A VARIANCE METHOD IN COMBINATORIAL NUMBER THEORY

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Let $s = s(a_1, a_2, \ldots, a_r)$ denote the number of integer solutions of the equation

$$u_1 + u_2 + \ldots + u_r = \left[\tfrac{1}{2}\sum_{i=1}^{r} a_i\right] = \lambda$$

subject to the conditions

$$0 \leqq u_i \leqq a_i \qquad (i = 1, \ldots, r),$$

the a_i being given positive integers, and square brackets denoting the integral part. Clearly $s(a_1, \ldots, a_r)$ is also the number $s = s(m)$ of divisors of $m = p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r}$ which contain exactly λ prime factors counted according to multiplicity, and is therefore, as is proved in **[1]**, the cardinality of the largest possible set of divisors of m, no one of which divides another.

In an earlier paper **[2]** we proved by means of contour integration that, under fairly general conditions,

$$s(m) \sim \sqrt{\left(\frac{2}{\pi}\right)} \frac{\tau(m)}{\sqrt{A(m)}}$$

as $\sum_{i=1}^{r} a_i \to \infty$, where $\tau(m) = \prod_{i=1}^{r} (1 + a_i)$ denotes the total number of divisors of m, and where $A(m) = \frac{1}{3}\sum_{i=1}^{r} a_i(a_i + 2)$. In the squarefree case, when $m = p_1 p_2 \ldots p_n$, this becomes

$$\binom{n}{[\frac{1}{2}n]} \sim \sqrt{\left(\frac{2}{\pi}\right)} \frac{2^n}{\sqrt{n}},$$

a result which checks with Stirling's formula. We now prove

THEOREM 1. *There exist constants* $C_1 > 0$, $C_2 > 0$ *such that*

$$C_1 \frac{\tau(m)}{\sqrt{A(m)}} \leqq s(m) \leqq C_2 \frac{\tau(m)}{\sqrt{A(m)}}$$

for all m.

If we define the degree n of an integer m to be the number of prime factors of m counted according to multiplicity, and let $N_l = N_l(m)$ denote the number of divisors of m of degree l, we then have $s = N_\lambda$, where $\lambda = [\frac{1}{2}n]$. Also let $\tau = \tau(m)$.

Let

$$\sigma^2 = \frac{2}{\tau}\sum_{l=0}^{\lambda}\left(\frac{n}{2} - l\right)^2 N_l = \frac{1}{2\tau}\sum_{l=0}^{\lambda}(n - 2l)^2 N_l \tag{1}$$

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denote the variance of the distribution of degrees among the divisors of m. Since the variance of $\{0, 1, 2, \ldots, a\}$ is $\frac{1}{12}a(a+2)$, and since the variance of a sum of independent distributions is equal to the sum of the variances, we have

$$\sigma^2 = \sum_{i=1}^{r} \tfrac{1}{12} a_i(a_i+2) = \tfrac{1}{4} A(m).$$

To prove Theorem 1, it therefore suffices to prove

THEOREM 2. *With the above notation, there exist constants* $C_3 > 0$, $C_4 > 0$ *such that*

$$C_3 \frac{\tau}{s} \leqq \sigma \leqq C_4 \frac{\tau}{s}.$$

2. The lower bound. Since

$$\binom{x+1}{3} - \binom{x-1}{3} = (x-1)^2,$$

we have

$$\begin{aligned} 2\tau\sigma^2 &= \sum_{l=0}^{\lambda} (n-2l)^2 N_l > \sum_{l=0}^{\lambda} (n-2l-1)^2 N_l \\ &= \sum_{l=0}^{\lambda} \left\{ \binom{n+1-2l}{3} - \binom{n-1-2l}{3} \right\} N_l \\ &= \sum_{l=1}^{\lambda} \binom{n+1-2l}{3} (N_l - N_{l-1}) + \binom{n+1}{3}. \end{aligned}$$

Now de Bruijn, Tengbergen and Kruyswijk [1] have shown that the divisors of m can be put into s disjoint chains ordered by divisibility, the number of chains containing $n+1-2l$ elements being precisely $N_l - N_{l-1}$. (Incidentally, the result

$$N_l \leqq N_{l+1} \quad \text{if} \quad l < \lambda \tag{2}$$

is implicit in this.) Thus $2\tau\sigma^2$ is essentially just the number of ways of selecting three divisors of m from the same chain.

If the s chains contain $x_1, x_2, \ldots, x_s$ elements, where $\sum_{i=1}^{s} x_i = \tau$, we have

$$2\tau\sigma^2 > \frac{1}{6} \sum_{i=1}^{s} x_i(x_i-1)(x_i-2).$$

This has a minimum if all the x_i are equal, when $x_i = \tau/s$. Thus, for sufficiently large n,

$$2\tau\sigma^2 \geqq (\tfrac{1}{6} - \varepsilon)\frac{\tau^3}{s^2},$$

since $s = o(\tau)$ (see [2]; alternatively, this follows from the second half of this theorem).

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Thus

$$\sigma \geqq \left(\frac{1}{2\sqrt{3}} - \varepsilon\right)\frac{\tau}{s}.$$

3. The upper bound. In view of (1), we have to estimate $\sum_{l=0}^{\lambda}(n-2l)^2 N_l$. To do this, we make use of the following lemmas, the first of which is proved by an elementary argument in [3].

LEMMA 1. *If* $0 < l < k < n$, *then* $N_l N_k \leqq N_{l+1} N_{k-1}$.

LEMMA 2. (*Reduction Formula*)

$$N_{\lambda-r}^2 \sum_{l=0}^{\lambda-r}(n-2l)^2 N_l \leqq N_{\lambda-r-1}^2 \sum_{l=0}^{\lambda-r-1}(n-2l)^2 N_l + 8N_{\lambda-r-1}^2 \sum_{l=0}^{\lambda-r-1}(n-2l)N_l$$
$$+16N_{\lambda-r-1}^2 \sum_{l=0}^{\lambda-r-1} N_l + 3(2r+5)^2 N_{\lambda-r}^3.$$

Proof. Throughout, we shall make repeated use of (2).

$$N_{\lambda-r}^2 \sum_{l=0}^{\lambda-r}(n-2l)^2 N_l \leqq N_{\lambda-r}^2 \sum_{l=0}^{\lambda-r-2}(n-2l)^2 N_l + 2(2r+1)^2 N_{\lambda-r}^3,$$

where the first term on the right is, by two applications of Lemma 1,

$$\leqq N_{\lambda-r-1}^2 \sum_{l=0}^{\lambda-r-2}(n-2l)^2 N_{l+2}$$
$$\leqq N_{\lambda-r-1}^2 \sum_{l=0}^{\lambda-r}(n-2l+4)^2 N_l$$
$$\leqq N_{\lambda-r-1}^2 \sum_{l=0}^{\lambda-r-1}(n-2l+4)^2 N_l + (2r+5)^2 N_{\lambda-r}^3$$
$$\leqq N_{\lambda-r-1}^2 \sum_{l=0}^{\lambda-r-1}(n-2l)^2 N_l + 8N_{\lambda-r-1}^2 \sum_{l=0}^{\lambda-r-1}(n-2l)N_l + 16N_{\lambda-r-1}^2 \sum_{l=0}^{\lambda-r-1} N_l + (2r+5)^2 N_{\lambda-r}^3.$$

The lemma now follows.

Starting with $r = 0$, we apply this reduction formula repeatedly to obtain

$$2\tau s^2\sigma^2 = N_\lambda^2 \sum_{l=0}^{\lambda}(n-2l)^2 N_l$$
$$\leqq 8\sum_{t=0}^{\lambda-1} N_{\lambda-t}^2\left(\sum_{l=0}^{\lambda-t}(n-2l)N_l\right) + 16\sum_{t=0}^{\lambda-1} N_{\lambda-t}^2\left(\sum_{l=0}^{\lambda-t} N_l\right) + 3\sum_{t=0}^{\lambda}(2t+5)^2 N_{\lambda-t}^3.$$

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But

$$\sum_{t=0}^{\lambda}(2t+5)^2N_{\lambda-t}^3 \ll \sum_{t=0}^{\lambda}N_{\lambda-t}(tN_{\lambda-t})^2$$
$$\ll \sum_{t=0}^{\lambda}N_{\lambda-t}(N_{\lambda-t}+\ldots+N_{\lambda-1})^2$$
$$\ll \left(\frac{\tau}{2}\right)^2\cdot\frac{\tau}{2} \ll \tau^3,$$

and

$$\sum_{t=0}^{\lambda-1}N_{\lambda-t}^2\left(\sum_{l=0}^{\lambda-t}N_l\right) \leqq \frac{\tau}{2}\sum_{t=0}^{\lambda-1}N_{\lambda-t}^2 \leqq \frac{\tau}{2}\left(\sum_{t=0}^{\lambda-1}N_{\lambda-t}\right)^2 \ll \tau^3.$$

Further, a reduction argument similar to the above [3] shows that

$$N_\lambda\sum_{l=0}^{\lambda}(n-2l)N_l \ll \tau^2.$$

Thus the remaining term in (3) is

$$\ll \tau^2\sum_{t=0}^{\lambda-1}N_{\lambda-t} \ll \tau^3.$$

Thus finally

$$2\tau s^2\sigma^2 \ll \tau^3,$$

whence

$$\sigma \ll \frac{\tau}{s}.$$

This completes the proof of Theorem 2. We remark, however, that a more careful estimation gives the result

$$\sigma \leqq \frac{\sqrt{11}}{2}\frac{\tau}{s}$$

provided the degree of m is big enough. Whereas the lower bound is the best possible, being attained when m is a prime power, our upper bound can certainly be improved upon. Perhaps its value in the case of m squarefree, namely $1/\sqrt{(2\pi)}$, is the true value.

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