Appendix F: Motion of ions in a combined electric and magnetic field

A simple theory of the motion of ions in a region with perpendicular electric and magnetic fields has been derived by Townsend [1]. Consider an ion with mass m and charge q. Let the electric field $\mathscr{E}$ lie along z and the magnetic field B lie along y. The equations of motion are

$$\ddot{x} = \omega \dot{z} \qquad \ddot{y} = 0 \qquad \ddot{z} = f - \omega \dot{x} \tag{F.1}$$

where dots denote time derivatives, $\omega = qB/m$, and $f = q\mathscr{E}/m$. If we assume that the ion is created with a small initial velocity and with a uniform distribution of angles, then the coupled $\ddot{x}$ and $\ddot{z}$ equations have the solutions

$$\begin{aligned} x(t) &= (f/\omega)t - (f/\omega^2)\sin \omega t \\ z(t) &= (f/\omega^2)(1 - \cos \omega t) \end{aligned} \tag{F.2}$$

Let $\{t_i\}$ be the sequence of time intervals between collisions and τ be the mean time interval. The mean displacement of the ion after N collisions is

$$\begin{aligned} \langle x \rangle &= (f/\omega) \sum_{i=1}^{N} t_i - (f/\omega^2) \sum_{i=1}^{N} \sin \omega t_i \\ \langle z \rangle &= (f/\omega^2) \sum_{i=1}^{N} 1 - (f/\omega^2) \sum_{i=1}^{N} \cos \omega t_i \end{aligned} \tag{F.3}$$

The ion traverses a portion of a circular arc between collisions. Townsend showed that the sine and cosine summations over these arcs have the values

$$\begin{aligned} \sum_{i=1}^{N} \sin \omega t_i &= \frac{N\omega\tau}{1 + \omega^2\tau^2} \\ \sum_{i=1}^{N} \cos \omega t_i &= \frac{N}{1 + \omega^2\tau^2} \end{aligned} \tag{F.4}$$

Substituting Eq. F.4 back into F. 3 and taking $N\tau \to t$, we find

$$\langle x(t)\rangle = \frac{q^2\mathscr{E}B\tau^2 t}{m^2\left(1+\frac{q^2B^2\tau^2}{m^2}\right)} \qquad \langle z(t)\rangle = \frac{q\mathscr{E}\tau t}{m\left(1+\frac{q^2B^2\tau^2}{m^2}\right)} \tag{F.5}$$

Note that both $\mathscr{E}$ and B must be nonzero to obtain a net motion along x. An electric field alone causes motion along z. If a magnetic field is also present, the motion along z is decreased. Measurements [2] of the displacement along x as a function of B in a spark chamber have shown that Eq. F.5 gives a reasonable fit to the data for $\mathscr{E} \leqslant 100$ V/cm.

The mean deflection angle and the mean drift velocity follow from Eq. F.5,

$$\tan\theta = \frac{\langle x\rangle}{\langle z\rangle} = \frac{qB\tau}{m} \qquad w = \frac{\langle z\rangle}{t} = \frac{q\mathscr{E}\tau}{m\left(1+\frac{q^2B^2\tau^2}{m^2}\right)} \tag{F.6}$$

Note that a simple estimate of the mean collision time τ can be obtained from a measurement of the drift velocity in a purely electric field.

References

[1] J. Townsend, *Electrons in Gases,* London: Hutchison, 1947.
[2] S. Korenchenko, A. Morozov, and K. Nekrasov, Displacement of spark chamber discharges in a magnetic field, Priboryi Tekhnika Eksperimenta, No. 5, 1966, p. 72.

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