) \leq k_d - 1$. Moreover, the subspace $W' = W_1(G, p) \cap W_2(G, p)$ consists of the infinitesimal motions of (G, p) that are supported on the middle $d-2$ coordinates. This implies $\dim(W') = k_{d-2}$. By basic linear algebra we have

$$\dim(W') + \dim(W(G, p)) = \dim(W_1(G, p)) + \dim(W_2(G, p)).$$

Substituting the above equalities and inequality and then rearranging gives

$$k_d - k_{d-1} \geq k_{d-1} - k_{d-2} + 1,$$

as desired. □

If G is $\mathcal{R}_{d-2}$-independent, then the conclusion of Lemma 3.10 does not hold. In this case, G is $\mathcal{R}_{d-1}$-independent and $\mathcal{R}_d$-independent as well, so we have $k_i = ni - |E|$ for $d-2 \leq i \leq d$ so that $k_d - k_{d-1} = k_{d-1} - k_{d-2}$.

The following combinatorial lemma lets us turn the recursive bound on k_i given in Lemma 3.10 into a lower bound that only depends on k_d and k_{d-1}.

Lemma 3.11. *Let $d \geq 2$ be an integer, and let $k_1, k_2, \ldots, k_d \in \mathbb{Z}$ be a sequence of integers with $k_d = \binom{d+1}{2} + x$ and $k_{d-1} = \binom{d}{2} + y$ for some $x, y \in \mathbb{Z}$. Suppose that for $1 \leq i \leq d-2$ we have $k_{i+2} - k_{i+1} \geq k_{i+1} - k_i + 1$. Then for $1 \leq i \leq d-1$ we have $k_i \geq \binom{i+1}{2} + (d-i)(y-x) + x$.*

Proof. Consider the numbers $l_i = k_i - \binom{i+1}{2}$ for $i = 1, \ldots, d$. Our goal is to prove $l_i \geq (d-i)(y-x) + x$. By definition, we have $l_{d-1} - l_d = y - x$. We claim that $l_i - l_{i+1} \geq l_{i+1} - l_{i+2}$ holds for all $i = 1, \ldots, d-2$.

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Indeed, using the bound on k_i we obtain

$$\begin{aligned} l_i = k_i - \binom{i+1}{2} &\geq 2k_{i+1} - k_{i+2} + 1 - \binom{i+1}{2} \\ &= (2\binom{i+2}{2} - \binom{i+3}{2} - \binom{i+1}{2} + 1) + (2l_{i+1} - l_{i+2}) \\ &= 2l_{i+1} - l_{i+2}, \end{aligned}$$

where we used the fact that for all $a \geq 1$, $\binom{a}{2} + \binom{a+2}{2} = 2\binom{a+1}{2} + 1$. This also gives

$$l_i - l_{i+1} \geq l_{i+1} - l_{i+2} \geq l_{i+2} - l_{i+3} \geq \cdots \geq l_{d-1} - l_d = y - x.$$

The statement now follows easily by induction on $j = d - i$. For $j = 1$, we need $l_{d-1} \geq y$, which is actually satisfied with equality. Now let us assume $1 < j < d$. Then by induction we have

$$l_i \geq l_{i+1} + y - x \geq (d - i - 1)(y - x) + x + y - x = (d - i)(y - x) + x,$$

as desired. □

If G is a connected graph, then its one-dimensional generic realizations have a one-dimensional space of infinitesimal motions. For such graphs, combining Lemma 3.10 with the bound on k_1 given by Lemma 3.11 gives the following corollary.

Corollary 3.12. *Let $d \geq 3$ be an integer and G a connected graph, and suppose that G is not $\mathcal{R}_{d-2}$-independent. Let k_{d-1} and k_d denote the dimension of the space of infinitesimal motions of a generic realization of G in $\mathbb{C}^{d-1}$ and in $\mathbb{C}^d$, respectively. Finally, let $\mathrm{dof}_{d-1}(G) = k_{d-1} - \binom{d}{2}$ and $\mathrm{dof}_d(G) = k_d - \binom{d+1}{2}$ denote the 'degrees of freedom' of G in $d - 1$ and d dimensions, respectively. Then we have*

$$\mathrm{dof}_d(G) \geq \frac{d-1}{d-2}\mathrm{dof}_{d-1}(G).$$

Now we are ready to prove our main theorem.

Proof of Theorem 3.6. The $d = 2$ case follows from Theorem 2.21, so we only prove for $d \geq 3$. Let H be a graph on n' vertices, without isolated vertices and such that $M_{d,G} =_\psi M_{d,H}$ under some edge bijection ψ. Then by Corollary 3.4, we also have $M_{d-1,H} \subseteq_\psi M_{d-1,G}$. Observe that, since G is globally rigid, it is $\mathcal{R}_d$-connected by Theorem 3.5, and thus, so is H by Theorem 2.14 and the equality of the measurement varieties. In particular, H is a connected graph, and it is not $\mathcal{R}_d$-independent and thus not $\mathcal{R}_{d-2}$-independent.

Let $s = n' - n$, and let k_i denote the dimension of the space of infinitesimal motions of a generic realization of H in $\mathbb{C}^i$ for $1 = 1, \ldots, d$. By considering the dimension of $M_{d,G}$ (using Lemma 2.13) and using the assumption that G is (globally) rigid, we get $n'd - k_d = nd - \binom{d+1}{2}$, implying $k_d = sd + \binom{d+1}{2}$. Similarly, from $M_{d-1,H} \subseteq_\psi M_{d-1,G}$, we have $n'(d-1) - k_{d-1} \leq n(d-1) - \binom{d}{2}$, so that $k_{d-1} \geq s(d-1) + \binom{d}{2}$. Note that here we used the fact that if G is (globally) rigid in $\mathbb{R}^d$, then it is rigid in $\mathbb{R}^{d-1}$; see, e.g., [16, Theorem 63.2.11].

Since $k_d \geq \binom{d+1}{2}$, we must have $s \geq 0$. On the other hand, since H is connected and not $\mathcal{R}_{d-2}$-independent, Corollary 3.12 implies that

$$sd = \mathrm{dof}_d(H) \geq \frac{d-1}{d-2}\mathrm{dof}_{d-1}(H) \geq \frac{d-1}{d-2}s(d-1).$$

Reordering, we get

$$sd(d-2) \geq s(d-1)^2,$$

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which is equivalent to $s \le 0$. It follows that $s = 0$, so G and H have the same number of vertices. By Theorem 2.20, G is strongly reconstructible in $\mathbb{C}^d$, so ψ is induced by a graph isomorphism $\varphi : G \to H$, as desired. □

Applying Theorem 3.6 to a globally rigid subgraph of a graph, we obtain the following corollary.

Corollary 3.13. *Let $d \ge 2$, and let (G, p) and (H, q) be generic frameworks in $\mathbb{C}^d$ that are length-equivalent under the edge bijection ψ. Let $G_0 = (V_0, E_0)$ be a globally rigid subgraph of $G = (V, E)$, and let H_0 denote the subgraph of H induced by $\psi(E_0)$. Then $\psi|_{E_0}$ is induced by an isomorphism $\varphi : V(G_0) \to V(H_0)$, and the frameworks $(G_0, p|_{V_0})$ and $(H_0, q|_{V(H_0)} \circ \varphi)$ are congruent.*

The $d = 2$ case of Corollary 3.13 can be found in [8, Corollary 5.2].

To close this section, let us briefly return to Lemma 3.10. Let G be a graph and, as before, let k_d denote the dimension of the set of infinitesimal motions of a generic realization of G in $\mathbb{C}^d$. Using the fact that, for a graph G on at least $d+2$ vertices we have $r_d(G) = nd - k_d$, we can rephrase the $i = d-2$ case of Lemma 3.10 in the following way.

Corollary 3.14. *Let $d \ge 3$, and let G be a graph that is not $\mathcal{R}_{d-2}$-independent. Then the rank of G satisfies*

$$r_d(G) - r_{d-1}(G) \le r_{d-1}(G) - r_{d-2}(G) - 1. \tag{3.3}$$

This bound is the best possible in the sense that if G is rigid in $\mathbb{R}^d$, then equation (3.3) is satisfied with equality. We note that Corollary 3.14 can be interpreted as a statement about the so-called secant defect of $M_{1,G}$, similar to Zak's theorem on superadditivity [7, 25]; see [10, Section 4.3] for a related discussion.

4. Examples and open questions

In this section, we examine various examples related to $\mathcal{R}_d$-connected and $\mathcal{R}_d$-separable graphs, as well as the unlabeled reconstruction problem.

4.1. New examples of H-graphs

Following [15], we say that a graph G is an *H-graph* in $\mathbb{R}^d$ if it is $(d+1)$-connected and redundantly rigid in $\mathbb{R}^d$ (i.e., it satisfies the necessary conditions of Theorem 2.1), but it is not globally rigid in $\mathbb{R}^d$. There are no H-graphs for $d = 1, 2$, but for $d \ge 3$ they exist, and finding more examples may lead to a better understanding of higher-dimensional global rigidity. For a long time, the complete bipartite graph $K_{5,5}$ was the only known H-graph in $\mathbb{R}^3$ (identified in [4]), until infinite families had been found in [15].

Theorem 3.5 can be used to give new examples of H-graphs which are $\mathcal{R}_3$-separable. These also demonstrate that redundant rigidity and $(d+1)$-connectivity together do not imply $\mathcal{R}_d$-connectivity in the $d \ge 3$ case.

Example 4.1. Consider the construction illustrated in Figure 2. It is easy to see that the graph G in the figure is 4-connected.

Claim. *G is redundantly rigid and $\mathcal{R}_3$-separable.*

Proof. We show that G is rigid by showing that a spanning subgraph of G can be reduced to K_4 by a sequence of the following operations: (i) deletion of a vertex of degree at least three, (ii) deletion of a vertex v of degree four and the addition of a new edge between two neighbours of v, (iii) the contraction of an edge uv for which u and v have exactly two common neighbours. It is well-known that the inverse operations (0- and 1-extension and vertex splitting) preserve rigidity in $\mathbb{R}^3$, see e.g. [23]. Thus, since K_4 is rigid, it will follow that G is also rigid.

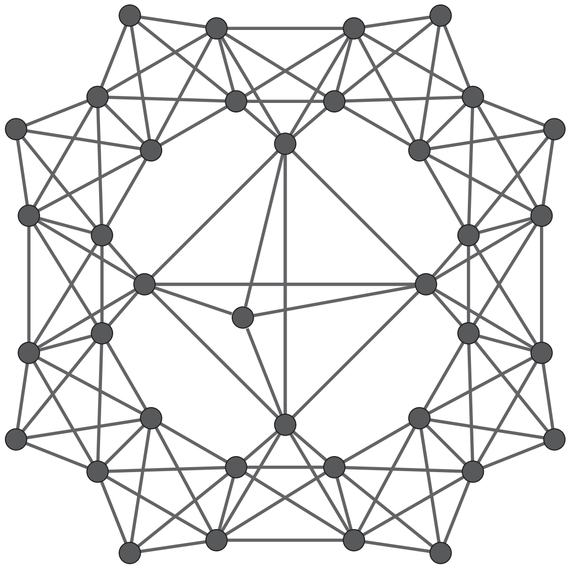

Figure 2. *This graph G is 4-connected, redundantly rigid in $\mathbb{R}^3$, and $\mathcal{R}_3$-separable. It satisfies $r_3(G) = 105 = 96 + 9 = r_3(G^o) + r_3(K_5)$, where G^o is the outer ring of K_5's.*

First, delete the nine vertices of degree four from G, and then delete one edge from each of the remaining copies of K_5. The resulting graph has 28 vertices and 78 edges. We shall reduce it to its internal K_4 subgraph. By the symmetry of the graph, we can perform the reduction steps in groups of four in a symmetric way. First, we contract four edges of the outer ring that do not belong to the four copies of $K_5 - e$, one from each 'corner'. These operations create a vertex of degree four in each corner, so we can apply operation (ii) at each of them to obtain a graph on 20 vertices and 54 edges, in which the four edges added form a four-cycle. After that we again apply operation (ii) in three rounds, decreasing the number of vertices by four in each round. If the added edges are chosen appropriately, we obtain a graph on 8 vertices, consisting of the internal K_4, a disjoint four-cycle C_4, and eight more edges that connect them (so that they span an 8-cycle). From here we apply operations (ii), (i), (ii) and then again (i) to get the K_4 subgraph.

Thus, G is indeed rigid, that is, $r_3(G) = 3|V(G)| - 6 = 105$. Note that G is also redundantly rigid, because every edge of G belongs to a K_5 subgraph. To see that G is $\mathcal{R}_3$-separable, first observe that if we remove one edge from each copy of K_5 in the outer ring G^o (say, one edge incident with each vertex of degree four), then we do not decrease its rank and obtain a spanning subgraph of G^o with 96 edges. Thus, $r_3(G^o) \leq 96$. The inner K_5 has rank 9. Hence, we must have $r_3(G) = r_3(G^o) + r_3(K_5)$, showing that G is indeed $\mathcal{R}_3$-separable. □

Since $\mathcal{R}_3$-separable graphs are not globally rigid in $\mathbb{R}^3$ by Theorem 3.5, G is indeed an H-graph in $\mathbb{R}^3$. We can obtain an infinite family of H-graphs in $\mathbb{R}^3$ from Example 4.1 by replacing the inner K_5 in Figure 2 with another 4-connected redundantly rigid graph K' in $\mathbb{R}^3$ on at least five vertices (as in Figure 3, where $K' = K_6 - e$).

We note that some of the H-graphs obtained in [15] (for example, the '6-ring' depicted in Figure 4) show that 4-connectivity, redundant rigidity and $\mathcal{R}_3$-connectivity together do not imply global rigidity in $\mathbb{R}^3$.

It is also interesting to note that every known H-graph in $\mathbb{R}^3$, except $K_{5,5}$, has a 4-separator.

Question 4.2. Is every 5-connected and redundantly rigid graph, other than $K_{5,5}$, globally rigid in $\mathbb{R}^3$?

The following related question also seems to be open.

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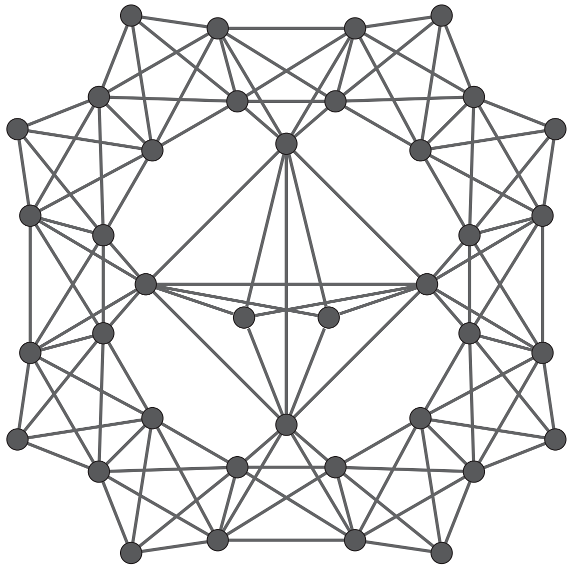

Figure 3. *This graph is also 4-connected, redundantly rigid in $\mathbb{R}^3$, and $\mathcal{R}_3$-separable.*

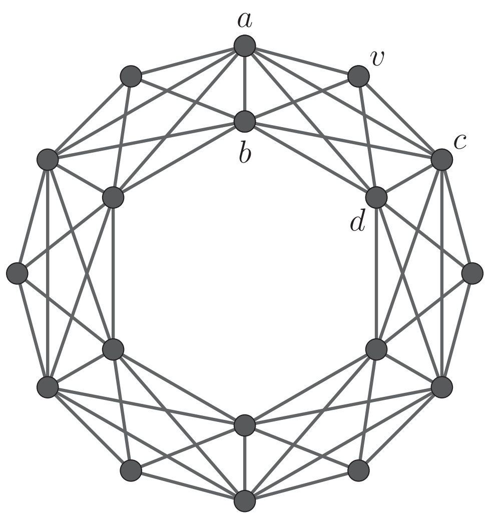

Figure 4. *A graph that is 4-connected, redundantly rigid in $\mathbb{R}^3$ and $\mathcal{R}_3$-connected but not globally rigid in $\mathbb{R}^3$.*

Question 4.3. Is every 5-connected and redundantly rigid graph $\mathcal{R}_3$-connected?

It is easy to see that coning increases vertex connectivity by one so that the cone of a $(d+1)$-connected graph is $(d+2)$-connected. It is also folklore (and follows from Theorem 2.4(a) and Lemma 5.3 in the next section) that if G is redundantly rigid in $\mathbb{R}^d$, then its cone graph G^v is redundantly rigid in $\mathbb{R}^{d+1}$. Together with Theorem 2.4(b) these imply that the cone of a H-graph in $\mathbb{R}^d$ is a H-graph in $\mathbb{R}^{d+1}$. We can use this fact to construct further families of H-graphs. However, these graphs will not be $\mathcal{R}_{d+1}$-separable (see Theorem 5.4). Instead, we can use higher-dimensional body-hinge graphs [15] to generalize the $\mathcal{R}_3$-separable construction of Figure 2 to $d \geq 4$. We omit the details.

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4.2. Unlabeled reconstructibility and small separators

In [12, Question 7.2], the authors asked whether every graph G that is 3-connected and redundantly rigid in $\mathbb{R}^d$ is determined by its measurement variety, that is, whether $M_{d,G} =_\psi M_{d,H}$ under some edge bijection ψ implies that G and H are isomorphic (note that here we do not require that the isomorphism induces ψ). Such a graph was called 'weakly reconstructible in $\mathbb{C}^d$' in [8]. In the other direction, in [8, Section 7], the authors asked whether every graph on at least $d+2$ vertices that is strongly reconstructible in $\mathbb{C}^d$ for some $d \geq 3$ is globally rigid in $\mathbb{R}^d$ or (more weakly) whether it is $(d+1)$-connected.

In this subsection, we provide negative answers to each of these questions for $d \geq 3$. Throughout this section, we shall use the following (folklore) result which also appears in the proof of [8, Theorem 5.21].

Lemma 4.4. *Let G and H be graphs on at least three vertices with G connected, and let $\varphi_1, \varphi_2 : V(G) \to V(H)$ be injective graph homomorphisms. Suppose that φ_1 and φ_2 induce the same edge map $\psi : E(G) \to E(H)$. Then $\varphi_1(v) = \varphi_2(v)$ for all $v \in V(G)$.*

Proof. Let $v \in V(G)$ be a vertex of degree at least two, and let $vu, vu' \in E(G)$ be a pair of edges incident to v. Now $\varphi_1(v)$ is the unique vertex in H that is an end-vertex of both $\psi(vu)$ and $\psi(vu')$. Since $\varphi_2(v)$ can be described in the same way, we have $\varphi_1(v) = \varphi_2(v)$. Note that in a connected graph on at least three vertices, every edge has at least one end-vertex with degree at least two. This shows that φ_1 and φ_2 send at least one vertex of each edge in G to the same vertex in H. Since they also send each edge in G to the same edge in H, they must agree on every vertex of G. □

Example 4.5. Consider again the graph G shown in Figure 2. As we have seen in Example 4.1, G is 4-connected, redundantly rigid and $\mathcal{R}_3$-separable.

Claim. *G is not strongly reconstructible in $\mathbb{C}^3$.*

Proof. By the $\mathcal{R}_3$-separability of G and Lemma 3.1, we have $M_{d,G} = M_{d,G^o} \times M_{d,K_5}$, where K_5 denotes the complete subgraph of G induced by the inner five vertices. Let ψ be a permutation of the edges of G that leaves the edges of G^o in place and permutes the edges of K_5 according to some permutation of its five vertices. Then $M_{d,G} =_\psi M_{d,G}$, i.e., $M_{d,G}$ is invariant under the permutation of the coordinate axes in $\mathbb{C}^{E(G)}$ induced by ψ. However, ψ is not induced by a graph automorphism of G: Since it leaves the edges of G^o in place, by Lemma 4.4, such an automorphism would have to leave the vertices of G^o in place and thus be the identity map on G, which does not induce ψ. By Theorem 2.19, this shows that G is not strongly reconstructible. □

It can be shown similarly that the graph G' shown in Figure 3 is not even weakly reconstructible in $\mathbb{C}^3$: The graph obtained by adding the missing edge to the inner K_6 and removing a different edge from it has the same measurement variety as G', even though the two graphs are not isomorphic. These examples can also be generalized to higher dimensions using the results on body-hinge graphs in [15].

The graphs considered in Example 4.5 are all $\mathcal{R}_3$-separable. The next example is of an $\mathcal{R}_3$-connected graph that is not strongly reconstructible in $\mathbb{C}^3$.

Example 4.6. Let G be the 6-ring of K_5's shown in Figure 4. As noted before, it is 4-connected, redundantly rigid in $\mathbb{R}^3$ and $\mathcal{R}_3$-connected.

Claim. *G is not strongly reconstructible in $\mathbb{C}^3$.*

Proof. Let v be a vertex of degree four, and let us denote the vertices of the K_5 subgraph that contains v by $\{v, a, b, c, d\}$, where the edges ab and cd are shared by neighbouring K_5's. Let $H = G - v$. It is easy to check that the edges ac, ad, bc, bd are all $\mathcal{R}_3$-bridges in H. Thus, by Lemma 3.1, we have $M_{d,H} = M_{d,H'} \oplus \mathbb{C}^4$, where $H' = H - \{ac, ad, bc, bd\}$. Let (G, p) be a generic realization of G, and (by a slight abuse of notation) let (H, p) be its restriction to H.

Consider the permutation ψ of the edges of H that leaves the edges of H' in place and maps ac, ad, bc, bd to bd, bc, ad, ac, respectively. Since $M_{d,H}$ is invariant under the permutation of coordinate axes induced by ψ, Lemma 2.12 implies that there exists a generic realization (H, q)

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of H in $\mathbb{C}^3$ such that $(H', q|_{V(H')})$ and $(H', p|_{V(H')})$ are equivalent and $m_{ac}(p) = m_{bd}(q)$, $m_{bd}(p) = m_{ac}(q)$, $m_{bc}(p) = m_{ad}(q)$, and $m_{ad}(p) = m_{bc}(q)$. This implies that the point configurations $(p(a), p(b), p(c), p(d))$ and $(q(b), q(a), q(d), q(c))$ are congruent, i.e., the points in them have the same pairwise squared distances. Since by genericity they have full affine span, this also implies that they are strongly congruent, in other words, there is a rigid motion of $\mathbb{C}^3$ that maps $p(a), p(b), p(c), p(d)$ to $q(b), q(a), q(d), q(c)$, respectively.

Let us extend (H, q) to a realization (G, q) by defining $q(v)$ to be the image of $p(v)$ under this rigid motion, and let us also extend ψ to all of $E(G)$ by mapping va, vb, vc, vd to vb, va, vd, vc, respectively. Then $m_{d,G}(p) = m_{d,G}(q)$ under the edge permutation ψ. On the other hand, ψ is not induced by a graph automorphism of G: Since it leaves the edges of H' in place, by Lemma 4.4 such an automorphism would leave the vertices of H' in place, and consequently, it would have to be the identity map on G, which does not induce ψ. By Theorem 2.19, this shows that G is not strongly reconstructible in $\mathbb{C}^3$, as desired. □

Finally, we construct examples of fully reconstructible graphs with small separators. In the next proof, we shall use the following fact. Let G_1, G_2 be rigid graphs in $\mathbb{R}^d$ on at least $d+1$ vertices, and let G be obtained from G_1 and G_2 by identifying k pairs of vertices. Then if $0 \le k \le d-1$, we have $r_d(G) = d|V(G)| - \binom{d+1}{2} - \binom{d-k+1}{2}$, and if $k \ge d$, then G is rigid. This follows, e.g., from the 'gluing lemma' [23, Lemma 11.1.9], or it can be seen directly by considering the infinitesimal motions of a generic realization of G in $\mathbb{R}^d$.

Theorem 4.7. *Let $G = (V, E)$ be a graph with induced subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ for which $V_1 \cup V_2 = V$ and $V_1 \cap V_2$ induces a connected subgraph of G on at least three vertices. Let $d \ge 1$. If G_1 and G_2 are fully reconstructible rigid graphs on at least $d+1$ vertices in $\mathbb{C}^d$, then G is fully reconstructible in $\mathbb{C}^d$.*

Proof. By Theorem 2.19, it suffices to show that if for some graph H we have $M_{d,G} =_\psi M_{d,H}$ under some edge bijection ψ, then ψ is induced by a graph isomorphism $\varphi : V(G) \to V(H)$. Let H_1, H_2 be the subgraphs of H induced by $\psi(E_1)$ and $\psi(E_2)$, respectively. Now for $i = 1, 2$, $M_{d,G_i} = M_{d,H_i}$, so by the full reconstructibility of G_i, there is a graph isomorphism $\varphi_i : V_i \to V(H_i)$ that induces $\psi|_{E_i}$. Since $V_1 \cap V_2$ induces a connected subgraph of G, Lemma 4.4 applies to $\varphi_1|_{V_1\cap V_2}$ and $\varphi_2|_{V_1\cap V_2}$, giving $\varphi_1(v) = \varphi_2(v)$ for all $v \in V_1 \cap V_2$.

It follows that H is the union of two subgraphs H_1, H_2 which are isomorphic to G_1, G_2, respectively, and have $k \ge |V_1 \cap V_2|$ vertices in common. Let $\ell = |V_1 \cap V_2| \ge 3$. We first show that $k = \ell$. Note that $|V(G)| = |V_1| + |V_2| - \ell$ and $|V(H)| = |V_1| + |V_2| - k$, and hence $|V(H)| - |V(G)| = l - k$.

Let us first consider the $k \le d-1$ case. Since by Theorem 2.14 the d-dimensional rigidity matroids of G and H are isomorphic, we obtain

$$d|V(G)| - \binom{d+1}{2} - \binom{d-\ell+1}{2} = r_d(G) = r_d(H) = d|V(H)| - \binom{d+1}{2} - \binom{d-k+1}{2}.$$

This gives

$$\binom{d+1}{2} + \binom{k}{2} = dk + \binom{d-k+1}{2} = d\ell + \binom{d-\ell+1}{2} = \binom{d+1}{2} + \binom{\ell}{2},$$

where the second equality comes from the previous equation and the first and third equalities from direct calculation. It follows that $k = \ell$. An analogous argument shows that $k = \ell$ holds in the $k \ge d$ case as well.

This implies that the only vertices of H that are in the image of both φ_1 and φ_2 are those in the image of $V_1 \cap V_2$, where φ_1 and φ_2 agree. Hence, we can 'glue' φ_1 and φ_2, i.e. the mapping $\varphi : V \to V(H)$ defined by $\varphi|_{V_i} = \varphi_i, i = 1, 2$ is well-defined and is an isomorphism (in particular, it is injective). Then φ induces ψ, as required. □

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In fact, the proof shows that we can slightly relax the conditions in Theorem 4.7 by only requiring that each connected component of the graph induced by $V_1 \cap V_2$ has at least three vertices.

Example 4.8. Let $d \geq 1$, and let G_d be the graph obtained by gluing two copies of the complete graph K_{d+2} along three pairs of vertices. Theorems 3.6 and 4.7 imply that G_d is fully reconstructible in $\mathbb{C}^d$. This example shows that fully (or strongly) reconstructible graphs need not be $(d+1)$-connected in the $d \geq 3$ case, which gives a negative answer to a question posed in [8, Section 7]. Also note that for $d \geq 4$, G_d is not even rigid in $\mathbb{R}^d$. It is unclear whether there exist nonrigid fully reconstructible graphs in $\mathbb{C}^3$.

The previous example also shows that a fully reconstructible graph in $\mathbb{C}^d$ need not be globally rigid in $\mathbb{R}^d$ in the $d \geq 3$ case. Another such example is given in [2], where it is shown, using a computer-assisted proof, that the complete bipartite graph $K_{5,5}$ is fully reconstructible in $\mathbb{C}^3$. A combinatorial characterization of fully reconstructible graphs in $\mathbb{C}^d$ seems elusive, even in the $d = 3$ case.

4.3. Monotonicity of unlabeled reconstructibility

The graph G_d of Example 4.8 also shows that for $d \geq 4$, edge addition does not necessarily preserve strong (or full) reconstructibility in $\mathbb{C}^d$. Indeed, G_d is fully reconstructible in $\mathbb{C}^d$, but for any pair of nonneighbouring vertices $u, v \in V(G_d)$, uv is an $\mathcal{R}_d$-bridge in $G_d + uv$. [8, Theorem 5.21] states that strongly reconstructible graphs in $\mathbb{C}^d$ (on at least $d+2$ vertices and without isolated vertices) do not contain $\mathcal{R}_d$-bridges so that $G_d + uv$ is not strongly reconstructible in $\mathbb{C}^d$. It would be interesting to see whether this phenomenon can only happen if the newly added edge is an $\mathcal{R}_d$-bridge.

Question 4.9. Let $d \geq 1$, and let $G = (V, E)$ be a graph on at least $d+2$ vertices that is strongly reconstructible in $\mathbb{C}^d$ (fully reconstructible in $\mathbb{C}^d$, respectively). Is it true that if for some pair of vertices $u, v \in V$ we have $uv \notin E$ and $r_d(G) = r_d(G + uv)$, then $G + uv$ is strongly reconstructible in $\mathbb{C}^d$ (fully reconstructible in $\mathbb{C}^d$, respectively)?

We can prove the following weaker result. We say that a pair $\{u, v\}$ of vertices in a graph G is *globally linked in G* in $\mathbb{C}^d$ if for every generic framework (G, p) in $\mathbb{C}^d$ and every equivalent realization (G, q), we have $m_{uv}(p) = m_{uv}(q)$.

Lemma 4.10. *Let $G = (V, E)$ be a strongly reconstructible graph in $\mathbb{C}^d$, and suppose that a pair of vertices $u, v \in V$ is globally linked in G in $\mathbb{C}^d$. Then $G' = G + uv$ is strongly reconstructible in $\mathbb{C}^d$. Moreover, if G is fully reconstructible in $\mathbb{C}^d$, then so is G'.*

Proof. By Theorem 2.19, it is sufficient to show that if $M_{d,G'} =_\psi M_{d,H'}$ under some edge bijection ψ, where H' is a graph on the same number of vertices as G', then ψ is induced by a graph isomorphism. Let H denote $H' - \psi(uv)$. Then $M_{d,G} =_\psi M_{d,H}$ under the edge bijection $\psi|_{E(G)}$, and thus again by Theorem 2.19, the strong reconstructibility of G implies that $\psi|_{E(G)}$ is induced by a graph isomorphism $\varphi : V(G) \to V(H)$. It is sufficient to show that $\psi(uv) = \varphi(u)\varphi(v)$. After composing ψ with the edge bijection induced by φ^{-1}, this amounts to showing that if $M_{d,G+uv} =_{\psi'} M_{d,G+u'v'}$ under the edge bijection ψ' that fixes the edges of G and sends uv to $u'v'$, then $\{u', v'\} = \{u, v\}$.

Let (G, p) be a generic realization of G in $\mathbb{C}^d$. By Lemma 2.12, there is a generic realization (G, q), equivalent to (G, p) and such that $m_{uv}(p) = m_{u'v'}(q)$. Since $\{u, v\}$ is globally linked in G, we must also have $m_{uv}(p) = m_{uv}(q)$. It follows from the genericity of q that $\{u, v\} = \{u', v'\}$, as required.

The same proof works when G is fully reconstructible in $\mathbb{C}^d$. □

We may also consider the effect of edge deletion on unlabeled reconstructibility. The analogue of Question 4.9 for deleting edges is not true: It is not difficult to find an example of a graph G that is strongly (or fully) reconstructible in $\mathbb{C}^d$ but for which $G - uv$ is not strongly (or fully) reconstructible for some edge $uv \in E$, even though $r_d(G - uv) = r_d(G)$. This can happen, e.g., if G is globally rigid and $G - uv$ contains an $\mathcal{R}_d$-bridge. However, it is possible that the analogue of Lemma 4.10 for edge deletions is true.

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Question 4.11. Let $d \geq 3$, and let $G = (V, E)$ be strongly reconstructible in $\mathbb{C}^d$ (fully reconstructible in $\mathbb{C}^d$, respectively). Is it true that if for some edge $uv \in E$ we have that $\{u, v\}$ is globally linked in $G - uv$, then $G - uv$ is strongly reconstructible in $\mathbb{C}^d$ (fully reconstructible in $\mathbb{C}^d$, respectively)?

The characterization of strong and full reconstructibility in $\mathbb{C}^1$ and $\mathbb{C}^2$ given by Theorem 2.21 shows that, for $d = 2$, the answer to Question 4.11 is positive, while for $d = 1$, it is negative: Let G be a 3-connected graph, and suppose that $G - uv$ is not 3-connected for some edge $uv \in E(G)$. Then G is fully reconstructible in $\mathbb{C}^1$ and $\{u, v\}$ is globally linked in $G - uv$ (in fact, $G - uv$ is globally rigid in $\mathbb{R}^1$), but $G - uv$ is not strongly reconstructible in $\mathbb{C}^d$.

5. Graphs with nonseparable rigidity matroids

In light of Theorem 3.5, the combinatorial properties of $\mathcal{R}_d$-connected graphs may be of interest in studying global rigidity. However, not much seems to be known about these graphs in the $d \geq 3$ case. In this section, we collect three results related to this notion.

Theorem 5.1. *Let $G = (V, E)$ be an $\mathcal{R}_d$-connected graph. Then G is $\mathcal{R}_{d'}$-connected for all $1 \leq d' \leq d$.*

Proof. It suffices to consider the $d \geq 2$ case and show that G is $\mathcal{R}_{d-1}$-connected. We may also assume that G is an $\mathcal{R}_d$-circuit. Consider a generic realization (G, p) of G in $\mathbb{R}^d$. Since G is an $\mathcal{R}_d$-circuit, there exists a unique, (up to scalar multiplication) nonzero equilibrium stress $\omega = (\omega_e)_{e \in E}$ of (G, p), which is nonzero on every edge $e \in E$.

For a contradiction, suppose that G is $\mathcal{R}_{d-1}$-separable, and let E_1, E_2 be a separation, that is, $E_1 \cup E_2 = E$ and $r_{d-1}(E_1) + r_{d-1}(E_2) = r_{d-1}(E)$.

Let (G, p_i) be the $(d-1)$-dimensional realization of G obtained from (G, p) by a projection along the i-axis for $1 \leq i \leq d$. These projected frameworks are also generic in $\mathbb{R}^{d-1}$, and ω is a stress on each (G, p_i). Let $R(G, p_i)$ be the matrix obtained from the rigidity matrix of (G, p) by replacing the $|V|$ columns corresponding to coordinate i by all-zero columns. Thus, the rigidity matrix of (G, p_i) can be obtained from $R(G, p_i)$ by removing these zero columns.

Since E_1, E_2 is a separation, we must have $\sum_{e \in E_1} \omega_e R_e(G, p_i) = 0$ for all $1 \leq i \leq d+1$, where $R_e(G, p_i)$ is the row of the edge e in $R(G, p_i)$. This gives

$$(d-1) \sum_{e \in E_1} \omega_e R_e(G, p) = \sum_{i=1}^{d} \sum_{e \in E_1} \omega_e R_e(G, p_i) = 0,$$

implying that the restriction of ω to E_2 is a nonzero stress on a proper subframework of (G, p). Then extending this restricted stress to all of E by setting it zero on every $e \in E_2$ gives another stress of (G, p), contradicting the uniqueness of ω. □

Our next result is a characterization of $\mathcal{R}_d$-connected cone graphs. We shall use the fact that G is an $\mathcal{R}_d$-circuit if and only if G^v is an $\mathcal{R}_{d+1}$-circuit. Although this result seems to be folklore, we could not find any proofs in the literature, so we provide one (due to W. Whiteley [24]) for completeness. For this, we shall need the following geometric result about coning, which is implicit in the work of Whiteley [22] (see also [20, Theorem 3]).

Theorem 5.2. *Let $d \geq 1$, and let $G = (V, E)$ be a graph. Let (G^v, p) be a realization of the cone graph G^v in $\mathbb{R}^{d+1}$, and let $H \subseteq \mathbb{R}^{d+1}$ be a hyperplane not containing $p(v)$ and not parallel to any of the lines $\overline{p(v)p(u)}, u \in V$. Finally, let (G, p_H) be a framework in $\mathbb{R}^d$ obtained by projecting $p(u), u \in V$ onto H from $p(v)$ and then identifying H with $\mathbb{R}^d$. Then the following holds:*

$$\operatorname{rank}(R(G^v, p)) = \operatorname{rank}(R(G^v, p_H)) + |V|.$$

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This also implies that

$$r_{d+1}(G^v) = r_d(G) + |V|.$$

Lemma 5.3. *Let $d \geq 1$, and let G be a graph and G^v its cone graph. Then G is an $\mathcal{R}_d$-circuit if and only if G^v is an $\mathcal{R}_{d+1}$-circuit.*

Proof. If G^v is an $\mathcal{R}_{d+1}$-circuit, then by Theorem 2.4(a) G is not $\mathcal{R}_d$-independent. On the other hand, for every edge $uw \in E(G)$ we have that $(G - uw)^v = G^v - uw$ is $\mathcal{R}_{d+1}$-independent, so by the same theorem $G - uw$ is $\mathcal{R}_d$-independent. This shows that G is an $\mathcal{R}_d$-circuit, as desired.

Now let G be an $\mathcal{R}_d$-circuit. Again by Theorem 2.4(a), $r_{d+1}(G^v) = |E(G^v)| - 1 = r_{d+1}(G^v - uw)$ for any edge $uw \in E(G)$. Thus, we only need to prove that $r_{d+1}(G^v - uv) = r_{d+1}(G^v)$ for any edge uv incident to the cone vertex.

Let $uv \in E(G^v)$ be such an edge and consider a framework (G^v, p) in which the vertices of G lie in the $x_{d+1} = 0$ hyperplane in a generic position (when viewed as a framework in $\mathbb{R}^d$) and $p(v)$ lies outside of this hyperplane. Let w be a neighbour of u in G. Theorem 5.2 and the fact that the subframework $(G, p|_{V(G)})$ is generic imply that

$$\operatorname{rank}(R(G^v, p)) = r_{d+1}(G^v) = r_{d+1}(G^v - uw) = \operatorname{rank}(R(G^v - uw, p)).$$

Consider a framework (G, q) obtained from (G, p) by changing the last coordinate of $p(w)$ by a sufficiently small amount so that $\operatorname{rank}(R(G^v - uw, q)) = r_{d+1}(G^v) = \operatorname{rank}(R(G^v, q))$ still holds. Then there is an equilibrium stress ω of (G^v, q) that is nonzero on uw. This stress must be nonzero on uv as well, since in this framework, the rest of the edges incident to u all lie in the $x_{d+1} = 0$ hyperplane. This shows that

$$\operatorname{rank}(R(G^v - uv, q)) = \operatorname{rank}(R(G^v, q)) = r_{d+1}(G^v),$$

which implies $r_{d+1}(G^v - uv) = r_{d+1}(G^v)$, as desired. □

Theorem 5.4. *Let G be a graph, and let G^v denote its cone graph. Then G^v is $\mathcal{R}_{d+1}$-connected if and only if G is connected and it has no $\mathcal{R}_d$-bridges.*

Proof. For the 'only if' direction, observe that coning takes an $\mathcal{R}_d$-bridge of G to an $\mathcal{R}_{d+1}$-bridge of G^v, which follows from Theorem 5.2 and that an $\mathcal{R}_d$-connected graph on at least two edges has no $\mathcal{R}_d$-bridges. Moreover, $\mathcal{R}_d$-connected graphs are 2-connected,[5] while the cone graph of a disconnected graph is not.

To prove the 'if' direction, let us first observe that for any edge xy of G, xy is in the same $\mathcal{R}_{d+1}$-connected component of G^v as vx and vy. Indeed, since xy is not an $\mathcal{R}_d$-bridge in G, it is contained in some subgraph of G that is an $\mathcal{R}_d$-circuit, and by Lemma 5.3, the cone of this subgraph is an $\mathcal{R}_{d+1}$-circuit which contains all three of these edges.

Thus, it is sufficient to prove that any pair of cone edges vx, vy is in the same $\mathcal{R}_{d+1}$-connected component of G^v. By assumption, there is a path $x = u_0, u_1, \dots, u_k = y$ in G. Now by the previous observation vu_i and vu_{i+1} are in the same $\mathcal{R}_{d+1}$-connected component of G^v, for all $0 \leq i < k$. By transitivity, we get that vx and vy are also in the same $\mathcal{R}_{d+1}$-connected component, as desired. □

For a graph $G = (V, E)$ let $\operatorname{dof}_d(G) = d|V| - \binom{d+1}{2} - r_d(G)$ denote its 'degrees of freedom' in the context of d-dimensional generic rigidity. The next theorem verifies a general combinatorial property of highly connected $\mathcal{R}_d$-separable graphs and may be useful in the construction of further families of examples.

[5]This follows, e.g., from Theorem 5.1 by recalling that a graph is $\mathcal{R}_1$-connected if and only if it is 2-connected.

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Theorem 5.5. *Let $d \geq 1$, and let G be a $(d+1)$-connected and redundantly rigid graph in $\mathbb{R}^d$. Suppose that G is $\mathcal{R}_d$-separable, and let $H_1, H_2, \ldots, H_q$ be the $\mathcal{R}_d$-connected components of G. Then*

$$\sum_{i=1}^{q} \mathrm{dof}_d(H_i) \geq \binom{d+1}{2}.$$

Proof. Let $X_i = V(H_i) - \cup_{j \neq i} V(H_j)$ denote the set of vertices belonging to no other $\mathcal{R}_d$-connected component than H_i, and let $Y_i = V(H_i) - X_i$ for $1 \leq i \leq q$. Let $n_i = |V(H_i)|$, $x_i = |X_i|$, $y_i = |Y_i|$. Clearly, $n_i = x_i + y_i$ and $|V| = \sum_{i=1}^{q} x_i + |\cup_{i=1}^{q} Y_i|$. Moreover, we have $\sum_{i=1}^{q} y_i \geq 2|\cup_{i=1}^{q} Y_i|$. Since G is redundantly rigid, every edge of G is in some $\mathcal{R}_d$-circuit. Every $\mathcal{R}_d$-circuit has at least $d+2$ vertices. Thus we have that $n_i \geq d+2$ for $1 \leq i \leq q$. Furthermore, since G is $(d+1)$-connected, $y_i \geq (d+1)$ holds for all $\mathcal{R}_d$-components.

Let us choose a base B_i in each rigidity matroid $\mathcal{R}(H_i)$. Using the above inequalities, we have

$$\begin{aligned}
d|V| - \binom{d+1}{2} &= |\cup_{i=1}^{q} B_i| = \sum_{i=1}^{q} |B_i| = \sum_{i=1}^{q} \left(dn_i - \binom{d+1}{2} - \mathrm{dof}_d(H_i) \right) \\
&= d \sum_{i=1}^{q} n_i - \binom{d+1}{2} q - \sum_{i=1}^{q} \mathrm{dof}_d(H_i) \\
&= \left(d \sum_{i=1}^{q} x_i + \frac{d}{2} \sum_{i=1}^{q} y_i \right) + \left(\frac{d}{2} \sum_{i=1}^{q} y_i - \binom{d+1}{2} q \right) - \sum_{i=1}^{q} \mathrm{dof}_d(H_i) \\
&\geq d|V| - \sum_{i=1}^{q} \mathrm{dof}_d(H_i)
\end{aligned}$$

Thus, we must have $\sum_{i=1}^{q} \mathrm{dof}_d(H_i) \geq \binom{d+1}{2}$, as claimed. □

The graph of Figure 2 shows that Theorem 5.5 is, in some sense, tight: It has a unique nonrigid $\mathcal{R}_3$-connected component with six degrees of freedom. Note that for $d = 1, 2$ the $\mathcal{R}_d$-connected components of a graph are rigid. Thus, the theorem implies that for $d \leq 2$ the $(d+1)$-connected redundantly rigid graphs are $\mathcal{R}_d$-connected, which was shown in [14, Theorem 3.2] by a similar argument.

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Conflicts of Interest. None.

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