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THE ASYMMETRIC PRODUCT OF THREE INHOMOGENEOUS LINEAR FORMS

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Abstract

Let Λ be a lattice in $\mathbf{R}_3$ of determinant 1. Define the homogeneous minimum of Λ as $m_h(\Lambda) = \inf |u_1u_2u_3|$ extended over all points (u_1, u_2, u_3) of Λ other than the origin. It is shown that for any given (c_1, c_2, c_3) in R_3 there exists a point (u_1, u_2, u_3) of Λ for which

$$-\rho \leq (u_1 + c_1)(u_2 + c_2)(u_3 + c_3) \leq \sigma, \qquad \rho, \sigma > 0,$$

provided that $\rho\sigma > 1/64$ if $m_h(\Lambda) = 0$, and $\rho\sigma \geq 1/16.81$ if $m_h(\Lambda) > 0$.

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1. Introduction

For $1 \leq i \leq n$, let $L_i = a_{i1}x_1 + \cdots + a_{in}x_n$ be n linear forms in the variables $x_1, \ldots, x_n$ with real coefficients a_{ij} satisfying $|\det(a_{ij})| = 1$. A classical conjecture of Minkowski asserts that, given n real numbers $c_1, \ldots, c_n$, the inequality

$$\prod_{i=1}^{n} |L_i + c_i| \leq \frac{1}{2^n} \tag{1}$$

has a solution in integral values of the variables. This has been proved for $n \leq 5$; see Bambah and Woods [1]. For the case $n = 2$, Davenport [5] generalised this is to the asymmetric case and proved that, given real numbers c_1, c_2, the inequality

$$-\sigma \leq (L_1 + c_1)(L_2 + c_2) \leq \rho, \qquad \rho, \sigma > 0, \tag{2}$$

has a solution in integral values of the variables provided that $\rho\sigma \geq 1/16$.

For the case $n = 3$, we obtain here a sufficient condition on ρ, σ for the asymmetric inequality corresponding to (2) to have a solution in integral values of the variables.

Let $A_j = (a_{1j}, \ldots, a_{nj})$, for $1 \leq j \leq n$, and denote by Λ the lattice with basis $A_1, \ldots, A_n$ such that $d(\Lambda) = 1$. Let $m_h(\Lambda)$ denote the infimum of $|x_1 \cdots x_n|$ extended over all points $(x_1, \ldots, x_n)$ of Λ other than the original 0. We have

THEOREM 1. *Let Λ be a lattice of determinant 1. For any point $C = (c_1, c_2, c_3) \in \mathbf{R}_3$, there exists a point $A = (a_1, a_2, a_3)$ of Λ such that*

$$-\rho < (a_1 + c_1)(a_2 + c_2)(a_3 + c_3) < \sigma, \qquad \rho, \sigma > 0, \tag{3}$$

provided that

(a) $\rho\sigma > 1/64$ *if* $m_h(\Lambda) = 0$, *and*

(b) $\rho\sigma \geq 1/16.81$ *if* $m_h(\Lambda) > 0$.

The method of proof is the projective one due to Birch and Swinnerton-Dyer [2]. The author [6] used the same method earlier to obtain the condition $\rho\sigma \geq (4\sqrt{5} - 5)/64 = 1/16.224\ldots$ to be a sufficient condition. Woods [9], using this method, proved that if in (3), $(a_3 + c_3)$ is replaced by $|a_3 + c_3|$ then the result holds for $\rho\sigma \geq 1/64$. Our method of proof is parallel to that of Woods [9].

2. Proof for case (a)

LEMMA 1. *If Λ is a lattice in $\mathbf{R}_2$ of determinant $d(\Lambda)$ and $\lambda > 0$, then given any point $c = (c_1, c_2) \in \mathbf{R}_2$, there exists a point $A = (a_1, a_2)$ of Λ such that*

$$\frac{-d(\Lambda)}{4\lambda} < (a_1 + c_1)(a_2 + c_2) \leq \frac{d(\Lambda)\lambda}{4}.$$

This result is due to Davenport [5].

We say that a lattice Λ in $\mathbf{R}_n$ is a covering lattice for a region S, if the translates of S by the points of Λ cover the whole space $\mathbf{R}_n$. It is clear that Λ is a covering lattice for S if and only if given any $C \in \mathbf{R}_n$, there exists $A \in \Lambda$ such that $A + C \in S$. We have

LEMMA 2. *Let S be an open set in $\mathbf{R}_n$, Λ a lattice and let ω_r be a sequence of automorphs of S such that $\{\omega_r\Lambda\}$ is a sequence of lattice converging to a lattice Γ. If Γ is a covering lattice for S, then so is Λ.*

PROOF. Since $\omega_r\Lambda = \Lambda^{(r)} \to \Gamma$, there exists a basis $A_1^{(r)}, \dots, A_n^{(r)}$ of $\Lambda^{(r)}$ and a basis $A_1, \dots, A_n$ of Γ such that $A_i^{(r)} \to A_i$, $1 \le i \le n$.

Let $C \in \mathbf{R}_n$ be arbitrary, let $C^{(r)} = \omega_r C$, and choose $\hat{C}(r)$ in the fundamental parallelopiped $\{\sum_{i=1}^n \alpha_i A_i^{(r)}; 0 \le \alpha_i < 1, i = 1, \dots, n\}$ of $\Lambda^{(r)}$ such that $C^{(r)} \equiv \hat{C}(r) \pmod{\Lambda^{(r)}}$, whence $\hat{C}^{(r)}$ is a bounded sequence. On replacing ω_r by a subsequence we can suppose that $\hat{C}^{(r)} \to \hat{C}$.

Since Γ is a covering lattice for S, there exists $B \in \Gamma$ such that $B + \hat{C} \in S$. Let $B^{(r)} \in \Lambda^{(r)}$ be such that $B^{(r)} \to B$. Then $B^{(r)} + \hat{C}^{(r)} \to B + \hat{C}$ and hence for larger r, $B^{(r)} + \hat{C}^{(r)} \in S$, and hence there exists $A \in \Lambda$ such that $A + C \in S$. This proves Lemma 2.

If Ω denotes the group of automorphs of the form $x_1 \cdots x_n$, generated by the permutations of x_i and the transformations of the type $x_i \to \lambda_i x_i$, $\lambda_i \in \mathbf{R}$ and $\prod \lambda_i = 1$, then we have following result due to Birch and Swinnerton-Dyer [2].

LEMMA 3. *Let Λ be a lattice in $\mathbf{R}_n$ with $m_h(\Lambda) = 0$. Then there exists a sequence ω_r in Ω such that $\{\omega_r\Lambda\}$ tends to a lattice Γ having a basis $A_1 \cdots A_n$ such that for some k, $1 \le k < n$, $A_1, \dots, A_k$ lie in a k-dimensional coordinate plane.*

Theorem 1(a) is a consequence of

THEOREM 1A. *Let Λ be a lattice in $\mathbf{R}_3$ of determinant 1 with $m_h(\Lambda) = 0$. For any given $\lambda > 0$, $\varepsilon > 0$ and $C = (c_1, c_2, c_3)$ in $\mathbf{R}_3$, there exists a point $A = (a_1, a_2, a_3)$ of Λ such that*

$$-\frac{(1+\varepsilon)}{8\lambda} < (a_1 + c_1)(a_2 + c_2)(a_3 + c_3) < \frac{(1+\varepsilon)\lambda}{8}. \tag{4}$$

PROOF. By Lemma 3, we can find a sequence $\omega_r \in \Omega$ such that $\omega_r\Lambda \to \Gamma = \{(L_1, L_2, L_3)$, for integral values of variables$\}$, where either

$$\begin{array}{lll} \text{either} & (1) & L_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ & & L_2 = \qquad\quad a_{22}x_2 + a_{23}x_3 \\ & & L_3 = \qquad\quad a_{32}x_2 + a_{33}x_3 \\ \text{or} & (2) & L_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ & & L_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ & & L_3 = \qquad\qquad\qquad a_{33}x_3. \end{array}$$

On replacing ω_r by $\omega \cdot \omega_r$ for suitable $\omega \in \Omega$, we can suppose that in case (1) $a_{11} = 1$ and in case (2) $a_{33} = 1$.

In view of Lemma 2, it is sufficient to prove the result for the lattice Γ.

Case 1. Since $a_{11} = 1$, the set $\mathscr{L} = (L_1, L_2)$ for integral values of variables is a lattice of determinant 1. By Lemma 1, with $\lambda = 1$, there exist integers u_2, u_3

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such that for $(x_2, x_3) = (u_2, u_3)$

$$|(L_2 + c_2)(L_3 + c_3)| \le \tfrac{1}{4}. \tag{5}$$

If in (5), $(L_2 + c_2)(L_3 + c_3) = 0$, then (4) holds for any choice of $u_1 \in \mathbb{Z}$.

Suppose that $(L_2 + c_2)(L_3 + c_3) \ne 0$. Let $\hat{c}_1 = a_{12}u_2 + a_{13}u_3 + c_1$. First suppose that $(L_2 + c_2)(L_3 + c_3) > 0$. In this case, for any given $\lambda > 0$, there exists $u_1 \in \mathbb{Z}$ such that

$$-\frac{1}{2\lambda} \le u_1 + \hat{c}_1 \le \frac{\lambda}{2}. \tag{6}$$

Since $0 < (L_2+c_2)(L_3+c_3) \le \frac{1}{4}$ and $L_1+c_1 = u_1+\hat{c}_1$, we have for $(u_1, u_2, u_3) \in \mathbb{Z}^3$ that

$$-\frac{1}{8\lambda} \le (L_1 + c_1)(L_2 + c_2)(L_3 + c_3) \le \frac{\lambda}{8},$$

and hence (4) holds for the lattice Γ and the point C. The case

$$(L_2 + c_2)(L_3 + c_3) < 0$$

follows on replacing λ by $1/\lambda$ in (6) and repeating the same argument as in the case of $(L_2 + c_2)(L_3 + c_3) > 0$.

Case 2. Since $a_{33} = 1$, there exists $u_3 \in \mathbb{Z}$ such that $|L_3 + c_3| \le 1/2$. As in Case 1, if $L_3 + c_3 = 0$, we are through. We can suppose that $L_3 + c_3 \ne 0$.

Let $\hat{L}_1 = a_{11}x_1 + a_{12}x_2, \hat{c}_1 = a_{13}u_3 + c_1$, and $\hat{L}_2 = a_{21}x_1 + a_{22}x_2, \hat{c}_2 = a_{23}u_3 + c_2$, so that the set of points $(\hat{L}_1, \hat{L}_2)$ for integral values of variables is a lattice of determinant 1.

Firstly suppose that $L_3 + c_3 > 0$. By Lemma 1, for $\lambda > 0$, we can find integral values u_1, u_2 of variables such that

$$-\frac{1}{4\lambda} < (L_1 + c_1)(L_2 c_2) \le \frac{\lambda}{4}. \tag{7}$$

Since $0 < L_3 + c_3 \le \frac{1}{2}$, we get

$$-\frac{1}{8\lambda} < (L_1 + c_1)(L_2 + c_2)(L_3 + c_3) \le \frac{\lambda}{8}.$$

When $L_3 + c_3 < 0$, the result follows on replacing λ by $\frac{1}{\lambda}$ in (7). This completes the proof of Theorem 1A and therefore of (a).

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3. Proof for case (b)

This is equivalent to proving

THEOREM 1B. *Lt Λ be a lattice of determinant* 4.1, *with $m_h(\Lambda) > 0$. Then for any given real numbers $\rho, \sigma > 0$, with $\rho\sigma = 1$, and any given point $C \in \mathbf{R}_3$, the grid $\Lambda + C$ has a point in the region*

$$S: \; -\rho < xyz < \sigma. \tag{8}$$

In order to obtain some restrictions on ρ and σ, we need

LEMMA 4. *Let L_1, L_2, L_3 be three real linear forms in three variables x, y, z of determinant $\Delta \neq 0$. Then given any real numbers c_1, c_2, c_3 there exist integers x, y, z such that*

$$|(L_1 + c_1)(L_2 + c_2)(L_3 + c_3)| \leq \frac{|\Delta|}{8}.$$

This result is due to Remark [8].

For the one sided inequality Chalk [3] proved

LEMMA 5. *If $L_1, \ldots, L_n$ are n real linear forms in n variables of determinant $\Delta \neq 0$, then for any real numbers $c_1, \ldots, c_n$, we can find integral values of variables such that*

$$0 < \prod_{i=1}^{n}(L_1 + c_i) \leq |\Delta|, \qquad L_i + c_i > 0, \qquad i = 1, \ldots, n.$$

COROLLARY. *It is sufficient to prove Theorem* 1B *for the case* $8/4.1 \leq \rho \leq 4.1$.

PROOF On replacing Λ by Λ', where $\Lambda' = \{(-x, y, z) | (x, y, z) \in \Lambda\}$, if necessary, we can suppose that $\rho \geq \sigma$. Now for $\rho < 8/4.1$, we have $\rho \geq \sigma > 4.1/8$, and the result follows by Lemma 4.

For $\rho > 4.1$, the result is a consequence of Lemma 5. Hence the Corollary follows.

It is enough to prove Theorem 1B, when $m_h(\Lambda)$ is attained, for otherwise as in Birch and Swinnerton-Dyer [2, Theorem 2], following Mahler [7, Theorem 20], there exists a sequence $\omega_r \in \Omega$ such that $\omega_r\Lambda \to \Gamma$, $m_h(\Lambda) = m_h(\Gamma)$ and the homogeneous minimum is attained for Γ. In view of Lemma 2, it is sufficient to prove Theorem 1B for the lattice Γ. Since Theorem 1B is invariant under Ω, we can further assume that $m_h(\Lambda)$ is attained at the point $P = (a, a, a) \in \Lambda$, $a > 0$. By a well known theorem of Davenport [4], it follows that $a^3 \leq 4.1/7$.

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LEMMA 6. *Let Λ be a lattice in $\mathbf{R}_3$ of determinant 4.1, with $P = (a, a, a) \in \Lambda$, $a > 0$, such that $a^3 = m_h(\Lambda)$. Let $\mathscr{L}$ be the projection of Λ on the x, y plane, parallel to the vector (a, a, a). For any given real numbers $\rho > 0$, $\sigma > 0$, with $\rho\sigma = 1$, and any point $x \in \mathbf{R}_2$, the two dimensional grid, $\mathscr{L} + X$ has a point in the region $\mathbf{K}$ in the plane, given as the set of points (x, y) such that*

$$(9) \qquad -\frac{(4.1)\sigma}{4a} \leq xy \leq \frac{(4.1)\rho}{4a}$$

and either $|x + y| \leq 12.3/8a^2$ or $|x - y| \leq 12.3/8a^2$.

PROOF. Woods [9, Theorem 1B] proved that for a lattice of determinant 8, the corresponding grid $\mathscr{L} + X$, for any given $X \in \mathbf{R}_2$, has a point in the region consisting of points (x, y) such that

$$(10) \qquad -\frac{2\sigma}{a} \leq xy \leq \frac{2\rho}{a}$$

and either $|x + y| \leq 3/a^2$ or $|x - y| \leq 3/a^2$.

Now, if Λ is a lattice of determinant 4.1, then for $d^3 = 8/4.1$, $d\Lambda$ is a lattice of determinant 8, with the point $Q = (ad, ad, ad) \in d\Lambda$, where $m_h(d\Lambda)$ is attained. Its projection on the x, y-plane parallel to the vector (ad, ad, ad) is $d\mathscr{L}$, so that for any $X \in \mathbf{R}_2$, $d\mathscr{L} + dX$ has a point in the region defined by (10) and hence $\mathscr{L} + X$ has a point in the region $\mathbf{K}$. This completes the proof of Lemma 6.

For any given ρ, $\sigma > 0$, $\rho\sigma = 1$. Let $\mathbf{K}_1$ be the set of points (x, y) such that for any given $t_0 \in \mathbf{R}$ there exists $t \equiv t_0 \pmod a$ satisfying

$$(11) \qquad -\rho < (x + t)(y + t)(t - b) < \sigma, \quad \text{where } b = 4a/4.1.$$

LEMMA 7. *Let $\mathbf{K}$ be defined by (9) and let $\mathbf{K}_1$ be as above. If for some ρ, σ, with $\rho\sigma = 1$, $\mathbf{K}_1 \supset \mathbf{K}$, then Theorem 1B is true for that choice of ρ and σ.*

PROOF. Let $C \in \mathbf{R}_3$ be any given point. Projection of the grid $\Lambda + C$ parallel to the vector (a, a, a) on the plane $Z = -b$ is a translate of the lattice $\mathscr{L}$ and hence in view of Lemma 6, has a point $(x^*, y^*, -b)$ in $\mathbf{K} + (0, 0, -b)$, so that $\Lambda + C$ has a point of the type $(x^* + t_0, y^* + t_0, t_0 - b)$, for $(x^*, y^*) \in \mathbf{K}$, for some $t_0 \in R$. Since $\mathbf{K} \subset \mathbf{K}_1$, there exists $t \equiv t_0 \pmod a$ satisfying (11). Also since $(a, a, a) \in \Lambda$, the point $(x^* + t, y^* + t, t - b) \in (\Lambda + C) \cap S$, which completes the proof of Lemma 7.

The corollary to Lemma 4 and Lemma 5, and Lemma 7 imply that Theorem 1B is consequence of

THEOREM 2. *Let $a > 0$ be a real number satisfying $a^3 \leq 4.1/7$. Let $\rho, \sigma > 0$, with $\rho\sigma = 1, 8/4.1 \leq \rho \leq 4.1$ be two given numbers. Then for any point (x, y) of* *the region* $\mathbf{K}$, *defined by* (9), *and any real number* t_0, *there exists* $t \equiv t_0 \pmod a$ *such that*

$$(12) \qquad -\rho < (x+t)(y+t)(t-b) < \sigma, \quad \textit{where } b = \frac{4a}{4.1}.$$

4. Proof of Theorem 2

From the symmetry in x, y we may assume that $|x| \le |y|$. Further for $(x, y) \in \mathbf{K}$, we have

$$(13) \qquad |xy| \le \frac{\rho}{b}, \quad |y| \le \min\left(\frac{12.3}{8a^2} + |x|, \frac{\rho}{b|x|}\right).$$

Also

$$(14) \qquad |x+y| \le \sqrt{\left(\frac{12.3}{8a^2}\right)^2 + \frac{4.1}{a}\rho}$$

and

$$(15) \qquad -\rho \le -xyb \le \sigma.$$

Let $f(t) = (t+x)(t+y)(t-b)$. For $X = (x, y) \in \mathbf{R}_2$, denote by S_X the set of all real numbers t satisfying (12). We have

LEMMA 8. *For all* $X \in \mathbf{K}$, $[b, a] \subseteq S_X$.

PROOF. For $b \le t \le a$, we have

$$\begin{aligned} |f(t)| &= |xy + t(x+y) + t^2||t-b| \\ &\le (|xy| + a|x+y| + a^2)(a-b) \\ &\le \frac{\rho}{40} + \frac{1}{41}\left(\left(\frac{12.3}{8}\right)^2 + (4.1)\rho a^3\right)^{1/2} + \frac{1}{41}a^3 \quad \text{(by (13) and (14))} \end{aligned}$$

if

$$\frac{\rho^2}{40} + \frac{\rho}{41}\left(\left(\frac{12.3}{8}\right)^2 + (4.1)\rho a^3\right)^{1/2} + \frac{1}{41}a^3 < 1 \quad (\text{since } \rho\sigma = 1).$$

Since $a^3 \le 4.1/7$ and $\rho \le 4.1$ the above holds and hence $[b, a] \subseteq S_X$ for all $X \in \mathbf{K}$.

REMARK 1. In view of Lemma 8, it is enough to prove that for $X \in \mathbf{K}$, we have either

$$(16) \qquad (0, b] \subseteq S_X,$$

or

(17) S_X contains a half open interval I of length a,

or

(18) there exist real numbers $t_1 < t_2 < t_3 < t_4$ such that $t_4 - t_1 \geq 2a, t_3 - t_2 < a$ and S_X contains $[t_1, t_2) \cup (t_3, t_4]$.

REMARK 2. If $g(t) = (t-\alpha)(t-\beta)(t-\gamma)$ is a polynomial with α, β, γ real and satisfying $\alpha \leq \beta \leq \gamma$ then it is easy to see that

(i) $g(t)$ is monotonically increasing for $t \leq \alpha$ and $t \geq \gamma$,

(ii) $g(t)$ is monotonically decreasing function for $(\alpha+\beta)/2 \leq t \leq (\beta+\gamma)/2$.

From now onward (x, y) will stand for a point in $\mathbf{K}$. We distinguish the following cases.

Case I: *(x, y) in the first quadrant.*

Subcase I (i): $x > b$. Since $y \geq x \geq b$, $f(t)$ is negative and has no root in the interval $(-x, b)$, so $f(t)$ is monotone in either the interval $[-x, 0)$ or the interval $(0, b]$. Since by (15), $f(0) = -xyb \geq -\rho$, either $(0, b]$ or $[-x, 0)$ is contained in S_X. Now if $x \geq a$, we are through. Otherwise for $-a \leq t \leq -x \leq -b$, we have

$$\begin{aligned} |f(t)| &= |(x+t)(y+t)(t-b)| \\ &< \left(\frac{1}{41}a\right)(2a)|y| \\ &\leq \frac{2}{41}\frac{a^2}{b^2}\rho \quad \left(\text{since } |y| \leq \frac{\rho}{b|x|} \leq \frac{\rho}{b^2}\right) \\ &< \sigma, \end{aligned}$$

since $\rho \leq 4.1$. So in this case either $[-a, 0)$ or $(0, b]$ is contained in S_X, the result follows in view of Remark 1.

Subcase I (ii): $x < b$ and $\rho \geq 2.11$. For $0 \leq t \leq b$, we have $f(t) \leq 0$ and

$$\begin{aligned} |f(t)| &= (b-t)(t+y)(t+x) \\ &\leq (b-t)\left(t + x + \frac{12.3}{8a^2}\right)(t+x) \quad \text{(by (13))} \\ &\leq (b^2 - t^2)(b+t) + \frac{12.3}{8}\left(\frac{b}{a}\right)^2 = g(t), \end{aligned}$$

say. Since $g(t)$ has maximum value at $t = b/3$ we have

$$|f(t)| \leq g\left(\frac{b}{3}\right) < 2.108\cdots < \rho \quad \left(\text{since } b = \frac{4a}{4.1}, a^3 \leq \frac{4.1}{7}\right)$$

so $(0, b] \subset S_X$, and the results follows by (16).

Subcase I (iii): $x \le b$, $a^3 \le 11/25, 8/4.1 \le \rho \le 2.11$. Again as in Case I (ii), we have, for $0 \le t \le b$, that

$$|f(t)| \le g(b/3) < 1.9477\cdots < 8/4.1 \le \rho \quad \left(\text{since } a^3 \le \frac{11}{25}\right).$$

So $(0, b] \subseteq S_X$ and the result follows by (16).

Subcase I (iv): $0 \le x \le (9.1)b$, $a^3 > 11/25$, $\rho \le 2.11$. We have, for $0 < t \le b$, that $f(t)$ is negative and

$$\begin{aligned}|f(t)| &= (b-t)(x+t)(y+t)\\ &\le (b-t)\left(\frac{91}{100}b+t\right)\left(\frac{91}{100}b+\frac{12.3}{8a^2}+t\right) \quad \text{(by (13))}\\ &= (b-t)\left(\frac{91}{100}b+t\right)^2+(b-t)\left(\frac{91}{100}b+t\right)\frac{12.3}{8a^2}\\ &\le (b-t)\left(\frac{91}{100}b+t\right)^2+\left(\frac{191}{200}\right)^2\left(\frac{12.3}{8}\right)\left(\frac{b^2}{a^2}\right) \quad \text{(A. G. mean)}\\ &= h(t),\end{aligned}$$

say. Since $h(t)$ has a maximum at $t = (109/300)b$, we have

$$|f(t)| \le h\left(\frac{109}{300}\right) < 1.9 < \rho,$$

so $(0, b] \subseteq S_X$ and the result follows by (16).

Subcase I (v): $(91/100)b \le x \le b, y \le (114/100)b, a^3 > 11/25$ and $\rho \le 2.11$. In this case, for $0 \le t \le b$, we have

$$\begin{aligned}|f(t)| &= -f(t) = (x+t)(y+t)(b-t)\\ &\le \left(\frac{x+b}{2}\right)^2\left(\frac{114}{100}b+b\right) \quad \text{(A. G. mean)}\\ &\le \left(\frac{214}{100}\right)b^3 < \rho,\end{aligned}$$

so that $(0, b] \subseteq S_X$. The result follows by (16).

Subcase I (vi): $(91/100)b \le x \le b, y \ge (114/100)b, a^3 \ge 11/25$ and $\rho \le 2.11$. By Remark 2, $f(t)$ is monotonically decreasing for $-a \le t \le -x$. Since $f(t)$ is positive for these values of t, to prove that $[-a, -x] \subset S_X$ it is enough to prove that $-a \in S_X$. We have

$$\begin{aligned}0 &< f(-a) = (-x+a)(b+a)(y-a)\\ &\le \left(\frac{41}{40}b-\frac{91}{100}b\right)\left(b+\frac{41}{40}b\right)\left(\frac{100^\rho}{91b^2}-\frac{41}{40}b\right) \quad \left(\text{since } |y| \le \frac{\rho}{|x|b}\right)\\ &< \sigma,\end{aligned}$$

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if

$$\left(\frac{23}{200}\right)\left(\frac{81}{40}\right)\left[\frac{100}{91}\rho^2-\left(\frac{40}{41}\right)^2 a^3\rho\right]<1 \quad (\text{since } \rho\sigma=1).$$

Since the L. H. S., as a function of ρ, is monotonically increasing, and for $\rho = 2.11$ and $a^3 \geq 11/25$, the above holds, we have $-a \in S_X$ and hence $[-a,-x] \subseteq S_X$. Now arguing as in Subcase I (i), we have either $[-a,0)$ or $(0,b]$ is contained in S_X. This completes the proof for Case I.

Case II: (x,y) *in the second quadrant.*

Subcase II (i): $|x| > b$. *Since* $y \geq |x| \geq b$, in view of Remark 2, $f(t)$ is positive and monotonically decreasing in the interval $(0,b]$. Since, by (15), $f(0) \leq \sigma$, we have $(0,b] \subseteq S_X$. The result follows from (16).

Subcase II (ii): $|x| \leq b$. As in Subcase II (i) above, $(0,-x] \subseteq S_X$. For $-x \leq t \leq b$, we have

$$\begin{aligned} 0 \leq -f(t) &= (b-t)(x+t)(x+t) \\ &\leq \frac{1}{4}(b+x)^2(y+t) \\ &\leq \frac{1}{4}(b+x)^2\left(\frac{12.3}{8a^2}+|x|+b\right) \\ &= \frac{1}{4}(1-\mu)^2\left(\frac{12.3}{8}\left(\frac{b}{a}\right)^2+\mu b^3+b^3\right) \quad (\text{where } |x| = b\mu) \\ &= g(\mu), \end{aligned}$$

say. Since $g(\mu)$ is a decreasing function of μ for $0 \leq \mu \leq 1$, we have $|f(t)| \leq g(0) < \rho$, so $(0,b] \subseteq S_X$. This completes Case II.

Case III: (x,y) *in the third quadrant.*

Subcase III (i): $|x| \geq b$. Since $|y| \geq |x| \geq b$, b is the smallest root of $f(t)$ and by Remark 2, $f(t)$ is negative and monotonically increasing for $0 \leq t \leq b$. Since by (15), $f(t) \geq -\rho$, we have $(0,b] \subseteq S_X$. This proves the result for Subcase III (i).

Subcase III (ii): $|x| \leq b$, $|y| \leq 4\sigma/b^2$. As in Subcase III (i), we have $(0,-x] \subseteq S_X$. For $-x \leq t \leq b$, $f(t)$ is positive and

$$\begin{aligned} |f(t)| &\leq (t+x)(b-t)(\max(|y|,|t|)) \\ &\leq \frac{b^2}{4}\max(|y|,b) < \sigma, \end{aligned}$$

since $b^3 < \frac{4}{7} < 4\sigma$. So $(0,b] \subseteq S_X$.

Subcase III (iii): $|x| \le b$, $|y| \ge 4\sigma/b^2$, $\rho < 2.733$, so $\sigma > .36598$. As before $(0, -x] \subseteq S_X$. Since $4\sigma/b^2 > b$, we have $|y| > b$ and thus $f(t)$ is positive for $-x \le t \le b$ and we have

$$\begin{aligned} 0 < f(t) &= (x+t)(b-t)(-y-t) \\ &\le (t-|x|)(b-t)\left(\frac{12.3}{8a^2} + |x| - t\right) \quad \text{(by (13))} \\ &= g(\mu, t), \end{aligned}$$

say, where $\mu = |x| \ge 0$. Since $g(\mu, t)$ as a function of μ is a decreasing function of μ, for all $t \in (0, b]$, we have

$$\begin{aligned} f(t) \le g(0,t) &= t(b-t)\left(\frac{12.3}{8a^2} - t\right) \\ &\le \frac{b^2}{4}\left(\frac{12.3}{8a^2}\right) < .3659 < \sigma \quad \text{(A. G. mean)}. \end{aligned}$$

Thus $(0, b] \subseteq S_X$ and the result follows from (16).

Subcase III (iv): $|x| \le b$, $|y| \ge 4\sigma/b^2$, $a^3 \le .4317$, $\rho \ge 2.733$.

As in Subcase III (iii), for $-x \le t \le b$, we have

$$|f(t)| \le g(0,t) \le t(b-t)\left(\frac{12.3}{8a^2}\right) = h(t),$$

say. Since $h(t)$ is monotonically decreasing for $t \ge b/2$, then for $t \ge (8/10)b$, we have

$$h(t) \le h\left(\frac{8}{10}b\right) < \frac{1}{4.1} < \sigma,$$

Now for $|x| \ge 8b/10$, we have $t \ge 8b/10$, so $[-x, b] \subset S_X$ and $(0, -x] \subseteq S_X$. As in the earlier case we have $(0, b]$, and in particular $[\frac{8}{10}b, b] \subseteq S_X$, for all X in this case.

Since $a^3 \le .4317$, $|y| \ge 4\sigma/b^2 > (73/40)b$, so $f(t)$ is negative for $b \le t \le (73/40)b$ and

$$\begin{aligned} -f(t) &= (x+t)(t-b)(-y-t) \\ &\le (t-|x|)(t-b)\left(\frac{12.3}{8a^2} + |x| - t\right) = g_1(\mu, t), \end{aligned}$$

say, where $\mu = |x| \ge 0$. Since $t < (73/40)b < 12.3/16a^2$, for $a^3 \le .4317$, we have $g_1(\mu, t)$ is a decreasing function of μ, and hence

$$\begin{aligned} -f(t) \le g_1(0,t) &= t(t-b)\left(\frac{12.3}{8a^2} - t\right) \\ &\le \left(\frac{73}{40}b\right)\left(\frac{33}{40}b\right)\left(\frac{12.3}{8a^2}\right) = 2.2033 < \rho, \end{aligned}$$

so $((8/10)b, (73/40)b] \subseteq S_X$. Since this interval is of length a, we are done in view of (17).

Subcase III (v): $|x| \le b$, $|y| \ge 4\sigma/b^2$, $\rho \ge 2.733$ and $a^3 \ge .4317$. Since $4\sigma/b^2 > (71/40)b$, we have $|y| \ge (71/40)b$.

Again as in Subcase III (iv), $[\frac{8}{10}b, b] \subseteq S_X$. Then for $(71/40)b \le |y| \le (73/40)b$ and for $(71/40)b \le t \le (73/40)b$, we have

$$|f(t)| = |t+x||t-b||t+y| \le \left(\frac{73}{40}b\right)\left(\frac{33}{40}b\right)\left(\frac{2}{40}b\right) < \sigma.$$

Also for $b \le t \le \min(|y|, (73/40)b)$, $f(t)$ is negative and

$$\begin{aligned} -f(t) &= (x+t)(t-b)(-y-t) \\ &\le (t-|x|)\left(\frac{12.3}{8a^2} + |x| - t\right)(t-b) \quad \text{(by (13))} \\ &\le \left(\frac{12.3}{16a^2}\right)^2 (t-b) \quad \text{(A. G. mean)} \\ &\le \left(\frac{12.3}{16}\right)^2 \frac{b}{a^4} < \rho \quad (\text{since } a^3 \ge .4317, \text{ and } \rho \ge 2.733) \end{aligned}$$

and so $[8b/10, (73/40)b] \subseteq S_X$. The result follows as in Subcase III (iv). This completes the proof for Case III.

Case IV: (x, y) *in the fourth quadrant.*

Subcase IV (i): $|x| \ge b$. By Remark 2, $f(t)$ is monotone in the interval $(0, b]$, so we have $(0, b] \subseteq S_X$.

Subcase IV (ii): $(25/40)b \le x \le b$. In this case, since $-x$ is the smallest root of $f(t)$, $f(t)$ is negative and a monotonically increasing function of t for $-a \le t \le -x$, and

$$\begin{aligned} |f(-a)| &= (a-x)(a+b)(-y+a) \\ &\le (a-x)(a+b)\left(\frac{12.3}{8a^2} + x + a\right) \quad \text{(by (13))} \\ &= h(x), \end{aligned}$$

say. Since $h(x)$ is monotonically decreasing, for $x \ge (25/40)b$, we have $|f(-a)| \le h(25b/40) < 1.92\cdots < \rho$, so $[-a, x] \subseteq S_X$.

If $|y| \ge b$, then $f(t)$ has a single extreme point between $-x$ and b, so either $[-x, 0)$ or $(0, b] \subseteq S_X$ and hence $[-a, 0)$ or $(0, b] \subseteq S_X$.

Otherwise, we have either $[-a, 0)$ or $(0, -y] \subseteq S_X$ as before, and for $-y \le t \le b$,

$$|f(t)| = -f(t) = (t+x)(b-t)(y+t) \le \left(\frac{b+x}{2}\right)^2 b \le b^3 < \rho.$$

So either $[-a,0)$ or $(0,b] \subseteq S_X$ and result follows by Remark 1.

Subcase IV (iii): $0 \leq x \leq (25/40)b$, $|y| \leq (59/40)b$, $\rho \geq 2.9$. In this case, we have

$$\begin{aligned} |f(-a)| = -f(-a) &= (a-x)(a+b)(-y+a) \\ &\leq (a-x)(a+b)\left(\frac{5}{2}a\right) \\ &\leq a(a+b)\left(\frac{5}{2}a\right) < 2.892\cdots < \rho. \end{aligned}$$

Arguing as in Case IV (ii), we have either $[-a,0)$ or $(0,b] \subseteq S_X$.

Subcase IV (iv): $0 \leq x \leq (25/40)b$, $|y| \leq b$, $\rho \leq 2.9$, so $\sigma > .344$.
For $b/10 \leq t \leq -y$, $f(t)$ is positive and

$$\begin{aligned} f(t) &= (t+x)(b-t)(-y-t) \\ &\leq \left(\frac{-y+x}{2}\right)^2 (b-t) \quad \text{(A. G. mean)} \\ &\leq \left(\frac{b+x}{2}\right)^2 (b-t) \quad (\text{since } |y| \leq b) \\ &\leq \left(\frac{65}{80}\right)^2 \left(\frac{9}{10}\right) b^3 < \sigma, \end{aligned}$$

so $[b/10, -y] \subseteq S_X$ or $[-y, b] \subseteq S_X$ as in Case IV (iii). Also for $b \leq t \leq (9/8)b$, we have

$$\begin{aligned} 0 \leq f(t) &= (t+s)(t-b)(t-|y|) \\ &\leq (t+x)(t-b)(t-x) \quad (\text{since } |y| \geq x) \\ &\leq t^2(t-b) \quad \text{(A. G. mean)} \\ &\leq \left(\frac{9}{8}\right)^2 \frac{1}{8} < .16 < \sigma, \end{aligned}$$

so $[b/10, (9/8)b] \subseteq S_X$. Since $(9/8)b - b/10 = a$, the result follows by (17).

Subcase IV (v): $0 \leq x \leq (25/40)b$, $b \leq |y| \leq (59/40)b$, $\rho \leq 2.9$.
For $|y| - (9/10)b \leq t \leq b$, we have

$$\begin{aligned} 0 \leq f(t) &= (t+x)(b-t)(|y|-t) \\ &\leq \left(\frac{b+x}{2}\right)^2 (|y|-t) \quad \text{(A. G. mean)} \\ &\leq \left(\frac{65}{80}\right)^2 \left(\frac{9}{10}\right) b^3 < \sigma, \end{aligned}$$

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so $[-y-(9/10)b, b] \subseteq S_X$. For $b \le t \le -y$, we have

$$\begin{aligned} 0 \le & -f(t) = (x+t)(t-b)(-y-t) \\ & \le \left(\frac{25}{40}b + \frac{59}{40}b\right)\left(\frac{19}{40}b\right)^2 < \rho, \end{aligned}$$

so $[b, -y] \subseteq S_X$. Also for $-y \le t \le -y + b/8$, we have

$$\begin{aligned} 0 \le & f(t) = (t+x)(t-b)(t+y) \\ & \le \left(\frac{64}{40}b + \frac{25}{40}b\right)\left(\frac{24}{40}b\right)\left(\frac{1}{8}b\right) < \sigma, \end{aligned}$$

so $[-y, -y+b/8] \subseteq S_X$ and hence $[-y-(9/10)b, -y+b/8] \subseteq S_X$. This is an interval of length a and the result follows by (17).

Subcase IV (vi): $0 \le x \le (25/40)b$, $|y| \ge (59/40)b$.

We have

$$\begin{aligned} 0 \le & -f\left(-\frac{25}{40}b\right) = \left(\frac{25}{40}b - x\right)\left(b + \frac{25}{40}b\right)\left(-y + \frac{25}{40}b\right) \\ & \le \left(\frac{25}{40}b - x\right)\left(\frac{12.3}{8a^2} + x + \frac{25}{40}b\right)\left(\frac{65}{40}b\right) \\ & = g(x), \end{aligned}$$

say. Since $g(x)$ is monotonically decreasing, for $x \ge 0$, we have

$$-f(\left(-\frac{25}{40}b\right) \le g(0) < 1.8316\cdots z\rho,$$

so $-(25/40)b \in S_X$ and hence $[-(25/40)b, -x] \subseteq S_X$. For $b \le t \le (57/40)b$, we have

$$\begin{aligned} 0 \le & -f(t) = (t+x)(t-b)(-y-t) \\ & \le (t+x)(t-b)\left(\frac{12.3}{8a^2} + x - t\right) \quad \text{(by (13))} \\ & \le (2a)\left(\frac{17}{40}b\right)\left(\frac{12.3}{8a^2}\right) \quad (\text{since } x < b \le t) \\ & < \rho, \end{aligned}$$

so $[b, (57/40)b] \subseteq S_X$.

Between $-x$ and b, $f(t)$ has a single extreme point, so $f(t)$ is monotone in either the interval $[-x, 0)$ or the interval $(0, b]$. In view of (15), either the interval $[-x, 0)$ or the interval $(0, b]$ is contained in S_X. If $(0, b] \subseteq S_X$, the result follows by (16). Otherwise we have $[-(25/40)b, 0) \cup [b, (57/40)b] \subseteq S_X$ and the result follows by (18). This completes the proof for Case IV and hence completes the proof of Theorem 2.

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