COMPACT LINEAR OPERATORS FROM AN ALGEBRAIC STANDPOINT

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1. Introduction. Let $B(X)$ denote the Banach algebra of all bounded linear operators on a Banach space X. Let t be an element of $B(X)$, and let e denote the identity operator on X. Since the earliest days of the theory of Banach algebras, it has been understood that the natural setting within which to study spectral properties of t is the Banach algebra $B(X)$, or perhaps a closed subalgebra of $B(X)$ containing t and e. The effective application of this method to a given class of operators depends upon first translating the data into terms involving only the Banach algebra structure of $B(X)$ without reference to the underlying space X. In particular, the appropriate topology is the norm topology in $B(X)$ given by the usual operator norm. Theorem 1 carries out this translation for the class of compact operators t. It is proved that if t is compact, then multiplication by t is a compact linear operator on the closed subalgebra of $B(X)$ consisting of operators that commute with t.

In §3 we exploit Theorem 1 by showing how the Riesz-Schauder spectral theory for a compact linear operator t may be obtained by applying the most elementary Banach algebra techniques to the least closed subalgebra of $B(X)$ containing t and e.

As a second application of Theorem 1, we prove a theorem which contains the Krein-Rutman theorem [**5**, Theorem 6.1] on positive compact linear operators. Let t be compact and have non-zero spectral radius ρ, and let A^+ denote the least closed semi-algebra in $B(X)$ containing t and e. Using entirely elementary arguments together with Theorem 1, we prove that if $A^+ \cap (-A^+) = (0)$, then there exists a nonzero element u of A^+ such that $tu = \rho u$. This result gives the Krein-Rutman theorem at once.

Theorem 1 may be regarded as an analogue of Schauder's theorem [**8**] on the compactness of the adjoint of a compact linear operator, and we give in §5 a theorem which includes both Theorem 1 and Schauder's theorem as special cases. As another special case of this theorem, we see that, if t is compact, the mapping $a \to tat$ is a compact linear operator on the whole of $B(X)$. This result is fundamental for the recent work of J. C. Alexander [**1**].

2. The compactness of multiplication operators.

THEOREM 1. *Let t be a compact linear operator on a Banach space X, and let Y be the centralizer of t. Then the mapping $a \to ta$ $(a \in Y)$ is a compact linear operator on Y.*

Proof. By definition, the *centralizer* Y of t is the set of all bounded linear operators that commute with t. It is clear that Y is a closed subalgebra of $B(X)$ and that the mapping $a \to ta$ is a bounded linear operator on the Banach space Y.

Let X_1 denote the closed unit ball in X, and let $E = \overline{tX_1}$. Then E is a compact subset of X in the norm topology. Given $a \in Y$ with $\| a \| \leq 1$, we have $atX_1 = taX_1 \subset tX_1$; and there-

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fore, by continuity, $aE \subset E$. Let $a_n \in Y$, $\| a_n \| \leqq 1$ $(n = 1, 2, \ldots)$. Then, for each $x \in E$, the set $\{a_n x : n = 1, 2, \ldots\}$ is contained in the compact subset E of the Banach space X. Also

$$\| a_n x - a_n x' \| \leqq \| x - x' \| \quad (x, x' \in E,\ n = 1, 2, \ldots),$$

which shows that the mappings $x \to a_n x$ $(x \in E,\ n = 1, 2, \ldots)$ form an equicontinuous sequence of mappings of the compact space E into the Banach space X. By Ascoli's theorem for Banach space valued functions, it follows that there exists a subsequence $\{a_{n_k}\}$ such that $\{a_{n_k} x\}$ converges uniformly for x in E. Consequently, $\{a_{n_k} tx\}$ converges uniformly for x in X_1, and so $\{a_{n_k} t\}$ converges with respect to the operator norm. Since $a_{n_k} t \in Y$ and Y is closed, this shows that $\{a_{n_k} t\}$ converges in Y. Finally, $ta_n = a_n t$.

Counter-example. Let X have infinite dimension, let A be a strictly irreducible subalgebra of $B(X)$, and let t be a nonzero element of A. Then the linear mapping $a \to at$ $(a \in A)$ is not compact on A. For, since X is an infinite dimensional normed space, there exist elements x_n of X such that $\| x_n \| = 1$ $(n = 1, 2, \ldots)$ and $\| x_k - x_j \| \geqq \frac{1}{2} (k \neq j)$. Since $t \neq 0$, there exists $x_0 \in X$ with $tx_0 \neq 0$. Then tx_0 is a strictly cyclic vector, and so (see [**4**], Proposition 17, Corollary 1) there exist a constant M and elements a_n of A such that

$$\| a_n \| \leqq M, \quad a_n t x_0 = x_n \quad (n = 1, 2, \ldots).$$

If the mapping $a \to at$ $(a \in A)$ is compact, there exists a subsequence $\{a_{n_k}\}$ such that $\{a_{n_k} t\}$ converges. But then $\{x_{n_k}\}$ converges, which is absurd.

Similarly, if the algebra A^* (the set of adjoints of elements of A) is strictly irreducible on the dual space X^* of X, then the mapping $a \to ta$ $(a \in A)$ is not compact. For, by what we have just proved, the mapping $a^* \to a^* t^*$ $(a^* \in A^*)$ is not compact, and the mapping $a \to a^*$ is an isometric anti-isomorphism of A on to A^*.

If A is dually strictly irreducible [**4**] on the pair of spaces X, X^*, then neither of the mappings $a \to at$, $a \to ta$ is compact on A. This is the case in particular for $A = B(X)$ or for any subalgebra A of $B(X)$ that contains all operators of finite rank.

3. Riesz-Schauder theory. We need to make use of two elementary propositions from the usual theory of compact operators [**6**].

PROPOSITION 1. *Let t be a compact linear operator on a Banach space X, and let $\{\lambda_n\}$ be a sequence of distinct eigenvalues of t. Then*

$$\lim_{n \to \infty} \lambda_n = 0.$$

PROPOSITION 2. *Let t be a compact linear operator on a Banach space X, λ a nonzero eigenvalue of t, and let*

$$N_k = \{x : (t - \lambda e)^k x = 0\} \quad (k = 1, 2, \ldots).$$

Then each N_k has finite dimension, and there exists a positive integer k such that $N_{k+1} = N_k$.

The well known proofs of both propositions depend on the fact that if X_0 is a proper closed linear subspace of X, then there exists $x \in X$ such that $\| x \| = 1$ and $\| x-y \| \geqq \frac{1}{2}$ $(y \in X_0)$.

For Theorems 2 and 3, we shift our attention to an abstract Banach algebra.

Notation. A will denote a commutative Banach algebra with unit element 1. We suppose that A contains an element t such that the mapping $a \to ta$ is a compact linear operator on the Banach space A. We denote the spectrum of an element a of A by $\mathrm{Sp}_A(a)$, i.e. $\mathrm{Sp}_A(a)$ is the set of complex numbers λ such that $\lambda - a$ has no inverse in A.

LEMMA 1. *Let λ be a nonzero frontier point of $\mathrm{Sp}_A(t)$. Then there exists $u \in A$ such that $u \neq 0$ and $tu = \lambda u$.*

Proof. $t-\lambda$ is a frontier point of the set of invertible elements of A, and is therefore a topological divisor of zero [7, Theorem 1.5.4]. Thus there exist elements a_n of A such that $\| a_n \| = 1$ $(n = 1, 2, \ldots)$, and $\lim_{n \to \infty} (t-\lambda)a_n = 0$. Since the mapping $a \to ta$ is compact, there exists a subsequence $\{a_{n_k}\}$ such that $\lim_{k \to \infty} ta_{n_k} = u$, say. Since $\lim_{k \to \infty} (ta_{n_k} - \lambda a_{n_k}) = 0$, we have in turn $\lim_{k \to \infty} \lambda a_{n_k} = u$, $\lim_{k \to \infty} \lambda t a_{n_k} = tu$, $tu = \lambda u$. Also, $\| u \| = \lim_{k \to \infty} | \lambda | \, \| a_{n_k} \| = | \lambda |$, and so $u \neq 0$.

THEOREM 2. 0 *is the only possible point of accumulation of* $\mathrm{Sp}_A(t)$; *and if* $\lambda \in \mathrm{Sp}_A(t)$ *and* $\lambda \neq 0$, *then there exists $u \in A$ such that $u \neq 0$ and $tu = \lambda u$.*

Proof. Lemma 1 and Proposition 1 show that 0 is the only possible point of accumulation of the frontier of $\mathrm{Sp}_A(t)$. Since $\mathrm{Sp}_A(t)$ is a compact subset of the complex plane, it follows that 0 is the only possible point of accumulation of $\mathrm{Sp}_A(t)$, and so all points of $\mathrm{Sp}_A(t)$ are frontier points. A second application of Lemma 1 completes the proof.

Notation. For Theorem 3, we suppose that $\lambda \in \mathrm{Sp}_A(t)$ and that $\lambda \neq 0$, and we define N_k, R_k by

$$N_k = \{a \in A \colon (t-\lambda)^k a = 0\}, \quad R_k = (t-\lambda)^k A.$$

By Proposition 2, there exists a least positive integer ν (the *index* of λ) such that $N_{\nu+1} = N_\nu$. It is easly verified that in fact $N_k = N_\nu$ $(k \geqq \nu)$.

THEOREM 3. (i) $A = N_\nu \oplus R_\nu$.

(ii) $1 = p + q$, *where p and q are idempotents that are unit elements for the subalgebras N_ν and R_ν respectively.*

(iii) $(t-\lambda)q$ *is invertible relative to the subalgebra R_ν, i.e. there exists $u \in R_\nu$ such that* $(t-\lambda)qu = q$.

(iv) $(t-\lambda)R_\nu = R_\nu$.

(v) *ν is the least positive integer for which* $(t-\lambda)^{\nu+1}A = (t-\lambda)^\nu A$.

Proof. (i) By a standard algebraic argument, we have $N_\nu \cap R_\nu = (0)$. For if $a \in N_\nu \cap R_\nu$, then $a = (t-\lambda)^\nu b$ for some $b \in A$. Then $(t-\lambda)^{2\nu} b = 0$, and so $b \in N_{2\nu} = N_\nu$, $a = 0$.

N_ν is a closed ideal of A. Let B denote the difference algebra $A - N_\nu$, the elements of which are the cosets $a' = a + N_\nu$. With the canonical norm $\| a' \| = \inf\{\| x \| : x \in a'\}$, B is a commutative Banach algebra with unit element $1'$. We prove next that the mapping $a' \to t'a'$ is a compact linear operator on B. In fact, given $b_n \in B$ with $\| b_n \| \leqq 1$ $(n = 1, 2, \ldots)$, there exist elements $a_n \in A$ with $\| a_n \| \leqq 2$ and $a_n' = b_n$. Then there exists a subsequence $\{a_{n_k}\}$ such that $\lim_{k \to \infty} ta_{n_k} = a$, say, and therefore $\lim_{k \to \infty} t'b_{n_k} = \lim_{k \to \infty} (ta_{n_k})' = a'$. This shows that Theorem 2 is applicable to B and t' in place of A and t, and it follows that $\lambda \notin \mathrm{Sp}_B(t')$. For otherwise there exists $u \in A$ such that $u' \neq 0$ and $t'u' = \lambda u'$. But then $(t-\lambda)u \in N_\nu$, $u \in N_{\nu+1} = N_\nu$, $u' = 0$, a contradiction. Therefore $(t-\lambda)'$ has an inverse a', say, in B, i.e. $(t-\lambda)'a' = 1'$. But then $((t-\lambda)^\nu a^\nu)' = 1'$, and so there exists $v \in N_\nu$ such that $1 = v + (t-\lambda)^\nu a^\nu$. This shows that $1 \in N_\nu \oplus R_\nu$, and, since $N_\nu \oplus R_\nu$ is an ideal, we have $A = N_\nu \oplus R_\nu$.

(ii) This is clear. We merely decompose 1 into its components p and q in N_ν and R_ν respectively.

(iii) We have seen that $(t-\lambda)'$ is invertible in $B = A - N_\nu$. However, the canonical mapping $a \to a'$ is an isomorphism of R_ν on to B, and $((t-\lambda)q)' = (t-\lambda)'$. Therefore $(t-\lambda)q$ is invertible relative to the subalgebra R_ν, i.e. there exists $u \in R_\nu$ such that $(t-\lambda)qu = q$.

(iv) $(t-\lambda)R_\nu = (t-\lambda)qR_\nu = R_\nu$.

(v) We have $(t-\lambda)^{\nu+1}A = (t-\lambda)R_\nu = R_\nu = (t-\lambda)^\nu A$. Also, if k is a positive integer for which $(t-\lambda)^{k+1}A = (t-\lambda)^k A$, then $(t-\lambda)^k = (t-\lambda)^{k+1}a$ for some $a \in A$. Therefore, if $(t-\lambda)^{k+1}x = 0$, we have $(t-\lambda)^k x = (t-\lambda)^{k+1}ax = 0$, and so $k \geqq \nu$.

We now apply Theorems 2 and 3 to obtain spatial properties of a bounded linear operator t on a Banach space X. We denote the spectrum of t by $\mathrm{Sp}(t)$, i.e. $\mathrm{Sp}(t) = \mathrm{Sp}_{B(X)}(t)$.

THEOREM 4. *Let $t \in B(X)$, let A be the least closed subalgebra of $B(X)$ containing t and e, and suppose that the mapping $a \to ta$ $(a \in A)$ is a compact linear operator on A. Then the following propositions hold.*

(i) $\mathrm{Sp}(t) = \mathrm{Sp}_A(t)$, *and each nonzero point of* $\mathrm{Sp}(t)$ *is an eigenvalue of t.*

(ii) *Let λ be a nonzero eigenvalue of t, let p, q be the projections given in Theorem* 3, *and let $Y = pX$, $Z = qX$. Then Y, Z are closed invariant subspaces for t,*

$$Y = \{x : (t-\lambda e)^\nu x = 0\}, \quad Z = (t-\lambda e)^\nu X,$$

$X = Y \oplus Z$, the restriction of $t - \lambda e$ to Z is invertible in $B(Z)$, and the index of λ for the linear operator t is equal to its index ν for t regarded as an element of A.

(iii) *Let t^* denote the adjoint of t, and A^* the least closed subalgebra of $B(X^*)$ containing t^* and e^*. Then the mapping $a^* \to t^*a^*$ $(a^* \in A^*)$ is a compact linear operator on A^*, and propositions* (i) *and* (ii) *hold for t^*, A^*, X^* in place of t, A, X. The index of a nonzero eigenvalue for the linear operator t^* is equal to its index for t, and for $k = 0, 1, \ldots, \nu-1$, the ranges of $(t-\lambda e)^k p$ and $(t^* - \lambda e^*)^k p^*$ have equal dimension.*

Remarks. (i) Theorem 1 shows that Theorem 4 is applicable in particular to a compact linear operator t on X.

(ii) The spectral projection p is an element of A. In fact, it belongs to the least closed subalgebra A_0 of $B(X)$ that contains t. For we have $(t-\lambda e)^\nu p = 0$, and $\lambda \neq 0$, from which $p \in tA \subset A_0$.

(iii) Theorem 4 is applicable to operators t that are not compact on X; for example the identity operator e.

Proof. (i) Since $A \subset B(X)$, it is obvious that $\mathrm{Sp}(t) \subset \mathrm{Sp}_A(t)$, and since all points of $\mathrm{Sp}_A(t)$ are frontier points, $\mathrm{Sp}_A(t) \subset \mathrm{Sp}(t)$, [7, Theorem 1.6.12]. It is now clear from Theorem 2 that all nonzero points of $\mathrm{Sp}(t)$ are eigenvalues.

(ii) It is clear that Y and Z are closed invariant subspaces for t and that $X = Y \oplus Z$. By Theorem 3(iii), $(t-\lambda e)q$ is invertible relative to R_ν, and so $(t-\lambda e)|_Z$ is invertible in $B(Z)$. We note next that

$$Y = \{x \in X: (t-\lambda e)^\nu x = 0\}, \quad Z = (t-\lambda e)^\nu X. \tag{1}$$

For, since $qA = R_\nu = (t-\lambda e)^\nu A$, there exist $a, b \in A$ with

$$q = a(t-\lambda e)^\nu, \quad (t-\lambda e)^\nu = qb. \tag{2}$$

It follows at once from (2) that $(t-\lambda e)^\nu X = qX$ and that $(t-\lambda e)^\nu x = 0$ if and only if $qx = 0$, which proves (1).

We prove that ν is the least positive integer k for which

$$\{x \in X: (t-\lambda e)^{k+1} x = 0\} = \{x \in X: (t-\lambda e)^k x = 0\}. \tag{3}$$

For if k satisfies (3) and $(t-\lambda e)^{k+1} a = 0$ for some $a \in A$, then we have in turn $(t-\lambda e)^{k+1} ax = 0$ $(x \in X)$, $(t-\lambda e)^k ax = 0$ $(x \in X)$, $(t-\lambda e)^k a = 0$; which shows that $k \geqq \nu$. On the other hand, if $(t-\lambda e)^{\nu+1} x = 0$ for some $x \in X$, then $(t-\lambda e)^{2\nu} x = 0$; and, by (1), $(t-\lambda e)^\nu x \in Y \cap Z$, $(t-\lambda e)^\nu x = 0$. Thus ν is an integer satisfying (3).

(iii) The mapping $a \to a^*$ of a on to its adjoint a^* is an isometric isomorphism between A and A^*. Thus the properties of (t^*, A^*, X^*) can be deduced from those of (t, A, X). The final statement is an elementary consequence of the duality between X and X^*. Let $a = (t-\lambda e)^k p$, and let $x_1, \ldots, x_n$ be linearly independent elements of aX. Then there exist $f_1, \ldots, f_n \in X^*$ and $y_1, \ldots, y_n \in X$ such that $f_i(x_j) = \delta_{ij}$ and $x_j = ay_j$ $(i, j = 1, \ldots, n)$. Then

$$(a^* f_i)(y_j) = f_i(ay_j) = f_i(x_j) = \delta_{ij},$$

showing that $a^*f_1, \ldots, a^*f_n$ are linearly independent elements of a^*X^*. Thus the dimension of aX does not exceed the dimension of a^*X^*, and similarly for the opposite inequality.

4. The Krein-Rutman theorem. We show how Theorem 1 may be applied to give a new proof of the following theorem of M. G. Krein and M. A. Rutman [5, Theorem 6.1].

THEOREM. *Let X^+ be a closed cone in a Banach space X such that X is the closed linear hull of X^+. Let t be a compact linear operator on X such that $tX^+ \subset X^+$ and the spectral radius ρ of t is not zero. Then there exist $x_0 \in X^+$ and $f_0 \in X^{*+}$ such that $x_0 \neq 0, f_0 \neq 0$, $tx_0 = \rho x_0$, $t^* f_0 = \rho f_0$.*

We first prove the following variant of the Krein-Rutman theorem.

THEOREM 5. *Let t be a compact linear operator on a Banach space X, and let the spectral radius ρ of t be nonzero. Let A^+ denote the least closed semi-algebra in $B(X)$ containing t and e. If $A^+ \cap (-A^+) = (0)$, then there exists $u \in A^+$ such that $u \neq 0$ and $tu = \rho u$.*

Proof. We use elementary techniques introduced in [2], but simplified by the fact that we are able to work throughout in $B(X)$. Suppose that $A^+ \cap (-A^+) = (0)$. We note first that if $\mu > \rho$, then

$$r_\mu = (\mu e - t)^{-1} = \frac{1}{\mu} e + \frac{1}{\mu^2} t + \frac{1}{\mu^3} t^2 + \ldots \in A^+.$$

We prove that there exists a sequence $\{\mu_n\}$ such that

$$\mu_n > \rho \quad (n = 1, 2, \ldots), \quad \lim_{n \to \infty} \mu_n = \rho, \quad \lim_{n \to \infty} \| r_{\mu_n} \| = \infty. \tag{4}$$

Suppose that there is no such sequence $\{\mu_n\}$. Then there exist positive constants ε, M such that $\| r_\mu \| \leqq M$ $(\rho < \mu < \rho + \varepsilon)$. We choose λ, μ such that

$$0 < \lambda < \rho < \mu < \rho + \varepsilon, \quad \mu - \lambda < M^{-1}.$$

Then we have

$$r_\lambda = (\lambda e - t)^{-1} = r_\mu + (\mu - \lambda) r_\mu^2 + \ldots \in A^+,$$

and

$$\left(e - \frac{1}{\lambda^{n+1}} t^{n+1}\right) r_\lambda = \frac{1}{\lambda} e + \frac{1}{\lambda^2} t + \ldots + \frac{1}{\lambda^{n+1}} t^n.$$

It follows from this that

$$r_\lambda = \frac{1}{\lambda^{n+1}} t^n + q_n, \tag{5}$$

with $q_n \in A^+$.

Let $\alpha_n = \lambda^{-n-1} \| t^n \|$. Since $0 < \lambda < \rho$, we have $\lim_{n \to \infty} \alpha_n = \infty$, and so there exists a subsequence $\{\alpha_{n_k}\}$ such that

$$\alpha_{n_k} \geqq \alpha_{n_k - 1} \quad (k = 1, 2, \ldots). \tag{6}$$

Let $a_k = \lambda^{-n_k} t^{n_k - 1}$. Since $\alpha_{n_k} \leqq \lambda^{-1} \| t \| . \| a_k \|$, we have $\lim_{k \to \infty} \| a_k \| = \infty$. By (5), we have

$$\| a_k \|^{-1} r_\lambda = \frac{1}{\lambda} t(\| a_k \|^{-1} a_k) + \| a_k \|^{-1} q_{n_k}. \tag{7}$$

By Theorem 1, there exists a subsequence of $\{\lambda^{-1} t(\| a_k \|^{-1} a_k)\}$ that converges to $b \in A^+$. Since the left-hand side of (7) tends to zero, the corresponding subsequence of $\{\| a_k \|^{-1} q_{n_k}\}$

also converges to $c \in A^+$, and $b+c=0$. Since $A^+ \cap (-A^+) = (0)$, we conclude that $b=0$. But this is a contradiction, for (6) gives

$$\left\| \frac{1}{\lambda} ta_k \right\| \geqq \| a_k \| \quad (k = 1, 2, \ldots),$$

and so $\| b \| \geqq 1$. This contradiction proves the existence of a sequence $\{\mu_n\}$ satisfying (4).

Let $u_n = \| r_{\mu_n} \|^{-1} r_{\mu_n}$. Then $u_n \in A^+$, $\| u_n \| = 1$, and

$$(\mu_n e - t)u_n = \| r_{\mu_n} \|^{-1} e. \tag{8}$$

By Theorem 1, there exists a subsequence $\{u_{n_k}\}$ such that $\lim_{k\to\infty} tu_{n_k} = u \in A^+$. Then, by (4) and (8), $\lim_{k\to\infty} \mu_{n_k} u_{n_k} = u$. Therefore $\lim_{k\to\infty} \mu_{n_k} tu_{n_k} = tu$, and, since $\lim_{k\to\infty} \mu_{n_k} = \rho$, we have $tu = \rho u$.

Proof of the Krein-Rutman theorem. Let X^+, t satisfy the conditions in the theorem. Each element a of A^+ is positive in the sense that $aX^+ \subset X^+$. If $a \in A^+ \cap (-A^+)$, then

$$aX^+ \subset X^+ \cap (-X^+) = (0), \quad aX^+ = (0).$$

Since the linear hull of X^+ is dense in X, this gives $a = 0$, $A^+ \cap (-A^+) = (0)$. Thus Theorem 5 is applicable, and there exists $u \in A^+$ such that $u \neq 0$ and $tu = \rho u$. Again, since the linear hull of X^+ is dense in X, there exists $x_1 \in X^+$ such that $ux_1 \neq 0$. Let $x_0 = ux_1$. Then $x_0 \in X^+$, $x_0 \neq 0$, and

$$tx_0 = tux_1 = \rho ux_1 = \rho x_0.$$

Also, $x_0 \notin -X^+$, and $-X^+$ is a closed positive homogeneous convex set. Therefore there exists $f_1 \in X^*$ such that $f_1(x_0) = 1$ and $f_1(x) \leqq 0$ $(x \in -X^+)$. This implies that $f_1 \in X^{*+}$. Let $f_0 = u^* f_1$. Then $f_0 \in X^{*+}$, and $f_0 \neq 0$, since

$$f_0(x_1) = (u^* f_1)(x_1) = f_1(ux_1) = f_1(x_0) = 1.$$

Finally, $t^* u^* = \rho u^*$ (since $ut = \rho u$), and so

$$t^* f_0 = t^* u^* f_1 = \rho u^* f_1 = \rho f_0.$$

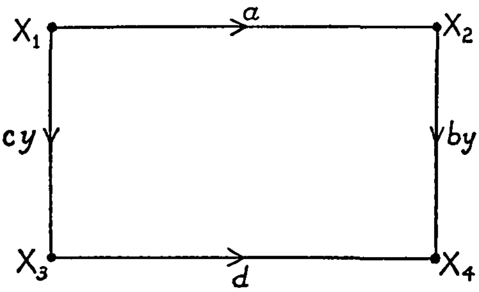
Remark. Let t satisfy the conditions of Theorem 5. If $A^+ \cap (-A^+) = (0)$, then A^+ is a locally compact semi-algebra. For, by Theorem 5, the spectral radius of t belongs to the spectrum of t, and so A^+ is locally compact [3, Theorem 5]. Conversely, if A is locally compact, then $A^+ \cap (-A^+)$ has finite dimension, since it is a normed linear space with a compact unit ball.

5. A general theorem. Given Banach spaces X, Y, we denote by $B(X, Y)$ the Banach space of all bounded linear mappings of X into Y with the usual norm.

THEOREM 6. *Let X_1, X_2, X_3, X_4, Y be Banach spaces, let a and d be compact linear mappings of X_1 into X_2 and X_3 into X_4 respectively, let b and c be bounded linear mappings of Y into $B(X_2, X_4)$ and $B(X_1, X_3)$ respectively, and let*

$$Z = \{y \in Y : (by) \circ a = d \circ (cy)\}.$$

Then the mapping $z \to (bz) \circ a$ is a compact linear mapping of Z into $B(X_1, X_4)$.



Proof. Let U, V, W be the closed balls given by

$$U = \{x \in X_1 : \| x \| \leqq 1\}, \quad V = \{z \in Z : \| z \| \leqq 1\}, \quad W = \{x \in X_3 : \| x \| \leqq \| c \|\},$$

and let $E = \overline{aU}$, $F = \overline{dW}$. Then E and F are compact subsets of X_2 and X_4 in their norm topologies, and we have

$$(cz)x \in W \quad (x \in U, \ z \in V),$$

from which

$$((bz) \circ a)x = (d \circ (cz))x \in dW \quad (x \in U, \ z \in V).$$

Since bz is continuous, it follows that

$$(bz)x \in F \quad (x \in E, \ z \in V).$$

Thus, for each $x \in E$, the image of V under the mapping $z \to (bz)x$ is contained in a compact subset of X_4. Also, $\{bz : z \in V\}$ is an equicontinuous family of mappings of the compact space E into the Banach space X_4. Therefore, by Ascoli's theorem for Banach space valued functions, given $z_n \in V$ $(n = 1, 2, \ldots)$, there exists a subsequence $\{z_{n_k}\}$ such that $\{(bz_{n_k})x\}$ converges uniformly on E. Since $aU \subset E$, it follows that $((bz_{n_k}) \circ a)x$ converges uniformly on U, i.e. $\{(bz_{n_k}) \circ a\}$ converges with respect to the norm in $B(X_1, X_4)$.

Example 1. Let t be a compact linear operator on a Banach space X over $\mathbf{F}$, where $\mathbf{F}$ is either $\mathbf{R}$ or $\mathbf{C}$. Take $X_1 = X_2 = X$, $X_3 = X_4 = \mathbf{F}$, $Y = X^*$, the dual space of X. Let $a = t$, $c = t^*$, and let b and d be the identity operators on X^* and $\mathbf{F}$ respectively. Since

$$(by) \circ a = y \circ t = t^*y = d \circ (cy) \quad (y \in Y),$$

we have $Z = Y = X^*$. Thus the mapping $y \to t^*y$ is a compact linear mapping of X^* into $B(X_1, X_4) = B(X, \mathbf{F}) = X^*$. This is Schauder's theorem.

Example 2. Let t be a compact linear operator on a Banach space X, let $X_1 = X_2 = X_3 = X_4 = X$, and let $Y = B(X, X)$. Let $a = d = t$, and let both b and c be the identity operator

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on Y. Then

$$Z = \{s \in B(X, X) : st = ts\},$$

and $s \to st$ is a compact linear mapping of Z into itself. This is Theorem 1.

Example 3. Let X_1, X_2, X_3, X_4 be Banach spaces, and let $Y = B(X_2, X_3)$. Let a and d be compact linear mappings of X_1 into X_2 and X_3 into X_4 respectively and let b and c be defined by $by = d \circ y$, $cy = y \circ a$. Then

$$(by) \circ a = d \circ y \circ a = d \circ (cy) \quad (y \in Y).$$

Therefore $Z = Y = B(X_2, X_3)$, and the mapping $y \to d \circ y \circ a$ is a compact linear mapping of $B(X_2, X_3)$ into $B(X_1, X_4)$. This result has been proved by K. Vala [9] as an application of his formulation of Ascoli's theorem.

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