J. Austral. Math. Soc. (Series A) 27 (1979), 371–377

LOCALLY UNIFORMLY ROTUND NORMS AND MARKUSCHEVICH BASES

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(Received 10 January 1977; revised 10 April 1978)

Communicated by E. Strzelecki

Abstract

If a Banach space E admits a Markuschevich basis, then E can be renormed to be locally uniformly rotund. When the coefficient space of the basis is 1-norming, and this norm is very smooth, E is weakly compactly generated.

Subject classification (Amer. Math. Soc. (MOS) 1970): 46 B 99.

1. Introduction

Dyer (1969), page 55, has shown that if E admits a Markuschevich basis, then there is a continuous one-to-one linear operator which maps E into some $c_0(\Gamma)$. Hence, by a result of Klee (1953), page 56, E can be equivalently renormed to be rotund. It will be shown here that if E admits a Markuschevich basis, then E can be renormed with a locally uniformly rotund norm which is $\sigma(E, Y)$ lower semi-continuous, where Y is the coefficient space of the basis. This norm is an equivalent norm for E if the coefficient space is norming (Zizler and John (1974b)). If the coefficient space is 1-norming, and this norm is very smooth, then the Markuschevich basis for E is shrinking, so E is weakly compactly generated. This improves a result of Zizler and John (1974b), page 687.

This paper is based on part of a doctoral thesis submitted to the University of Newcastle under the supervision of Associate Professor J. R. Giles.

2. Notation and definitions

Let E be a real Banach space, E^* its dual, and $\hat{E}$ the canonical embedding of

E in E^{**}. The unit spheres of E and E^* will be denoted by $S(E)$ and $S(E^*)$, respectively. The term 'subspace' will always mean norm closed linear subspace. The *density character* of a subspace $Y \subset E$, denoted dens Y ,is the minimum cardinality of a norm dense subset of Y. The *weak—* density character* of a subspace $F \subset E^*$, denoted $\sigma(E^*, E)\,\text{dens}\,F$, is the minimum cardinality of a weak—* dense subset of F. The norm closed linear span of a set $A \subset E$ will be denoted by $\overline{\text{sp}}\,A$.

The set $\{(x_i, f_i) : i \in I\}$ contained in $E \times E^*$ is said to be a *Markuschevich basis* (*M-basis*) for E if

(1) $\{(x_i, f_i)\}$ is *biorthogonal*; that is, $f_i(x_i) = 1$, while $f_i(x_j) = 0$ if $i \neq j$ for all $i \in I$.

(2) $\{x_i\}$ is *fundamental* in E; that is, $\overline{\text{sp}}\,\{x_i\} = E$.

(3) $\{f_i\}$ is *total* over E; that is, $\bigcap_{i \in I} f^{-1}(0) = \{0\}$.

The subspace $\overline{\text{sp}}\,\{f_i\}$ of E^* is called the *coefficient space* of the M-basis.An M-basis is *shrinking* if $\overline{\text{sp}}\,\{f_i\} = E^*$; *boundedly complete* if whenever $\{y_\delta : \delta \in D\}$ is a bounded net in E with the property that $\lim_\delta f_i(y_\delta)$ exists for each $i \in I$, there is an $x \in E$ such that $\lim_\delta f_i(y_\delta) = f_i(x)$ for each i.

For any subspace $Y \subset E^*$, $\|x\|_Y = \sup\{|f(x)| : f \in S(Y)\}$ defineds a semi-norm on E. If Y is total over E, then $\|\cdot\|_Y$ is norm. In general, $\|x\|_Y \leqslant \|x\|$ for all $x \in E$; that is, the $\|\cdot\|_Y$ topology is a weaker topology than the $\|\cdot\|$ topology. The space Y is said to be *norming* if the $\|\cdot\|$ and $\|\cdot\|_Y$ topologies are equivalent. Y is 1-*norming*- if $\|x\| = \|x\|_Y$ for every $x \in E$.

E is *locally uniformly rotund* (*LUR*) at $x \in S(E)$ if every sequence (or net) $\{x_n\}$ in $S(E)$ with $\|x_n + x\| \to 2$, has $\|x_n - x\| \to 0$. E is LUR if it is LUR at every $x \in S(E)$.

The set valued mapping D_E of E into 2^{E^*} which assigns to each $x \in E$ the $\{f \in E^* : f(x) = \|f\|\,\|x\| \text{ and } \|x\| = \|f\|\}$ is called the *duality mapping*. The mapping $x \to f_x$ which sends each $x \in S(E)$ to an $f_x \in D_E(x)$, and has the property that, for $\lambda > 0$, $f_{\lambda x} = \lambda f_x$ is called a *support mapping* on E. The norm of E is said to be *very smooth* if every support mapping $x \to f_x$ on E is continuous when E has the norm topology and E^* has the $\sigma(E^*, E^{**})$ topology (Giles (1975), page 72). The norm of E is *Fréchet differentiable* (*F-differentiable*) if every support mapping on E is continuous when both E and E^* have the norm topology (see Giles (1971), page 107). When the norm of E is very smooth (F-differentiable), the space E is said to be a very smooth (F-differentiable) space.

If E contains a $\sigma(E, E^*)$ compact fundamental subset, then E is *weakly compactly generated* (*WCG*).

3. Main result and applications

This section will state the Main Result and give several applications. The proof of the Main Result will be given in the next section.

MAIN RESULT *Let E admit an M-basis $\{(x_i, f_i): i \in I\}$ with coefficient space Y. Then E can be renormed with a LUR norm which is $\sigma(E, Y)$ lower semi-continuous.*

COROLLARY 1. (Zizler and John (1974b)) *Let E admit an M-basis with a norming coefficient space. Then E can be equivalently renormed to be LUR.*

COROLLARY 2. *Let E admit a boundedly complete M-basis. Then E can be equivalently renormed to be LUR.*

PROOF. If the M-basis is boundedly complete, then the coefficient space is norming (Johnson (1970a), page 175).

COROLLARY 3. *Let E^* admit an M-basis $\{(f_i, F_i): i \in I\}$ with the property that its coefficient space Y is contained in $\hat{E}$. Then E^* can be renormed with a LUR dual norm.*

PROOF. Since $\{F_i: i \in I\}$ is total over E^*, $\operatorname{sp}\{F_i\}$ is $\sigma(E^{**}, E^*)$ dense in E^{**}. If $Y \subset \hat{E}$, then each $F_i = \hat{x}_i$ for some $x_i \in E$. Thus $\operatorname{sp}\{x_i\}$ is $\sigma(E, E^*)$ dense in E, hence $\overline{\operatorname{sp}}\{x_i\} = E$. Now by the Main Result, E^* admits a LUR norm which is $\sigma(E^*, E)$ lower semi-continuous.

COROLLARY 4. *Let E^* be as in Corollary 3. Then E is WCG.*

PROOF. By Corollary 3, E^* admits an M-basis $\{(f_i, \hat{x}_i): i \in I\}$ with coefficient space $\overline{\operatorname{sp}}\{\hat{x}_i\} = E$. Thus $\{(x_i, f_i)\}$ is a shrinking M-basis for E, so E is WCG (see Zizler and John (1974a), page 10).

This recovers a result of Vašak (1974), page 221.

4. Proof of the main result

Let E admit an M-basis $\{(x_i, f_i): i \in I\}$ with coefficient space Y. Assume $\|x\|_Y = 1$ for each $i \in I$, and let $K = \{x_i: i \in I\}$. Clearly, $K \cup \{0\}$ is $\sigma(E, Y)$ compact.

Let X denote the $\|\cdot\|_Y$ completion of E. Since E and X have (essentially) the same dual, $K \cup \{0\}$ is $\sigma(X, Y)$ compact, as well. Now since X is a Banach space, the $\sigma(X, Y)$ closed convex hull of $K \cup \{0\} \cup \{-K\}$ is also $\sigma(X, Y)$ compact (Dunford and Schwartz (1958), page 434). Denote this set by K_1, and the gauge of K_1 by $|||\cdot|||$. Since K_1 is $\sigma(X, Y)$ compact it is $\|\cdot\|_Y$ bounded and so it may be assumed that $\|x\|_Y \leqslant |||x|||$ for all $x \in X$.

The proof of the Main Result involves modifying a sequence of lemmas due to Amir and Lindenstrauss (1968).

https://doi.org/10.1017/S1446788700012489 Published online by Cambridge University Press

LEMMA 1. *Let E admit an M-basis with coefficient space Y, and X be its $\|\cdot\|_Y$ completion. Then, given $\varepsilon > 0$, an integer $n > 0$, m elements $f_1, f_2, \ldots, f_m$ of $S(Y)$, and a finite dimensional subspace $B \subset X$, there is a $\|\cdot\|_Y$ separable subspace C of X containing B such that for every subspace Z, with $B \subset Z \subset X$ and $\dim Z/B = n$, there is a linear operator $T: Z \to C$ with $\|T\|_Y$, $|||T||| \leqslant 1+\varepsilon$, $Tb = b$ for all $b \in B$, and $|f_k(z) - f_k(Tz)| \leqslant \varepsilon \|z\|$ for every $z \in Z$ and each k, $1 \leqslant k \leqslant m$.*

PROOF. Proceed exactly as in Lemma 3 of Amir and Lindenstrauss (1968). Note that in this case K_1 has the (relative) $\sigma(X, Y)$ topology and so the operator T is the homogeneous extension of a $\sigma(X, Y)$ pointwise cluster point of a net on the compact Hausdorff space $(2K_1)^{K_1}$.

LEMMA 2. (Amir and Lindenstrauss (1968, page 43) *Let E and X be as in Lemma 1, $\mathcal{M}$ an infinite cardinal number, G a subspace of X with $\|\cdot\|_Y$ dens $G \leqslant \mathcal{M}$, and F a subspace of Y with $\sigma(Y, X)$ dens $F \leqslant \mathcal{M}$. Then there is a projection P on X with the following properties*:

(1) $\|P\|_Y = |||P||| = 1$;
(2) *$Pg = g$ for all $g \in G$*;
(3) *$P^*f = f$ for all $f \in F$*;

and

(4) *$\|\cdot\|_Y$ dens $PX \leqslant \mathcal{M}$.*

LEMMA 3. (Amir and Lindenstrauss (1968), page 44) *Let E and X be as in Lemma 1, and μ be the first ordinal of cardinality $\|\cdot\|_Y$ dens X. Then there is a transfinite sequence of projections $\{P_\alpha: \omega \leqslant \alpha \leqslant \mu\}$ on X such that*

(1) *$\|P_\alpha\|_Y = |||P_\alpha||| = 1$ for each α*;
(2) *$\|\cdot\|_Y$ dens $P_\alpha X \leqslant$ cardinality of α, for each α*;
(3) *$P_\alpha P_\beta = P_\beta P_\alpha = P_\beta$ whenever $\beta < \alpha$*;

and

(4) *$\bigcup_{\beta<\alpha} P_{\beta+1} X$ is $\|\cdot\|_Y$ dense in $P_\alpha X$ for every $\alpha > \omega$.*

This next lemma is due to Troyanski (1971), page 175.

LEMMA 4. *Let E be a Banach space which satisfies the following properties*:

(A) *There is a continuous one-to-one linear operator T which maps E into $c_0(\Gamma)$, for some set Γ.*

(B) *There is a transfinite sequence of bounded linear operators $\{T_\delta: \delta \in D\}$ on E such that*

(1) *for each $x \in E$ and each $\varepsilon > 0$, the set*

$$\Lambda(x, \varepsilon) = \{\delta: \|T_{\alpha+1}x - T_\alpha x\| > \varepsilon(\|T_{\alpha+1}\| + \|T_\alpha\|)$$

is finite;

(2) *for each $x \in E$, $x \in \overline{\mathrm{sp}}\,[(\|T_1 x\| T_1 E) \cup \bigcup_{\delta\in\Lambda(x)} (T_{\alpha+1} - T_\alpha) E]$,*

$$\text{where } \Lambda(x) = \bigcup_{\varepsilon>0} \Lambda(x, \varepsilon),$$

(3) *dens sp* $[(T_{\alpha+1}-T_\alpha)E] \leqslant$ *dens* $T_1 E = \aleph_0$.

Then E can be equivalently renormed to locally uniformly rotund.

In order to use Lemma 4 in the proof of the Main Result, one more lemma is needed.

LEMMA 5. *Let X admit an M-basis* $\{(x_i, f_i): i \in I\}$. *Then* $(P_{\alpha+1}-P_\alpha)X$ *admits an M-basis for each* α, $\omega \leqslant \alpha < \mu$.

PROOF Let $x_i^\alpha = (P_{\alpha+1}-P_\alpha)\ x_i$ and $f_i^\alpha = j_\alpha^* f_i$, where j_α^* is the mapping which restricts each f_i to $(P_{\alpha+1}-P_\alpha)X$. After deleting all x_i^α and f_i^α which are zero, it must be shown that $\{(x_j^\alpha, f_i^\alpha): i \in I_\alpha\}$ is an M-basis for $(P_{\alpha+1}-P_\alpha)X$. Firstly, $f_j^\alpha(x_i^\alpha) = f_j^\alpha\ (P_{\alpha+1}-P_\alpha)x_i = (P_{\alpha+1}-P_\alpha)^* f_i^\alpha(x_i) = f_j(x_i)$, since $(P_{\alpha+1}-P_\alpha)^*$ is the inverse of j_α^* for each α. Thus if $\{(x_i, f_i)\}$ is biorthogonal, so is $\{(x_i^\alpha, f_i^\alpha)\}$ foreach α. Next, for each $0 \neq y \in P_{\alpha+1}X$ there is an $x \in X \backslash \{0\}$ such that $P_{\alpha+1}x = y$, and so if $\{x_i\}$ is fundamental in X, then $(x_i^\alpha\}$ must be fundamental in $(P_{\alpha+1}-P_\alpha)X$. Finally, if $x \in (P_{\alpha+1}-P_\alpha)X$ and $f_i^\alpha(x) = 0$ for all $i \in I$, then $j_\alpha^* f_i(x) = 0$ for all i. But this can not happen since $\{f_i\}$ is total over X. Thus $\{x_i^\alpha, f_i^\alpha): i \in I\}$ is an M-basis for $(P_{\alpha+1}-P_\alpha)X$ for each α, $\omega \leqslant \alpha < \mu$.

PROOF OF THE MAIN RESULT The aim is to show that X satisfies the conditions of Lemma 4. Since E is $\|\cdot\|_Y$ dense in X, it is clear that if E admits an M-basis, then this biorthogonal set is also an M-basis for X. By a result of Dyer (1969), page 55, if X admits an M-basis, then X satisfies condition (A) of Lemma 4. Therefore it remains to construct a set of operators $\{T_\delta: \delta \in D\}$ on E which satisfies (B1), (B2) and (B3).

If X is separable, then $T_\delta = I$, the identity operator on X, for each $\delta \in D$. Therefore assume that X is non-separable and proceed by transfinite induction on dens X.

Assume that the Main Result is true for all cardinal numbers less than dens X. By Lemma 3, X admits a transfinite sequence of projections $\{P_\alpha: \omega \leqslant \alpha \leqslant \mu\}$, where μ is the first ordinal number of cardinality dens X and, by Lemma 5, each $(P_{\alpha+1}-P_\alpha)X$ also admits an M-basis. Hence, by the inductive hypothesis, there is a transfinite sequence of linear operators $\{S_\beta^\alpha : \omega \leqslant \beta \leqslant \Gamma_\alpha\}$, where Γ_α is the first ordinal of cardinality dens $(P_{\alpha+1}-P_\alpha)X$, which maps $(P_{\alpha+1}-P_\alpha)X$ into itself and satisfies (B1), (B2) and (B3) of Lemma 4. Let Λ_α denote the set of ordinal numbers γ, with $\omega \leqslant \gamma \leqslant \Gamma_\alpha$, and D the set of ordered pairs of ordinal numbers (α, β), where $\beta \in \Lambda_\alpha \cup \{0\}$ and $\omega \leqslant \alpha \leqslant \mu$. Order this set lexiographically. For each $\delta = (\alpha, \beta) \in D$ define

$$T_\delta = \begin{cases} S_\beta^\alpha (P_{\alpha+1}-P_\alpha)+P_\alpha & \text{if } \alpha < \mu \quad (S_0^\alpha = 0), \\ P_\mu = I & \text{if } \alpha = \mu. \end{cases}$$

https://doi.org/10.1017/S1446788700012489 Published online by Cambridge University Press

As in Troyanski (1971), pages 177–178, the set $\{T_\delta : \delta \in D\}$ satisfies the conditions (B1), (B2) and (B3) of Lemma 4. By Lemma 3, the operators P_α and S_β^α are $\sigma(E, Y)$ continuous on X for each α, $\omega \leqslant \alpha \leqslant \mu$, and each $\beta \in \Lambda_\alpha$. Thus, by Lemma 4, X can be renormed with an LUR norm which is $\sigma(X, Y)$ lower semi-continuous. Now restrict this norm to E to get the Main Result.

THEOREM. *Let E admit an M-basis with a* 1*-norming coefficient space Y. If $\|\cdot\|_Y$ is a very smooth norm, then E admits a shrinking M-basis, and so E is WCG.*

PROOF. The fact that Y is norming gives $E = X$, so proceed by transfinite induction on $\operatorname{dens} E$. If E is separable the result follows by applying Theorem III.1 of Johnson (1970b). Now assume the result is true for all cardinal numbers less than $\operatorname{dens} E$. By Lemma 3 there is a transfintie sequence of projections $\{P_\alpha : \omega \leqslant \alpha \leqslant \mu\}$ defined on E with $\operatorname{dens}(P_{\alpha+1} - P_\alpha)E < \mu$ for all α, $\omega \leqslant \alpha < \mu$, where μ is the first ordinal number of cardinality $\operatorname{dens} E$. Since the coefficient space is 1-norming, $\|\cdot\| = \|\cdot\|_2$, so E has a very smooth norm. Hence, by Tacon (1970) and Zizler and John (1974), page 3, there is a transfinite sequence of projections $\{P_\alpha^* : \omega \leqslant \alpha \leqslant \mu\}$ on E^* which are continuous when E^* has the weak-* topology. As in Tacon (1970), page 419, each $(P_\alpha E)^*$ may be identified (isometrically isomorphically) with $P\ {}^*E^*$. Now continue with the inductive procedure.

Since dens $(P_{\alpha+1} - P_\alpha)E < \mu$, the inductive hypothesis gives that $(P_{\alpha+1} - P_\alpha)E$ admits a shrinking M-basis $\{(x_i^\alpha, \tilde{f}_i^\alpha) : i \in I_\alpha\}$ for each α, $\omega \leqslant \alpha < \mu$. Since $(P_{\alpha+1} - P_\alpha)$ maps E onto $(P_{\alpha+1} - P_\alpha)E$, the operator $(P_{\alpha+1} - P_\alpha)^*$ maps $[(P_{\alpha+1} - P_\alpha)E]^*$ isomorphically onto $(P_{\alpha+1}^* - P_\alpha^*)E^*$. Let $f_i^\alpha = (P_{\alpha+1}^* - P_\alpha^*)\tilde{f}_i^\alpha$ for each $i \in I$ and each α, $\omega \leqslant \alpha < \mu$. Now it must be shown that $\{(x_i^\alpha, f_i^\alpha) : i \in I_\alpha, \omega \leqslant \alpha < \mu\}$ is a shrinking M-basis for E.

Clearly, $\{x_i^\alpha\}$ is fundamental in E since, by Lemma 3,

$$\overline{\operatorname{sp}}\{x_i^\alpha\} = \bigcup_{\alpha < \mu} (P_{\alpha+1} - P_\alpha)E = E.$$

Also,

$$\overline{\operatorname{sp}}\{f_i^\alpha\} = \bigcup_{\alpha < \mu} (P_{\alpha+1}^* - P_\alpha^*)E^* = E^*,$$

since E is very smooth (Tacon (1970), page 421). The set $\{f_i^\alpha\}$ is total over E, since $\{\tilde{f}_\alpha^i\}$ is total over $(P_{\alpha+1} - P_\alpha)\ E$ for each α, $\omega \leqslant \alpha < \mu$. Therefore, it only remains to show that $\{(x_i^\alpha, f_i^\alpha)\}$ is a biorthogonal set.

If $\alpha \neq \beta$, then

$$\begin{aligned} f_i^\alpha(x_i^\beta) &= (P_{\alpha+1}^* - P_\alpha^*)\tilde{f}_i^\alpha(x_i^\beta) \\ &= (P_{\alpha+1}^* - P_\alpha^*)(\tilde{f}_i^\alpha)(P_{\beta+1} - P_\beta)(x_i^\beta) \\ &= \tilde{f}_i^\alpha(P_{\alpha+1} - P_\alpha)(P_{\beta+1} - P_\beta)x_i^\beta. \end{aligned}$$

But, by Lemma 3, $P_\alpha P_\beta = P_\beta P_\alpha = P_\beta$ whenever $\beta < \alpha$ and so

$$(P_{\alpha+1} - P_\alpha)(P_{\beta+1} - P_\beta) = 0.$$

Hence $f_i^\alpha(x_i^\beta) = 0$ for all x_i^β with $\beta \neq \alpha$. By construction $f_i^\alpha(x_j^\alpha) = 1$ if $i = j$, and $f_i^\alpha(x_j^\alpha) = 0$ otherwise. Therefore $\{x_i^\alpha, f_j^\alpha\}$ is biorthogonal, and this completes the proof.

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https://doi.org/10.1017/S1446788700012489 Published online by Cambridge University Press