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MAPPINGS RELATED TO CONTRACTIONS

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Summary. Some results generalizing a fixed point theorem due to R. Kannan are presented.

0. Let $\mathscr{F}$ be a family of self-mappings of a metric space (X, d). In [1], [2], Kannan considered the following conditions:

A mapping $f: X \to X$ is said to satisfy *condition* (a) if there exists β with $0<\beta<1/2$ such that $d[f(x), f(y)] \leq \beta\{d[x, f(x)]+d[y, f(y)]\}$ for every $x, y \in X$.

Two mappings $f, g: X \to X$ are said to satisfy *condition* (b) if there exists β with $0<\beta<1/2$, such that $d[f(x), g(y)] \leq \beta\{d[x, f(x)]+d[y, g(y)]\}$ for every $x, y \in X$.

We first obtain the conclusion of [2] under considerably weaker hypotheses. Also considered are variants of the above condition.

1. For a mapping $f: X \to X$, points $x, y \in X$ and $\beta_1, \beta_2 \in \mathbb{R}$. Let

$$m(x, y; \beta_1, \beta_2) = \beta_1\, d[x, f(x)]+\beta_2\, d[y, f(y)]$$

and

$$M(x, y; \beta_1, \beta_2) = \max\{m(x, y; \beta_1, \beta_2), m(x, y; \beta_2, \beta_1)\}.$$

The following conditions on the mapping f will be considered:

$(A, <, \leq)$: there exist $\beta_1 \geq 0$ and $\beta_2 \geq 0$ with $\beta_1+\beta_2<1$ such that for every $x, y \in X$, $d[f(x), f(y)] \leq M(x, y; \beta_1, \beta_2)$;

$(A, \leq, <)$: there exist $\beta_1 \geq 0$, and $\beta_2 \geq 0$, with $\beta_1+\beta_2 \leq 1$ such that for every $x, y \in X$, and $x \neq y$, $d[f(x), f(y)] < M(x, y; \beta_1, \beta_2)$;

$(A, \leq, \leq)$: there exist $\beta_1 \geq 0$ and $\beta_2 \geq 0$, with $\beta_1+\beta_2 \leq 1$ such that for every $x, y \in X$, $d[f(x), f(y)] \leq M(x, y; \beta_1, \beta_2)$.

For two mappings, $f, g: X \to X$, points $x, y \in X$ and $\beta_1, \beta_2 \in \mathbb{R}$. Let

$$m(x, y; f, g; \beta_1, \beta_2) = \beta_1\, d[x, f(x)]+\beta_2\, d[y, g(y)]$$

and

$$M(x, y; f, g; \beta_1, \beta_2) = \max\{m(x, y; f, g; \beta_1, \beta_2), m(x, y; f, g; \beta_2, \beta_1)\}.$$

The following conditions on the mappings f and g will be considered:

$(B, <, \leq)$: there exist $\beta_1 \geq 0$, and $\beta_2 \geq 0$ with $\beta_1+\beta_2<1$ such that for every $x, y \in X$, $d[f(x), g(y)] \leq M(x, y; f, g; \beta_1, \beta_2)$;

$(B, \leq, <)$: there exist $\beta_1 \geq 0$ and $\beta_2 \geq 0$, with $\beta_1+\beta_2 \leq 1$ such that for every $x, y \in X$ and $x \neq y$, $d[f(x), g(y)] < M(x, y; f, g; \beta_1, \beta_2)$;

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$(B, \leq, \leq)$: there exist $\beta_1 \geq 0$ and $\beta_2 \geq 0$, with $\beta_1+\beta_2 \leq 1$ such that for every x, $y \in X$, $d[f(x), g(y)] \leq M(x, y; f, g; \beta_1, \beta_2)$.

Clearly conditions (a) and (b) of Kannan are stronger than conditions $(A, <, \leq)$ and $(B, <, \leq)$ respectively.

EXAMPLE 1. Let $X=[0, 1]$ with the usual metric. Suppose $f: X \to X$ is defined by: $f(x)=x/3$ where $x \in [0, 1)$ and $f(1)=0$.

It is easily checked that f fails condition (a) if $x=1/3$, $y=0$ and satisfies condition $(A, <, \leq)$ when $\beta_1=5/9$ and $\beta_2=1/3$.

It is also clear that f fails to be a contraction.

EXAMPLE 2. Let $X=[0, 1]$ with the usual metric. Let $f(x)=x/3$ where $x \in [0, 1)$ and $f(1)=0$, while $g(x)=x/4$, where $x \in [0, 1)$ and $g(1)=0$. Clearly f, g fail condition (b) when $x=1/3$ and $y=0$. However, f, g satisfy condition $(B, <, \leq)$ if $\beta_1=5/8$ and $\beta_2=1/4$.

EXAMPLE 3. While Example 1 shows that a mapping satisfying condition $(A, <, \leq)$ may fail to be a contraction, this example establishes the independence of the two notions. Let $X=[0, 1]$ with the usual metric. Suppose $f: X \to X$ is defined by $f(x)=9x/10$ for $x \in [0, 1]$. Clearly f is a contraction. However, f fails condition $(A, <, \leq)$ if $x=1$ and $y=0$.

EXAMPLE 4. Let $X=[0, 1]$ with the usual metric. Let $\mathscr{F}=\{f_n : n=3, 4, \ldots\}$ where each f_n is defined by $f_n(x)=x/n$ where $x \in [0, 1)$ and $f_n(1)=0$.

It can be easily checked that:

(1) f_3 satisfies condition $(A, <, \leq)$ when $\beta_1=5/9$ and $\beta_2=1/3$, (2) for $n \geq 4$, each f_n satisfies condition $(A, <, \leq)$ when $\beta_1=3/8$ and $\beta_2=1/2$, (3) condition $(B, <, \leq)$ is satisfied by every distinct pair of mappings in $\mathscr{F}$ when $\beta_1=5/8$ and $\beta_2=1/4$.

LEMMA 1. *Let f be a mapping of (X, d) into itself.*

(i) If f satisfies $(A, <, \leq)$, then there is an $r \in \mathbb{R}$, with $0 \leq r < 1$, such that

$$d[f^{n+1}(x), f^n(x)] \leq r\, d[f^n(x), f^{n-1}(x)] \quad \text{for all } x \in X,$$

and

$$n \geq 1.$$

(ii) If f satisfies $(A, \leq, <)$, then there is an $r \in \mathbb{R}$, with $0 \leq r \leq 1$, such that

$$d[f^{n+1}(x), f^n(x)] < r\, d[f^n(x), f^{n-1}(x)] \quad \text{for all } x \in X,$$

with

$$f^n(x) \neq f^{n-1}(x),$$

where $n \geq 1$.

(iii) If f satisfies $(A, \leq, \leq)$, then there is an $r \in \mathbb{R}$, with $0 \leq r \leq 1$ such that

$$d[f^{n+1}(x), f^n(x)] \leq r\, d[f^n(x), f^{n-1}(x)] \quad \text{for all } x \in X,$$

and

$$n \geq 1.$$

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Proof. (i) Since f satisfies $(A, <, \leq)$, there exist β_1 and β_2 ($\beta_1 \geq 0, i=1, 2$) with $\beta_1+\beta_2<1$ such that $d[f(x), f(y)] \leq M(x, y; \beta_1, \beta_2)$. Let $r=\max\{\beta_1/(1-\beta_2), \beta_2/(1-\beta_1)\}$. Then $r<1$, and $d[f^{n+1}(x), f^n(x)] \leq r\, d[f^n(x), f^{n-1}(x)]$. The conclusion of (ii) and (iii) can be obtained similarly.

PROPOSITION 1. *Let f be a mapping on (X, d) into itself which satisfies $(A, <, \leq)$. Suppose for some $p \in X$, the sequence $\{f^n(p): n=1, 2, \ldots\}$ contains a convergent subsequence, then f has a unique fixed point.*

Proof. By Lemma 1, the sequence $\{f^n(p): n=1, 2, \ldots\}$ is Cauchy. Hence $u=\lim_{n\to\infty} f^n(p)$ exists. Let $\beta=\max\{\beta_1, \beta_2\}$; where β_1, β_2 are as guaranteed by $(A, <, \leq)$. Then $d[f(u), f^n(p)] \leq \beta\{d[u, f(u)]+d[f^{n-1}(p), f^n(p)]\}$. As $n\to\infty$, $d[f(u), u] \leq \beta\, d[u, f(u)]$. Since $\beta<1$, $d[f(u), u]=0$ showing that u is a fixed point of f. Suppose $w \in X$ satisfies $f(w)=w$. Since $d(w, u)=d[f(w), f(u)] \leq \beta\{d[w, f(w)]+d[u, f(u)]\}=0$, we have $w=u$, showing the uniqueness of u.

COROLLARY. *Let f be a mapping of (X, d) into itself which satisfies condition $(A, <, \leq)$. If X is complete, or if for some $p \in X$, the sequence $\{f^n(p): n=1, 2, \ldots\}$ contains a convergent subsequence, then f has a unique fixed point.*

Proof. This follows immediately from Lemma 1 and Proposition 1.

REMARK. It is immediate that Theorem 1 [2] follows from the above Corollary. It is also noted that f is not assumed to be continuous at p in our result.

PROPOSITION 2 (*cf. Theorem* 1 *of* [1]). *Let (X, d) be a complete metric space and $\mathscr{F}$ an arbitrary nonempty family of mappings of X into itself. Suppose $(B, <, \leq)$ is satisfied by every $f, g \in \mathscr{F}$. Then $\mathscr{F}$ has a unique common fixed point.*

Proof. It is clear that by the Corollary to Proposition 1, each $f \in \mathscr{F}$ has a unique fixed point in X. Let $f, g \in \mathscr{F}$ with $f \neq g$. Suppose u and z are the unique fixed points of f and g respectively. By condition $(B, <, \leq)$, $d(u, z)=d[f(u), g(z)] \leq 0$, Hence $u=z$ and $\mathscr{F}$ has a unique common fixed point.

COROLLARY. *Let (X, d) be a metric space and $\mathscr{F}$ an arbitrary nonempty family of mappings of X into itself. Suppose* (i) *condition $(B, <, \leq)$ is satisfied by every $f, g \in \mathscr{F}$ and* (ii) *for every $f \in \mathscr{F}$, there exists $p \in X$ such that $\{f^n(p); n=1, 2, \ldots\}$ has a convergent subsequence. Then $\mathscr{F}$ has a unique common fixed point.*

Proof. Use the Corollary to Proposition 1, and the proof of Proposition 2.

LEMMA 2. *Let f be a continuous mapping of (X, d) into itself satisfying $(A, \leq, \leq)$. Suppose there exists $z \in X^f$. Then f is an isometry on each pair $f^n(z), f^{n-1}(z)$, where $n=1, 2, \ldots$. (X^f is the set of points $x \in X$ such that there exists $p \in X$ and a sequence of integers m_i with $\lim_{i\to\infty} f^{m_i}(p)=x$).*

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Proof. For every point $x \in X$, and any $n=1, 2, \ldots$, by Lemma 1, $d[f^n(x), f^{n+1}(x)] \le d[f^n(x), f^{n-1}(x)]$. Hence, for any $m \ge n+2$, $d[f^{m+1}(x), f^m(x)] \le d[f^{n+1}(x), f^n(x)]$. Since $z \in X^f$ there exists $p \in X$ and a sequence of integers m_i such that $\lim_{i\to\infty} f^{m_i}(p) = \lim_{i\to\infty} f^{m_{3i}}(p) = z$. Hence,

$$d[f^{n+m_{3i}-1}(p), f^{n+m_{3i}}(p)] \le d[f^{n+m_i+1}(p), f^{n+m_i}(p)].$$

As $i \to \infty$, $d[f^{n-1}(z), f^n(z)] \le d[f^{n+1}(z), f^n(z)]$, and we have $d[f^{n+1}(z), f^n(z)] = d[f^{n-1}(z), f^n(z)]$ for $n=1, 2, \ldots$.

PROPOSITION 3. *Let f be a continuous mapping of (X, d) into itself satisfying $(A, \le, \le)$. Suppose for every $x \in X$ with $x \ne f(x)$, there exists an integer K, $K \ge 1$, such that $d[f^K(x), f^{K+1}(x)] < d[f^{K-1}(x), f^K(x)]$ whenever $d[f^{K-1}(x), f^K(x)] > 0$, and there exists $z \in X^f$. Then z is a unique fixed point of f.*

Proof. By the previous lemma and the hypotheses, $f(z)=z$. The uniqueness of Z follows by a similar argument as in the proof of Proposition 1.

The following corollaries are immediate consequences of Lemma 1 and Proposition 3.

COROLLARY 1. *Let f be a continuous mapping of (X, d) into itself satisfying $(A, \le, <)$. Suppose there exists $u \in X^f$. Then u is a unique fixed point of f.*

COROLLARY 2. *Let $\mathscr{F}$ be an arbitrary nonempty family of continuous mappings of (X, d) into itself such that $(B, \le, \le)$ is satisfied by every two members $f, g \in \mathscr{F}$. Suppose each $f \in \mathscr{F}$ satisfies condition $(A, \le, <)$. If for each $f \in \mathscr{F}$ there exists $z \in X^f$, then z is a unique common fixed point of $\mathscr{F}$.*

2. Condition (a) is generalized in another direction by Reich [3] to obtain the following condition on a mapping f from (X, d) into itself: (R): $d[f(x), f(y)] \le \beta_1 d[x, f(x)] + \beta_2 d[y, f(y)] + \beta_3 d(x, y)$, where $\beta_i \ge 0$, $i=1, 2, 3$ and $\beta_1+\beta_2+\beta_3<1$. However, by interchanging the role of x and y, condition (R) actually reduces to condition (R'): $d[f(x), f(y)] \le \alpha\{d[x, f(x)] + d[y, f(y)]\} + \beta\, d(x, y)$ where $2\alpha+\beta<1$.

As a result, when $\beta_3=0$, condition (R) reduces to condition (a) of Kannan, but not to condition $(A, <, \le)$.

Using the idea of involving the term $d(x, y)$, conditions $(A, <, \le)$ $(A, \le, <)$, and $(A, \le, \le)$ can be further generalized as follows:

$(A^*, <, \le)$: there exist β_1, β_2 and $\beta_3 (\beta_i \ge 0, i=1, 2, 3)$ with $\beta_1+\beta_2+\beta_3<1$ such that for every $x, y \in X$, $d[f(x), f(y)] \le M(x, y; \beta_1, \beta_2) + \beta_3 d(x, y)$;

$(A^*, \le, <)$: there exist β_1, β_2 and $\beta_3 (\beta_i \ge 0, i=1, 2, 3)$ with $\beta_1+\beta_2+\beta_3 \le 1$, such that for every $x, y \in X$, $(x \ne y)$, $d[f(x), f(y)] < M(x, y; \beta_1, \beta_2) + \beta_3 d(x, y)$;

$(A^*, \le, \le)$: there exist β_1, β_2 and β_3 $(\beta_i \ge 0, i=1, 2, 3)$ with $\beta_1+\beta_2+\beta_3 \le 1$, such that for every $x, y \in X$, $d[f(x), f(y)] \le M(x, y; \beta_1, \beta_2) + \beta_3 d(x, y)$.

Previous results involving conditions $(A, <, \le)$, $(A, \le, <)$ and $(A, \le, \le)$ remain valid when these conditions are replaced by $(A^*, <, \le)$, $(A^*, \le, <)$ and $(A^*, \le, \le)$ respectively. These follow readily after observing that the assertions of Lemma 1 remain valid when the conditions $(A, <, \leq)$, $(A, \leq, <)$ and $(A, \leq, \leq)$ are replaced by $(A^*, <, \leq)$, $(A^*, \leq, <)$ and $(A^*, \leq, \leq)$ respectively.

3. In this section we introduce conditions related to those discussed previously. As before, let f be a mapping of (X, d) into itself. For every $x, y \in X$, $\beta_1, \beta_2 \in \mathbb{R}$, let

$$k(x, y; \beta_1, \beta_2) = \beta_1\{d[x, f(x)]+d[y, f(y)]\}+\beta_2\{d[f(x), f^2(x)]+d[f(y), f^2(y)]\}.$$

We consider the following condition:

(GL): there exist β_1 and β_2 ($\beta_i \geq 0$, $i=1, 2$), with $\beta_1+\beta_2<1$ and for every $x, y \in X$, there exists a nonnegative integer N such that for $n \geq N$,

$$d[f^{n+1}(x), f^{n+1}(y)]+d[f^{n+2}(x), f^{n+2}(y)] \leq k(f^n(x), f^n(y); \beta_1, \beta_2).$$

PROPOSITION 4. *Let f map (X, d) into itself and satisfy condition (GL). Suppose X is either complete, or for some $\bar{x} \in X$, the sequence $\{f^n(\bar{x}): n=1, 2, \ldots\}$ contains a convergent subsequence. If f is continuous, then f has a unique fixed point.*

Proof. This result is immediate since it can be easily shown that for every $x \in X$, the sequence $\{f^n(x): n=1, 2, \ldots\}$ is Cauchy.

4. Let $\mathscr{F}$ be a commutative semigroup of self-mappings on (X, d). For every $x, y \in X, f, g \in X$, and $\beta_1, \beta_2 \in \mathbb{R}$; let

$$\begin{aligned} k(x, y; f, g, n; \beta_1, \beta_2) &= \beta_1\{d[f^n g(x), f^{n+1}g(x)]+d[f^n g(y), f^{n+1}g(y)]\} \\ &\quad +\beta_2\{d[f^{n+1}g(x), f^{n+2}g(x)]+d[f^{n+1}g(y), f^{n+2}g(y)]\} \end{aligned}$$

The following condition on $\mathscr{F}$ is introduced:

(SL^*) there exist β_1, β_2 ($\beta_i \geq 0$, $i=1, 2$) and $\lambda>0$, with $\beta_1+\beta_2<\lambda$; and for every $x, y \in X$, there exists $g \in \mathscr{F}$ such that for all $f \in \mathscr{F}$, $n = 0, 1, \ldots,$

$$\lambda\{d[f^{n+2}g(x, f^{n+2}g(y)]+d[f^{n+1}g(x), f^{n+1}g(y)]\} \leq k(x, y; f, g, n; \beta_1, \beta_2)$$

PROPOSITION 5. *Let (X, d) be a complete metric space and $\mathscr{F}$ a commutative semigroup of continuous mappings of X into itself satisfying (SL^*). Suppose each $f \in \mathscr{F}$ satisfies $(A, \leq, \leq)$, then $\mathscr{F}$ has a unique common fixed point.*

Proof. Let β_1 and β_2 be as in condition (SL^*). For every $x \in X$, if $h \in \mathscr{F}$ is such that $h(x) \neq x$ then there exists $g \in \mathscr{F}$ such that

$$\begin{aligned} &d[h^{n+1}g(x), h^{n+2}g(x)]+d[h^{n+2}g(x), h^{n+3}g(x)] \\ &\qquad \leq (\beta_1/(1-\beta_2))\{d[h^n g(x), h^{n+1}g(x)]+d[h^{n+1}g(x), h^{n+2}g(x)]\} \end{aligned}$$

Since $\beta_1/(1-\beta_2)<1$ the sequence $\{h^n g(x)\}$ is Cauchy, and the conclusion of the proposition is immediate.

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