THE BRACKET FUNCTION AND COMPLEMENTARY SETS OF INTEGERS

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1. Introduction. The following result is well known (as usual, $[x]$ denotes the integral part of x):

(A) *Let α and β be positive irrational numbers satisfying*

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1. \tag{1}$$

Then the sets $[n\alpha]$, $[n\beta]$, $n = 1, 2, \ldots$, are complementary with respect to the set of all positive integers; see, e.g. (**1**; **2**; **4**; **5**; **6**; **7**; **8**; **10**; **13**; **14**; **15**; **16**). In some of these references the result, or a special case thereof, is mentioned in connection with *Wythoff's game*, with or without proof. It appears that Beatty (**4**) was the originator of the problem.

The theorem has a converse, and the following holds:

(B) *Let α and β be positive. The sets $[n\alpha]$ and $[n\beta]$, $n = 1, 2, \ldots$, are complementary with respect to the set of all positive integers if and only if α and β are irrational, and* (1) *holds.*

In this paper we deal with the *non-homogeneous* case, that is, we investigate conditions for sets of the form $[n\alpha + \gamma]$, $[n\beta + \delta]$ to be complementary. The integer n runs over all integers or is restricted to subsets $n \geqq N$ or $n < N$, where N is a fixed integer. The numbers α and β are either both rational or both irrational; γ and δ are real.

More specifically, let α and β be positive numbers, γ and δ real, N integral. Let S, S_N', S_N be the sets of all integers from the sequences $\phi_n = [n\alpha + \gamma]$, where $n = 0, \pm 1, \pm 2, \ldots$ for S, $n < N$ for S_N', $n \geqq N$ for S_N. Similarly, let T, T_N', and T_N be the sets of all integers from the sequences $\psi_n = [n\beta + \delta]$, for $n = 0, \pm 1, \pm 2, \ldots, n < N, n \geqq N$, respectively.

Without loss of generality, we shall assume that $\phi_N \leqq \psi_N$ throughout.

Definition 1. We say that S and T are *complementary* (S_N' and T_N' are *N-lower complementary*) [S_N and T_N are *N-upper complementary*] if

(i) no integer appears more than once in ϕ_n and no integer appears more than once in ψ_n, $n = 0, \pm 1, \pm 2, \ldots$ ($n < N$) [$n \geqq N$];

(ii) $S \cap T = \emptyset$ ($S_N' \cap T_N' = \emptyset$) [$S_N \cap T_N = \emptyset$];

(iii) $S \cup T = Z$ ($S_N' \cup T_N' = Z_N'$) [$S_N \cup T_N = Z_N$], where Z (Z_N') [Z_N] is the set of all integers ($<\phi_N$) [$\geqq \phi_N$].

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Section 2 contains preliminary results and lemmas. In § 3 we give necessary and sufficient conditions for sets to be lower and upper complementary. We prove the following results.

THEOREM I. *Let* α *and* β *be positive irrational numbers. Then* S_N' *and* T_N' *are* N*-lower complementary if and only if* (1) *holds and*

$$\frac{\gamma}{\alpha} + \frac{\delta}{\beta} = \phi_N - 2N + 1, \tag{2}$$

$$n\beta + \delta = K, \qquad n, K \text{ integral implies } n \geqq N. \tag{3}$$

THEOREM II. *Let* α *and* β *be positive irrational numbers. Then* S_N *and* T_N *are* N*-upper complementary if and only if* (1) *and* (2) *hold and*

$$n\beta + \delta = K, \qquad n, K \text{ integral implies } n < N. \tag{4}$$

Example 1. $\alpha = \sqrt{2}$, $\beta = 2 + \sqrt{2}$, $\gamma = -8 + 3\sqrt{2}/2$, $\delta = -5 + 3\sqrt{2}/2$. It is easily verified that (1) is satisfied and that the left-hand side of (2) is -5. The same value is obtained on the right-hand side for $N = 0, -1$ (see Table 1). Furthermore, there exist no integers n and K satisfying $n\beta + \delta = K$. Hence, S_N' and T_N' (S_N, T_N) are N-lower (N-upper) complementary for $N = -1, 0$.

Example 2. $\alpha = \sqrt{5}$, $\beta = (5 + \sqrt{5})/4$, $\gamma = 2 - 2\sqrt{5}$, $\delta = (3 - \sqrt{5})/4$. Equation (1) is satisfied and the left-hand side of (2) is -1. The right-hand side has the same value for $2 \leqq N \leqq 6$. Thus, S_N' and T_N' would be N-lower complementary for these values of N, except for the fact that (3) is violated, namely, $\psi_1 = \beta + \delta = 2$.

TABLE 1

n	ϕ_n	ψ_n
−7	−16	−27
−6	−15	−24
−5	−13	−20
−4	−12	−17
−3	−11	−14
−2	−9	−10
−1	−8	−7
0	−6	−3
1	−5	0
2	−4	3
3	−2	7
4	−1	10
5	1	14
6	2	17
7	4	21

TABLE 2

n	ϕ_n	ψ_n	ψ_n^*
−3	−10	−6	−8
−2	−7	−4	−6
−1	−5	−2	−4
0	−3	0	−2
1	−1	2	0
2	2	3	2
3	4	5	3
4	6	7	5
5	8	9	7
6	10	11	9
7	13	12	11
8	15	14	12
9	17	16	14
10	19	18	16
11	22	20	18

Table 2 shows that $\phi_2 = \psi_1 = 2$. Moreover, since $\alpha > 2, f(N) = \phi_N - 2N + 1$ is a non-decreasing function of N. Since $f(1) = -2$, we have that S_N' and T_N' are n-lower complementary for no N.

Suppose now that we replace δ by

$$\delta^* = \delta - \beta = -\frac{1+\sqrt{5}}{2}.$$

This has the effect of shifting the values of ψ_n one place down, and results in the column $\psi_n^* = [n\beta + \delta^*]$ of Table 2. Now, $\gamma/\alpha + \delta^*/\beta = -2$, and the right-hand side of (2) has the same value for $-2 \leqq N \leqq 1$. For these values of N, S_N' and $T_N^{*\prime}$ are N-lower complementary, where $T_N^{*\prime}$ is the set of all integers from the sequence ψ_n^*, $n < N$. The same considerations show that S_N and T_N are N-upper complementary for $2 \leqq N \leqq 6$, but S_N and T_N^* are N-upper complementary for no N.

Skolem investigated the non-homogeneous case (for $N = 1$, α and β irrational only) in (**12**). In his *Satz* 10, he states (using the language of the present paper) that if α and β are positive irrational, then S_1 and T_1 are 1-upper complementary if and only if (1) holds and

$$\frac{\gamma}{\alpha} + \frac{\delta}{\beta} \equiv 0 \pmod{1}. \tag{5}$$

However, in view of Theorem II, these conditions are not quite sufficient. (In both Examples 1 and 2, (1) and (5) are satisfied. But S_1 and T_1 of Example 1 and S_1 and T_1^* of Example 2 are not 1-upper complementary. In Example 1, (2) does not hold, and in Example 2, (4) does not hold.)

Now suppose that α and β are rationals of the form

$$\begin{aligned} \alpha &= a/c, \qquad (a, c) = 1, \\ \beta &= b/d, \qquad (b, d) = 1, \end{aligned} \tag{6}$$

where a, b, c, d are positive integers. We write

$$\frac{\gamma}{\alpha} \equiv \eta \pmod{a^{-1}}, \qquad \frac{\delta}{\beta} \equiv \rho \pmod{b^{-1}},$$

meaning that

$$\frac{\gamma}{\alpha} = ka^{-1} + \eta, \quad 0 \leqq \eta < a^{-1}, \qquad \frac{\delta}{\beta} = lb^{-1} + \rho, \quad 0 \leqq \rho < b^{-1}, \tag{7}$$

k and l are integers. We prove the following theorem.

THEOREM III. *Let α and β be positive rational numbers of the form* (6). *Then S_N and T_N are N-upper complementary if and only if* (1) *holds and*

$$\frac{\gamma}{\alpha} + \frac{\delta}{\beta} = \phi_N - 2N + 1 - a^{-1} + \eta + \rho. \tag{8}$$

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Theorem IV shows that for the rational case, S_N and T_N are N-upper complementary if and only if S_N' and T_N' are N-lower complementary. This is in contrast to the irrational case, where it can easily happen that S_N and T_N are N-upper complementary without S_N' and T_N' being N-lower complementary (e.g., if α and β satisfy (1), $\gamma = \delta = 0$, $N = 1$).

Example 3. $\alpha = 5/3$, $\beta = 5/2$, $\gamma = -1$, $\delta = \sqrt{2}$. Then

$$\frac{\gamma}{\alpha} + \frac{\delta}{\beta} + a^{-1} - \eta - \rho = 0,$$

and also, $\phi_N - 2N + 1 = 0$ for $-2 \leqq N \leqq 0$. Hence, S_N and T_N are N-upper and S_N' and T_N' are N-lower complementary for these values of N (see Table 3).

TABLE 3

n	ϕ_n	ψ_n
−6	−11	−14
−5	−10	−12
−4	− 8	− 9
−3	− 6	− 7
−2	− 5	− 4
−1	− 3	− 2
0	− 1	1
1	0	3
2	2	6
3	4	8
4	5	11
5	7	13
6	9	16

The following theorem is an immediate application of the rational case.

THEOREM V. *Let a and b be positive integers, $a > 1$. The set of all numbers*

$$(9) \qquad \left[n\frac{a}{a-1} + \delta\right], \qquad n \geqq 0,$$

is identical with all integers m such that $m > b$ and $m \not\equiv b$ (mod a) *if and only if*

$$(10) \qquad \delta = b + 1 + \rho a/(a-1), \qquad 0 \leqq \rho < a^{-1}.$$

Theorems VI and VII establish equivalence of sets of conditions, which permit alternative formulations of Theorems I and II. Section 3 concludes with a brief survey of the homogeneous case, i.e., $\gamma = \delta = 0$. We prove the following results.

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THEOREM VIII. *Let* α *and* β *be positive numbers,* $\gamma = \delta = 0$. *Then* S_N' *and* T_N' *are N-lower complementary if and only if* α *and* β *are irrational,* (1) *holds,* $\phi_N = 2N - 1$, *and* $N \leqq 0$.

THEOREM IX. *Let* α *and* β *be positive numbers,* $\gamma = \delta = 0$. *Then* S_N *and* T_N *are N-upper complementary if and only if* α *and* β *are irrational,* (1) *holds,* $\phi_N = 2N - 1$, *and* $N > 0$.

These theorems follow directly from Theorems I and II. The special case, $N = 1$, of the sufficiency part of Theorem IX is Beatty's result A. The special case $N = 1$ is the result B. The results of this section show, among other things, that the rational case vanishes only if $\gamma = \delta = 0$. For the non-homogeneous case, non-trivial results hold also when α and β are rational. Note also that S and T can never be complementary in the homogeneous case.

Section 4 contains results on complementary sets. Theorem X demonstrates a connection between lower and upper complementary sets and complementary sets for both the rational and the irrational case. By means of Theorems I and II we prove also the following general result.

THEOREM XI. *Let* α *and* β *be positive irrational numbers. Then* S *and* T *are complementary if and only if* (1) *and* (5) *hold and there exist no integers* n *and* K *satisfying* $n\beta + \delta = K$.

Similarly, by Theorems III and IV we prove the following result.

THEOREM XII. *Let* α *and* β *be positive rational numbers of the form* (6). *Then* S *and* T *are complementary if and only if* (1) *holds and*

$$\frac{\gamma}{\alpha} + \frac{\delta}{\beta} + a^{-1} - \eta - \rho \equiv 0 \pmod 1. \tag{11}$$

Section 4 concludes with a theorem about almost complementary sets, i.e., sets which are complementary except for a finite number of irregularities. The conditions of this theorem are precisely those of Skolem (**12**, *Satz* 10) mentioned above.

2. Preliminary results.

LEMMA 1. *Suppose that* α *and* β *are positive irrational and satisfy* $r\alpha + s\beta = t$, *where* r, s, *and* t *are integers satisfying* $(r, s, t) = 1$. *Then the points*

$$(n\alpha + h, n\beta + l), \qquad n = -1, -2, \ldots, \quad h = 0, \pm 1, \pm 2, \ldots, \quad l = 0, \pm 1, \pm 2, \ldots, \tag{12}$$

and the points

$$(n\alpha + h, n\beta + l), \qquad n = 0, 1, 2, \ldots, \quad h = 0, \pm 1, \pm 2, \ldots, \quad l = 0, \pm 1, \pm 2, \ldots, \tag{13}$$

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lie on all the straight lines $rx + sy \equiv 0 \pmod 1$ *and only on them. Moreover, the points* (12), *as well as the points* (13), *are everywhere dense on each of these lines.*

This lemma is essentially the same as Theorem 3.6 of (**9**) and the proof is therefore omitted.

We shall now characterize the set R of all points (12) for the case where α and β are rational. Incidentally, the sets (12) and (13) are identical in this case. Specifically, let α and β be of the form (6). If $(x, y) \in R$, then necessarily, $x = e/c$, $y = f/d$, where e and f are integers. But it is easy to see that in general not every point $(e/c, f/d)$ belongs to R.

Let

$$M = [c, d] = k_1 c = k_2 d, \qquad (k_1, k_2) = 1. \tag{14}$$

There clearly exist infinitely many integers s, t, u such that

$$sak_1 + tbk_2 = uM, \qquad (s, t, u) = 1. \tag{15}$$

Note that (α, β) lies on $sx + ty = u$. For fixed s, t we prove the following auxiliary result.

LEMMA 2. (i) $(e/c, f/d) \in R$ *if and only if*

$$(c, d) \mid (a_c^{-1} e - b_d^{-1} f), \tag{16}$$

where a_c^{-1} *denotes a residue of* $a^{-1} \pmod c$, b_d^{-1} *a residue of* $b^{-1} \pmod d$.

(ii) *The points of R lie on all and only on all straight lines* $sx + ty \equiv 0 \pmod 1$.

(iii) *The projections on the x-axis and y-axis of the distances between adjacent points lying on the straight lines* $sx + ty \equiv 0 \pmod 1$, *are* t/M *and* s/M, *respectively.*

(iv) *Suppose that* $s_1 x + t_1 y \equiv 0 \pmod 1$ *and* $s_2 x + t_2 y \equiv 0 \pmod 1$ *are two distinct families on each of which lie all points of R. Then R is identical with all meeting points of the two families if and only if* $|D| = M$, *where* $D = s_1 t_2 - s_2 t_1$.

Proof. The proof can be based on the theory of point lattices, but we use a different argument.

(i) $(e/c, f/d) = (n\alpha + h, n\beta + l)$ if and only if

$$(e, f) = (na + hc, nb + ld),$$

which holds if and only if

$$n \equiv a^{-1} e \pmod c, \qquad n \equiv b^{-1} f \pmod d.$$

By the generalized Chinese Remainder Theorem, these congruences have a solution if and only if (16) holds.

(ii) If $(n\alpha + h, n\beta + l)$ is any point of R, it lies on $sx + ty = q$ for some integer q. Indeed,

$$s(n\alpha + h) + t(n\beta + l) = nu + sh + tl = q.$$

Conversely, let q be any integer. Since $(u, s, t) = 1$, the linear diophantine equation

$$ux + sy + tz = q$$

has an integral solution

$$(x, y, z) = (n, h, l).$$

For this solution, we have that

$$s(n\alpha + h) + t(n\beta + l) = q.$$

Thus, all the points of R lie on all the lines $sx + ty \equiv 0 \pmod 1$.

(iii) From (15), $k_2|sak_1$. By (14), $k_2|c$. Since $(a, c) = 1$, $(k_2, a) = 1$. Therefore, $k_2|s$. Similarly, $k_1|t$. Letting

$$\sigma = s/k_2, \qquad \tau = t/k_1,$$

we obtain

$$\sigma a + \tau b = uM/k_1k_2 = u(c, d), \qquad (\sigma, \tau, u) = 1. \tag{17}$$

Suppose that $(\xi/c, \eta/d)$ is a point of R lying on $sx + ty = q$, q an arbitrary integer. The point

$$Q = \left(\frac{\xi + \lambda}{c}, \frac{\eta - \omega}{d}\right)$$

lies on $sx + ty = q$ if and only if

$$\frac{s\lambda}{c} - \frac{t\omega}{d} = 0,$$

which is equivalent to $\sigma\lambda = \tau\omega$. Thus, Q lies on $sx + ty = q$ if and only if

$$\lambda = k\tau/(\sigma, \tau), \quad \omega = k\sigma/(\sigma, \tau), \qquad k \text{ integral.}$$

Let $\Delta = a_c^{-1}\lambda + b_d^{-1}\omega$. By (i), $Q \in R$ if and only if $(c, d)|\Delta$. For suitable integers A and B, we have that

$$\Delta = \frac{b\lambda(cA + 1) + a\omega(dB + 1)}{ab}$$

$$= \frac{k}{ab(\sigma, \tau)}\,(u(c, d) + \tau Abc + \sigma Bad),$$

by (17). Thus,

$$\frac{\Delta}{(c, d)} = \frac{k}{ab(\sigma, \tau)}\left(u + \tau Ab\,\frac{c}{(c, d)} + \sigma Ba\,\frac{d}{(c, d)}\right).$$

Since $(\sigma, \tau, u) = 1$, we must have that $(\sigma, \tau)|k$, if $\Delta/(c, d)$ is to be an integer. Hence, $\lambda = l\tau$, $\omega = l\sigma$, and the points

$$\left(\frac{\xi + l\tau}{c}, \frac{\eta + l\sigma}{d}\right) = \left(\frac{\xi}{c} + l\frac{t}{M}, \frac{\eta}{d} + l\frac{s}{M}\right), \qquad l = 0, \pm 1, \pm 2, \ldots$$

and only these points lie on $sx + ty = q$ and belong to R.

(iv) Every point of R is a meeting point of the two families. For, let $(x_0, y_0) \in R$. Then (x_0, y_0) lies on $s_i x + t_i y = q_i$ for some integers q_i, $i = 1, 2$. Since the two families are distinct, $D \neq 0$, and the two straight lines meet at a single point which is (x_0, y_0).

It remains to show that there are no extraneous points if and only if $|D| = M$.

Suppose that the point (α, β) lies on $s_i x + t_i y = u_i$, $i = 1, 2$, where α, β are given by (6). Let $d_i = (s_i, t_i, u_i)$. By (15) and by (ii), all points of R lie on each of the two families

$$\frac{s_i}{d_i} x + \frac{t_i}{d_i} y \equiv 0 \pmod 1,$$

and these families contain the families $s_i x + t_i y \equiv 0 \pmod 1$. Hence, $d_i = 1$, $i = 1, 2$.

Similar to (17), we obtain for the two families

$$\sigma_i a + \tau_i b = u_i (c, d), \qquad (\sigma_i, \tau_i, u_i) = 1,$$

where $\sigma_i = s_i/k_2$, $\tau_i = t_i/k_1$, $i = 1, 2$. Thus,

$$(\sigma_1 \tau_2 - \sigma_2 \tau_1) a = (u_1 \tau_2 - u_2 \tau_1)(c, d).$$

Since $(a, (c, d)) = 1$, $\sigma_1 \tau_2 - \sigma_2 \tau_1 = k(c, d)$, k integral. Now,

$$D = s_1 t_2 - s_2 t_1 = k_1 k_2 (\sigma_1 \tau_2 - \sigma_2 \tau_1) = k_1 k_2 k (c, d) = kM \neq 0.$$

Let

$$s_1 x + t_1 y = q_1, \qquad s_2 x + t_2 y = q_2$$

be two members of the two families. Their meeting point (x, y) is given by

$$x = \frac{q_1 \tau_2 - q_2 \tau_1}{kc}, \qquad y = \frac{q_2 \sigma_1 - q_1 \sigma_2}{kd}.$$

Suppose that there are no extraneous points, i.e., $(x, y) \in R$. Then certainly,

$$k | (q_1 \tau_2 - q_2 \tau_1), \qquad k | (q_2 \sigma_1 - q_1 \sigma_2)$$

for all q_1, q_2. This implies that $k | \sigma_i$, $k | \tau_i$, $i = 1, 2$. Thus,

$$\sigma_i = \sigma_i' k, \quad \tau_i = \tau_i' k, \quad x = (q_1 \tau_2' - q_2 \tau_1')/c, \quad y = (q_2 \sigma_1' - q_1 \sigma_2')/d.$$

A simple computation shows that

$$a_c^{-1}(q_1 \tau_2' - q_2 \tau_1') - b_d^{-1}(q_2 \sigma_1' - q_1 \sigma_2') =$$
$$\frac{(q_1 u_2 - q_2 u_1)(c, d)/k + Abc(q_1 \tau_2' - q_2 \tau_1') + Bad(q_1 \sigma_2' - q_2 \sigma_1')}{ab} = \frac{N}{ab}, \quad \text{say.}$$

By (i), $(c, d) | N$, i.e.,

$$(c, d) | (q_1 u_2 - q_2 u_1) \frac{(c, d)}{k}.$$

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If $|k| = 1$ (and $|D| = M$), this condition is satisfied. If $|k| > 1$ (and $|D| \neq M$), then since the condition has to hold for all q_1, q_2, we must have that $k|u_1$ and $k|u_2$. But this is impossible, since $(\sigma_i, \tau_i, u_i) = 1$.

Now suppose that $|D| = M$. Then $|k| = 1$, and the above computation shows that $(x, y) \in R$.

LEMMA 3. *Let α and β be positive numbers, γ and δ real, and suppose that* (1) *holds.*

(i) *Suppose that*

$$\frac{\gamma}{\alpha} + \frac{\delta}{\beta} = q, \qquad q \text{ integral.} \tag{22}$$

There exists an integer m such that $m\alpha + \gamma = K$, K integral, if and only if there exists an integer n such that $n\beta + \delta = K$.

(ii) *If* (2) *holds, then* $(N - 1)\beta + \delta \leqq \phi_N$.

(iii) *Equation* (2) *and*

$$m\alpha + \gamma = K, \quad m, K \text{ integral implies } m \leqq N, \tag{23}$$

imply that $N\beta + \delta$ is non-integral.

Proof. (i) By (1), $\beta = \alpha/(\alpha - 1)$; by (22), $\delta = (q\alpha - \gamma)/(\alpha - 1)$. Hence,

$$n\beta + \delta = \frac{n\alpha + q\alpha - \gamma}{\alpha - 1} = K,$$

which implies that $(K - n - q)\alpha + \gamma = K$. The converse implication follows in the same way.

(ii) Suppose that $(N - 1)\beta + \delta > \phi_N$. By (2),

$$\frac{\gamma}{\alpha} + \frac{(N-1)\beta + \delta}{\beta} = \phi_N - N.$$

Using (1),

$$(N - 1)\beta + \delta = \left(\phi_N - N - \frac{\gamma}{\alpha}\right)\frac{\alpha}{\alpha - 1} = \frac{(\phi_N - N)\alpha - \gamma}{\alpha - 1} > \phi_N,$$

which implies that $\phi_N > N\alpha + \gamma$, a contradiction.

(iii) Suppose that $N\beta + \delta$ is integral. By (1) and (2), and since $\phi_N \leqq \psi_N$,

$$\phi_N + 1 = \frac{N\alpha + \gamma}{\alpha} + \frac{N\beta + \delta}{\beta} < \frac{N\beta + \delta + 1}{\alpha} + \frac{N\beta + \delta}{\beta} = \psi_N + \frac{1}{\alpha} < \psi_N + 1. \tag{24}$$

Thus, $\phi_N < \psi_N$. In particular, $N\alpha + \gamma < \psi_N$. This shows that (24) can be strengthened, i.e.,

$$\phi_N + 1 = \frac{N\alpha + \gamma}{\alpha} + \frac{N\beta + \delta}{\beta} < \frac{N\beta + \delta}{\alpha} + \frac{N\beta + \delta}{\beta} = \psi_N.$$

By the proof of (i),

$$(\psi_N - \phi_N + N - 1)\alpha + \gamma = \psi_N,$$

which contradicts (23).

3. Theorems on lower and upper complementary sets.

Proof of Theorem I. Suppose that conditions (1), (2), and (3) are satisfied. Let ζ be the sequence of all numbers of the form $n\alpha + \gamma, n\beta + \delta, n < N$. It suffices to show that exactly one element of ζ is in $[h, h + 1)$ for every integer $h < \phi_N$. Hence, it suffices to show that for every integer $M < \phi_N$, the number L of elements of ζ which are greater than or equal to M satisfies

$$L = \phi_N - M.$$

Let

$$\mu = \frac{M - \gamma}{\alpha}, \qquad \nu = \frac{M - \delta}{\beta}. \tag{25}$$

We consider the following two cases:

(1) ν *non-integral.* By Lemma 3(i), also μ is non-integral. Then $M \leqq n\alpha + \gamma < \phi_N$ for $n = N - 1, N - 2, \ldots, [\mu] + 1$. Similarly, $M \leqq n\beta + \delta < \phi_N$ for $n = N - 1, N - 2, \ldots, [\nu] + 1$. The latter holds since Lemma 3(ii) and (3) imply that $(N - 1)\beta + \delta < \phi_N$. Hence,

$$L = 2N - [\mu] - [\nu] - 2.$$

But

$$\mu - 1 < [\mu] < \mu, \qquad \nu - 1 < [\nu] < \nu.$$

Adding and using (25), (1) and (2), we have that

$$M - \phi_N + 2N - 3 < [\mu] + [\nu] < M - \phi_N + 2N - 1.$$

The only possibility is that $[\mu] + [\nu] = M - \phi_N + 2N - 2$, which implies that $L = \phi_N - M$, as required.

(2) $\nu = n$ *is integral.* Then $n\beta + \delta = M$. By (3), $n \geqq N$. Hence,

$$N\beta + \delta \leqq n\beta + \delta = M < \phi_N \leqq \psi_N \leqq N\beta + \delta.$$

This contradiction completes the proof of the sufficiency part of the theorem.

Now suppose that S_N' and T_N' are N-lower complementary. For $M < \min(\phi_{N-1}, \psi_{N-1}, 0)$, let $D_\alpha(M)$ be the number of elements of S_N' which are greater than or equal to M. Define m by

$$(m - 1)\alpha + \gamma < M \leqq m\alpha + \gamma.$$

Then $\mu \leqq m < \mu + 1$. Since there is no repetition in ϕ_n, $D_\alpha(M) = N - m$. Thus,

$$N - \mu - 1 < D_\alpha(M) \leqq N - \mu. \tag{26}$$

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Dividing by M, which is negative, and using (25), we have that

$$\frac{N}{M} - \frac{1-\gamma/M}{\alpha} \leqq \frac{D_\alpha(M)}{M} < \frac{N-1}{M} - \frac{1-\gamma/M}{\alpha}.$$

Hence,

$$\lim_{M\to-\infty} \frac{D_\alpha(M)}{M} = -\frac{1}{\alpha}.$$

Similarly,

$$\lim_{M\to-\infty} \frac{D_\beta(M)}{M} = -\frac{1}{\beta},$$

where $D_\beta(M)$ is the number of elements of T_N' which are greater than or equal to M. Also,

$$L = D_\alpha(M) + D_\beta(M) = \phi_N - M.$$

Dividing by M, we have that

$$\frac{D_\alpha(M)}{M} + \frac{D_\beta(M)}{M} = \frac{\phi_N}{M} - 1.$$

Letting $M \to -\infty$, this equality implies (1). Similarly to (26) one obtains

(27) $$N - \nu - 1 < D_\beta(M) \leqq N - \nu.$$

Suppose that there is equality at the right-hand side in both (26) and (27). By (25),

$$(N - D_\alpha(M))\alpha + \gamma = (N - D_\beta(M))\beta + \delta = M.$$

But $D_\alpha(M) \geqq 1$ and $D_\beta(M) \geqq 1$. Since S_N' and T_N' are N-lower complementary, the last equality is impossible, and at least in one of (26), (27) there must be strict inequality on the right. Adding (26) and (27), we obtain

$$2N - M + \frac{\gamma}{\alpha} + \frac{\delta}{\beta} - 2 < \phi_N - M < 2N - M + \frac{\gamma}{\alpha} + \frac{\delta}{\beta},$$

$$\phi_N - 2N < \frac{\gamma}{\alpha} + \frac{\delta}{\beta} < \phi_N - 2N + 2.$$

Thus, (2) must hold if we can show that (5) holds, which we shall do now.

By Lemma 1, the points

$$(n\alpha^{-1} + h, n\beta^{-1} + l), \qquad n = -1, -2, \ldots, \quad h = 0, \pm 1, \pm 2, \ldots,$$
$$l = 0, \pm 1, \pm 2, \ldots,$$

are everywhere dense on all the lines $x + y \equiv 0 \pmod 1$. Suppose that the semi-closed rectangle Q (Figure 1) defined by

(28) $$\frac{\gamma-1}{\alpha} < x \leqq \frac{\gamma}{\alpha}, \qquad \frac{\delta-1}{\beta} < y \leqq \frac{\delta}{\beta}$$

contains an interval of one of these lines.

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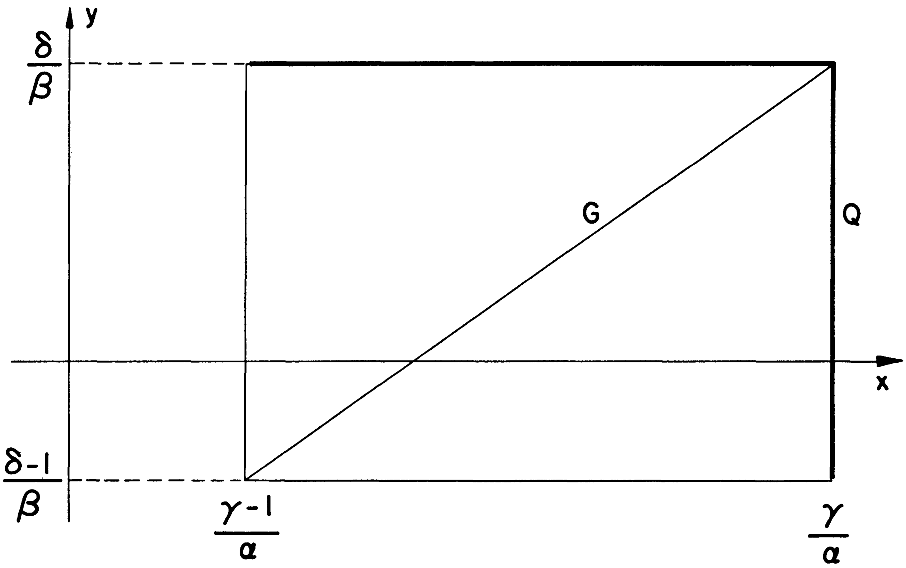

FIGURE 1

Then there exists an infinity of integral triplets (K, m, n), $K < 0$, such that

$$\frac{\gamma-1}{\alpha} < K\alpha^{-1} - m \leqq \frac{\gamma}{\alpha}, \qquad \frac{\delta-1}{\beta} < K\beta^{-1} - n \leqq \frac{\delta}{\beta}. \tag{29}$$

This is equivalent to

$$K \leqq m\alpha + \gamma < K + 1, \qquad K \leqq n\beta + \delta < K + 1,$$

which means that $S_N' \cap T_N'$ is infinite. Therefore, Q cannot contain an interval of any of the lines $x + y \equiv 0 \pmod 1$.

On the other hand, if $x + y = q$, then

$$(x \pm \alpha^{-1}) + (y \pm \beta^{-1}) = q \pm 1.$$

This shows that any straight line parallel to the diagonal G of Q is cut by the family $x + y \equiv 0 \pmod 1$ into segments of length G. Hence, Q does not contain an interval of the family $x + y \equiv 0 \pmod 1$ if and only if the point $(\gamma/\alpha, \delta/\beta)$ lies itself on one of the lines of the family; which means that $\gamma/\alpha + \delta/\beta \equiv 0 \pmod 1$.

It remains to prove (3). Suppose that $n\beta + \delta = K$, $n < N$. By Lemma 3(i), $m\alpha + \gamma = K$; hence $m \geqq N$. Thus,

$$\phi_N \leqq N\alpha + \gamma \leqq m\alpha + \gamma = n\beta + \delta \leqq (N-1)\beta + \delta < \phi_N,$$

a contradiction. (In terms of Figure 1, condition (3) means that the corner point $(\gamma/\alpha, \delta/\beta)$ is not of the form $(K\alpha^{-1} - m, K\beta^{-1} - n)$, although it lies on $x + y \equiv 0 \pmod 1$.)

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Theorem II can be proved by the same method with slight technical differences. The proof is therefore omitted.

Proof of Theorem III. The proof is similar to that of Theorem I, but sufficiently different to warrant a separate account. Suppose that (1) and (8) hold. Then $\alpha = a/c$, $\beta = a/(a-c)$, $a > c > 0$, $(a, c) = 1$. Let ζ be the sequence of all numbers of the form $n\alpha + \gamma$, $n\beta + \delta$, $n \geqq N$. It suffices to show that for every integer $M > \phi_N$, the number L of elements of ζ which are less than M satisfies

$$L = M - \phi_N.$$

(1) *μ and ν* (see (25)) *are both non-integral.* Then

$$L = [\mu] + [\nu] - 2N + 2,$$

and (see Figure 2 and the definitions of η and ρ in (7)),

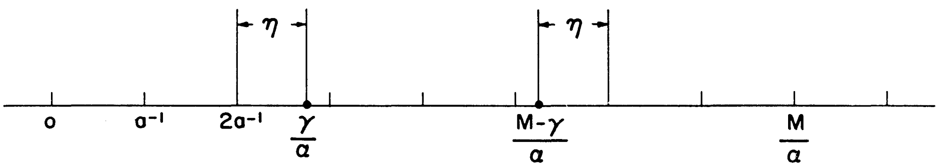

FIGURE 2

$$\mu - 1 + \eta \leqq [\mu] \leqq \mu - (a^{-1} - \eta), \qquad \nu - 1 + \rho \leqq [\nu] \leqq \nu - (a^{-1} - \rho).$$

Adding, we obtain

$$M - \phi_N + 2N - 3 + a^{-1} \leqq [\mu] + [\nu] \leqq M - \phi_N + 2N - 1 - a^{-1};$$

hence, $[\mu] + [\nu] = M - \phi_N + 2N - 2$ and therefore, $L = M - \phi_N$, as required.

(2) *μ non-integral, ν integral.* Then $\rho = 0$ and $L = [\mu] + \nu - 2N + 1$. Also,

$$M - \phi_N + 2N - 2 + a^{-1} \leqq [\mu] + \nu \leqq M - \phi_N + 2N - 1,$$

which implies that $[\mu] + \nu = M - \phi_N + 2N - 1$ and $L = M - \phi_N$. The case μ integral, ν non-integral is dealt with in the same way.

(3) *$\mu = m$, $\nu = n$ are both integral.* Then $\eta = \rho = 0$. By (8),

$$\frac{\gamma}{\alpha} + \frac{\delta}{\beta} \equiv -a^{-1} \not\equiv 0 \pmod 1.$$

By (25) and (1),

$$M - \left(\frac{\gamma}{\alpha} + \frac{\delta}{\beta}\right) = m + n,$$

which implies that $\gamma/\alpha + \delta/\beta \equiv 0 \pmod 1$. Therefore, this case is impossible.

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Now, suppose that S_N and T_N are N-upper complementary. For $M > \max(\psi_N, 0)$, let $D_\alpha(M)$ and $D_\beta(M)$ be the number of elements of S_N and T_N, respectively, which are less than M. Define m by

$$m\alpha + \gamma < M \leqq (m + 1)\alpha + \gamma.$$

Then $D_\alpha(M) = m - N + 1$, and similarly to (26) and (27),

$$(30) \quad \mu - N \leqq D_\alpha(M) < \mu - N + 1, \qquad \nu - N \leqq D_\beta(M) < \nu - N + 1,$$

where equality on the left cannot hold in both expressions. Also,

$$\lim_{M\to\infty} \frac{D_\alpha(M)}{M} = 1/\alpha, \qquad \lim_{M\to\infty} \frac{D_\beta(M)}{M} = 1/\beta,$$

and

$$L = D_\alpha(M) + D_\beta(M) = M - \phi_N$$

are obtained, leading to (1). We consider two cases:

(1) *Strict inequality in both expressions of* (30). Then

$$\mu - N + \eta \leqq D_\alpha(M) \leqq \mu - N + 1 - (a^{-1} - \eta),$$
$$\nu - N + \rho \leqq D_\beta(M) \leqq \nu - N + 1 - (a^{-1} - \rho).$$

Adding, we obtain

$$M - \left(\frac{\gamma}{\alpha} + \frac{\delta}{\beta}\right) - 2N + \eta + \rho \leqq M - \phi_N \leqq$$
$$M - \left(\frac{\gamma}{\alpha} + \frac{\delta}{\beta}\right) - 2N + \eta + \rho + 2 - 2a^{-1},$$

$$\phi_N - 2N + \eta + \rho \leqq \frac{\gamma}{\alpha} + \frac{\delta}{\beta} \leqq \phi_N - 2N + \eta + \rho + 2 - 2a^{-1}.$$

Suppose that

$$(31) \qquad \frac{\gamma}{\alpha} + \frac{\delta}{\beta} = q - a^{-1} + \eta + \rho, \qquad q \text{ integral.}$$

Then the last inequality implies that $q = \phi_N - 2N + 1$, which establishes (8);

(2) *Inequality in the first and equality in the second expression of* (30). A similar computation shows that in this case

$$\phi_N - 2N + \eta \leqq \frac{\gamma}{\alpha} + \frac{\delta}{\beta} \leqq \phi_N - 2N + \eta + 1 - a^{-1}.$$

Under assumption (31) this implies, again, (8). The case where there is equality in the first and inequality in the second expression of (30) is dealt with in the same manner.

We shall now establish the truth of (31). Consider the rectangle Q defined by (28) (Figure 3). Now $S_N \cap T_N \neq \emptyset$ (in fact, $S_N \cap T_N$ is infinite) if and only if there exist integers, K, m, n, satisfying (29). By Lemma 2(i) or (iii), a necessary condition for (29) to hold is that

$$x = K\alpha^{-1} - m = (k - \mu)a^{-1}, \qquad 0 \leqq \mu \leqq c - 1,$$
$$y = K\beta^{-1} - n = (l - \nu)a^{-1}, \qquad 0 \leqq \nu \leqq a - c - 1,$$

where k and l are as defined in (7). Adding, we obtain

$$x + y = (k + l - \mu - \nu)a^{-1}, \qquad 0 \leqq \mu + \nu \leqq a - 2.$$

Suppose that $k + l \equiv -1 \pmod{a}$. Then

$$x + y = q - (\mu + \nu + 1)a^{-1}, \qquad 1 \leqq \mu + \nu + 1 \leqq a - 1,$$

for some integer q. Since $x + y \not\equiv 0 \pmod 1$, Lemma 2(ii) shows that (x, y) is not of the form (29).

If, on the other hand, $k + l \not\equiv -1 \pmod{a}$, i.e.,

$$k + l = qa + p, \qquad 0 \leqq p \leqq a - 2,$$

then $x + y \equiv 0 \pmod 1$ for $\mu + \nu = p$. In this case, (x, y) is of the form (29). It follows that $S_N \cap T_N = \emptyset$ if and only if $k + l \equiv -1 \pmod{a}$. By (7), the latter condition is equivalent to (31).

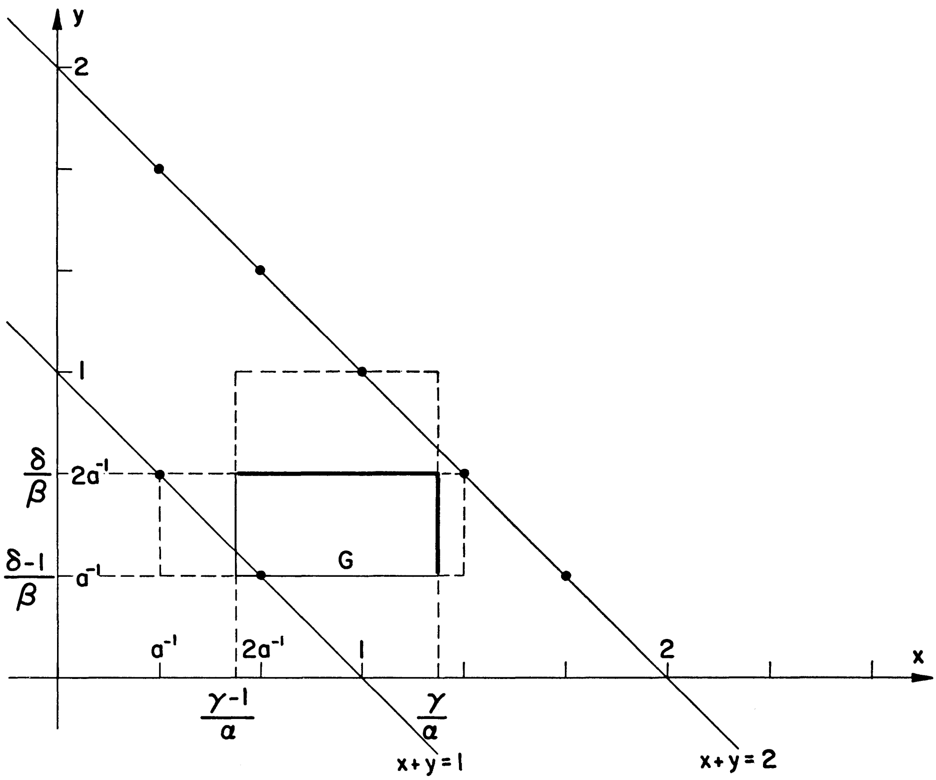

FIGURE 3 (for the case $\alpha = 3/2$, $\beta = 3$)

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By the same method we can prove the following result.

THEOREM IV. *Let* α *and* β *be positive rational numbers of the form* (6). *Then* S_N' *and* T_N' *are N-lower complementary if and only if* (1) *and* (8) *hold.*

These are the same conditions as for Theorem III.

Proof of Theorem V. Let $\alpha = a, \beta = a/(a-1), \gamma = b$. All conditions of Theorem III are satisfied for $N = 0$ if and only if δ is given by (10). Hence, the sets

$$S_0 = an + b, \quad T_0 = \left[n\frac{a}{a-1} + \delta\right], \qquad n \geqq 0,$$

are 0-upper complementary if and only if δ satisfies (10).

COROLLARY. *Let a and b be positive integers,*

$$a > 1, \quad (a, b) = 1, \quad \delta = kb + 1 + \rho a/(a-1), \quad 0 \leqq \rho \leqq a^{-1},$$

k integral. Then the complement of the set of numbers (9) *contains an infinity of primes, all of which are congruent to b* (mod *a*).

The next two theorems establish equivalence of various conditions. These equivalences permit obvious alternative formulations of Theorems I and II.

THEOREM VI. *If* $\alpha, \beta > 0$ *satisfy* (1), *the following three sets of conditions are equivalent:*

(I):

(2) $$\frac{\gamma}{\alpha} + \frac{\delta}{\beta} = \phi_N - 2N + 1,$$

(32) $$m\alpha + \gamma = K, \qquad m, K \text{ integral implies } m > N;$$

(II):

(2) $$\frac{\gamma}{\alpha} + \frac{\delta}{\beta} = \phi_N - 2N + 1,$$

(3) $$n\beta + \delta = K, \qquad n, K \text{ integral implies } n \geqq N;$$

(III):

(5) $$\frac{\gamma}{\alpha} + \frac{\delta}{\beta} \equiv 0 \pmod 1,$$

(33) $$(N-1)\beta + \delta < \phi_N < \frac{\phi_{N-1} + 1}{\alpha} + \frac{N\beta + \delta}{\beta},$$

(3) $$n\beta + \delta = K, \qquad n, K \text{ integral implies } n \geqq N.$$

Proof. (I) *implies* (II): Suppose that $n\beta + \delta = K, n < N$. By Lemma 3(i) and (32), $m\alpha + \gamma = K, m > N$. By Lemma 3(ii),

$$\phi_N \leqq N\alpha + \gamma < m\alpha + \gamma = n\beta + \delta \leqq (N-1)\beta + \delta \leqq \phi_N,$$

a contradiction.

(II) *implies* (I): Suppose that $m\alpha + \gamma = K, m \leqq N$. Then $n\beta + \delta = K, n \geqq N$. There are two possibilities:

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(1) $m = N$. Then

$$K = N\alpha + \gamma \leqq N\beta + \delta \leqq n\beta + \delta = K,$$

i.e., $N\alpha + \gamma = N\beta + \delta$. By (1) and (2),

$$\frac{N\alpha + \gamma}{\alpha} + \frac{N\beta + \delta}{\beta} = N\alpha + \gamma = \phi_N + 1,$$

a contradiction.

(2) $m < N$. Since (1) implies $\alpha > 1$, we have that

$$K = m\alpha + \gamma \leqq (N-1)\alpha + \gamma < N\beta + \delta + 1 - \alpha < N\beta + \delta \leqq n\beta + \delta = K,$$

a contradiction.

(II) *implies* (III): Lemma 3(ii) and (3) imply the left-hand side of (33). (2) implies that

$$\phi_N = \frac{(N-1)\alpha + \gamma}{\alpha} + \frac{N\beta + \delta}{\beta} < \frac{\phi_{N-1} + 1}{\alpha} + \frac{N\beta + \delta}{\beta}$$

(III) *implies* (II): Let $q = \gamma/\alpha + \delta/\beta$. Then

$$\phi_N < \frac{\phi_{N-1} + 1}{\alpha} + \frac{N\beta + \delta}{\beta} \leqq \frac{(N-1)\alpha + \gamma + 1}{\alpha} + \frac{N\beta + \delta}{\beta} = q + 2N - 1 + \frac{1}{\alpha}.$$

Thus, $q \geqq \phi_N - 2N + 1$. Also,

$$\frac{\phi_{N-1} + 1}{\alpha} + \frac{N\beta + \delta}{\beta} < \frac{\phi_N}{\alpha} + \frac{\phi_N + \beta}{\beta} = \phi_N + 1.$$

On the other hand,

$$\frac{\phi_{N-1} + 1}{\alpha} + \frac{N\beta + \delta}{\beta} > \frac{(N-1)\alpha + \gamma}{\alpha} + \frac{N\beta + \delta}{\beta} = q + 2N - 1.$$

Hence, $q \leqq \phi_N - 2N + 1$, establishing (2).

THEOREM VII. *If* $\alpha, \beta > 0$ *satisfy* (1), *the following three sets of conditions are equivalent:*

(I):

(2) $$\frac{\gamma}{\alpha} + \frac{\delta}{\beta} = \phi_N - 2N + 1,$$

(23) $m\alpha + \gamma = K$, $\quad m, K$ *integral implies* $m \leqq N$;

(II):

(2) $$\frac{\gamma}{\alpha} + \frac{\delta}{\beta} = \phi_N - 2N + 1,$$

(4) $n\beta + \delta = K$, $\quad n, K$ *integral implies* $n < N$;

(III):

(5) $$\frac{\gamma}{\alpha} + \frac{\delta}{\beta} \equiv 0 \pmod 1,$$

(34) $$\frac{N\alpha + \gamma}{\alpha} + \frac{\psi_N}{\beta} < \phi_N + 1,$$

(4) $n\beta + \delta = K$, $\quad n, K$ *integral implies* $n < N$.

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Proof. (I) *implies* (II): Suppose that $n\beta + \delta = K$. By Lemma 3(iii), $n \neq N$. If $n > N$, then

$$K = m\alpha + \gamma \leqq N\alpha + \gamma < N\beta + \delta + 1 \leqq (n-1)\beta + \delta + 1 < n\beta + \delta = K,$$

a contradiction. Hence, $n < N$.

(II) *implies* (I): Suppose that $m\alpha + \gamma = K$, $m > N$. Then

$$\phi_N \leqq N\alpha + \gamma < m\alpha + \gamma = n\beta + \delta \leqq (N-1)\beta + \delta,$$

contradicting Lemma 3(ii).

(II) *implies* (III): By (4), $N\beta + \delta$ is non-integral. Therefore,

$$\frac{N\alpha + \gamma}{\alpha} + \frac{\psi_N}{\beta} < \frac{N\alpha + \gamma}{\alpha} + \frac{N\beta + \delta}{\beta} = \phi_N + 1.$$

(III) *implies* (II): Let $q = \gamma/\alpha + \delta/\beta$. Then

$$2N + q - \frac{1}{\beta} = \frac{N\alpha + \gamma}{\alpha} + \frac{N\beta + \delta - 1}{\beta} < \frac{N\alpha + \gamma}{\alpha} + \frac{\psi_N}{\beta} < \phi_N + 1.$$

Thus, $q \leqq \phi_N - 2N + 1$. Also,

$$\phi_N = \frac{\phi_N}{\alpha} + \frac{\phi_N}{\beta} \leqq \frac{N\alpha + \gamma}{\alpha} + \frac{N\beta + \delta}{\beta} = 2N + q.$$

Equality would imply that $\phi_N = N\beta + \delta$, contradicting (4). Hence, $\phi_N < 2N + q$, i.e., $q \geqq \phi_N - 2N + 1$, establishing (2).

We shall now establish the homogeneous case.

Proof of Theorem VIII. Suppose that the four conditions are satisfied. The only solution of

$$n\beta = K, \qquad n, K \text{ integers},$$

is $n = K = 0$. Since $N \leqq 0$, (3) is satisfied and S_N' and T_N' are N-lower complementary by Theorem I.

Now, suppose that S_N' and T_N' are N-lower complementary. If α and β were rational, (8) would have to hold by Theorem IV, which is impossible for $\gamma = \delta = 0$. Hence α and β are irrational, and the other three conditions must hold by Theorem I.

Note. The condition $\phi_N = 2N - 1$ is not satisfied for $N = 0$. Hence, the condition $N \leqq 0$ could be replaced by $N < 0$. This is also what is obtained directly if conditions (I) (rather than (II)) of Theorem VI are used.

The proof of Theorem IX follows in the same way. (The last condition of Theorem IX can be replaced by $N \geqq 0$.)

4. Theorems on complementary sets and related results.

THEOREM X. *S and T are complementary if and only if there exists an integer N such that* S_N *and* T_N *are N-upper complementary and* S_N' *and* T_N' *are N-lower*

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complementary, except that if $\alpha = \beta = 2$, *there are complementary sets* S *and* T, *such that* S_N *and* T_N *are* N*-upper complementary for no* N.

Proof. From Definition 1, it follows immediately that if S_N and T_N are N-upper complementary and S_N' and T_N' are N-lower complementary for some N, then $S = S_N' \cup S_N$ and $T = T_N' \cup T_N$ are complementary.

Now, suppose that S and T are complementary. By renaming α and β if necessary, we may assume that $\alpha \leqq \beta$. We consider two cases:

(1) $\alpha < \beta$. There clearly exists an integer N such that $\phi_n > \psi_n$ for all $n < N$, and $\phi_N < \psi_N$. In particular, $\psi_{N-1} < \phi_{N-1} < \phi_N < \psi_N$. This immediately implies that S_N' and T_N' are N-lower complementary and S_N and T_N are N-upper complementary;

(2) $\alpha = \beta$. Then $\gamma < \delta$ and $\phi_n < \psi_n$ for all n. Let N be any integer. For $M > \max(\psi_N, 0)$, define $D_\alpha(M)$ and $D_\beta(M)$ as in the proof of Theorem III. Then

$$\lim_{M\to\infty} \frac{D_\alpha(M)}{M} = \frac{1}{\alpha}, \qquad \lim_{M\to\infty} \frac{D_\beta(M)}{M} = \frac{1}{\beta}$$

is obtained as before.

Let U be the set of all $\phi_n, \psi_n, n \geqq N, \phi_n < M, \psi_n < M$. Since S and T are complementary, every integer in the interval $[\psi_N, M)$ is in U. In addition, U contains the L integers ϕ_n satisfying $\phi_N \leqq \phi_n < \psi_N$. Hence,

$$D_\alpha(M) + D_\beta(M) = M - \psi_N + L,$$

which, upon division by M, leads to (1). Hence, $\alpha = \beta = 2$.

The exceptional case does indeed happen, e.g., for $\phi_n = 2n$, $\psi_n = 2n + 3$. But $\phi_n = 2n$, $\psi_n = 2n + 1$ induces the normal case of the theorem.

Proof of Theorem XI. Suppose that the conditions are satisfied. By renaming α and β if necessary, we may assume that $\alpha < \beta$. Hence, $1 < \alpha < 2$, and therefore, $f(N) = \phi_N - 2N + 1$ is a non-increasing function of N. Let N be any integer satisfying $f(N) < q$, where $q = \gamma/\alpha + \delta/\beta$. Let $k = q - f(N)$, $\delta^* = \delta - k\beta$. Then $\gamma/\alpha + \delta^*/\beta = q - k = f(N)$. Also, $\gamma - \alpha < \gamma - 1 < \phi_N - N\alpha$. Since $\alpha = \beta/(\beta - 1)$,

$$\beta\left(\frac{\gamma}{\alpha} - 1\right) < \phi_N(\beta - 1) - N\beta.$$

Thus,

$$\phi_N < \beta\left(\phi_N - N - \frac{\gamma}{\alpha} + 1\right) = \beta(\phi_N - N - q + 1) + \delta < [(N - k)\beta + \delta] + 1.$$

Hence, $\phi_N \leqq \psi_N^* = [N\beta + \delta^*]$, and the sets, S_N' and $T_N^{*\prime}$, of all integers from the sequences $\phi_n, \psi_n^* = [n\beta + \delta^*], n < N$, are N-lower complementary by Theorem I, and S_N and T_N^* are N-upper complementary by Theorem II. Hence, S and T^* are complementary by Theorem X. Since the sequence ψ_n is nothing but a translation of the sequence ψ_n^* by k places, S and T are complementary.

Now, suppose that S and T are complementary. By Theorem X, there exists an integer N such that S_N', T_N' and S_N, T_N are N-lower and N-upper complementary, respectively. Hence, the conditions of the theorem must hold by Theorems I and II.

Proof of Theorem XII. Suppose that the conditions are satisfied. By renaming α and β if necessary, we assume that $\alpha \leqq \beta$.

(1) $\alpha < \beta$. Letting

$$q = \frac{\gamma}{\alpha} + \frac{\delta}{\beta} + a^{-1} - \eta - \rho$$

and using (since $\alpha = a/c \geqq a/(a-1)$)

$$\gamma - \alpha(1 - a^{-1} + \eta + \rho) \leqq \gamma - 1 < \phi_N - N\alpha,$$

the condition $\phi_N \leqq \psi_N^*$ is obtained as in the proof of Theorem XI.

(2) $\alpha = \beta$. Actually, $\alpha = \beta = 2$. Also, $\gamma \leqq \delta$. Now

$$f(N) = \phi_N - 2N + 1 = [2N + \gamma] - 2N + 1.$$

If $f(N) < q$ for some N, the above argument is valid. If $f(N) \geqq q$ for all N, let $l = f(N) - q$, $\delta^* = \delta + l\beta$. Then

$$\frac{\gamma}{\alpha} + \frac{\delta^*}{\beta} = q - a^{-1} + \eta + \rho + l = f(N) - a^{-1} + \eta + \rho,$$

and certainly, $\phi_N \leqq \psi_N^*$.

Thus, in any case, $\phi_N \leqq \psi_N^*$. Hence, S_N', $T_N^{*\prime}$ and S_N, T_N^* are N-lower and N-upper complementary by Theorems III and IV, respectively. Hence, S, T^* and, hence S, T are complementary.

Now, suppose that S and T are complementary. If $\alpha < \beta$, the result follows from Theorem X. If $\alpha = \beta$, (1) is proved as in the proof of Theorem X, and (11) follows by the same arguments used in proving (31).

It is natural to introduce the following concept.

Definition 2. We say that *S and T* are *almost complementary* (*S_N'* and *T_N'* are *almost N-lower complementary*) [S_N and T_N are *almost N-upper complementary*] if

(i) no integer appears more than once in ϕ_n and no integer appears more than once in ψ_n, $n = 0, \pm 1, \pm 2, \ldots$ $(n < N)$ $[n \geqq N]$;

(ii) $S \cap T = \Phi$ $(S_N' \cap T_N' = \Phi)$ $[S_N \cap T_N = \Phi]$, where Φ denotes a set containing at most one element;

(iii) $S \cup T = \bar{Z}$ $(S_N' \cup T_N' = \bar{Z}_N')$ $[S_N \cup T_N = \bar{Z}_N]$, where $\bar{Z}$ $(\bar{Z}_N')$ $[\bar{Z}_N]$, denotes the set of all integers with the possible exception of a single integer (all integers less than ϕ_N and a finite number of integers greater than or equal to ϕ_N) [all integers greater than or equal to ϕ_N except for a finite number of these integers].

THEOREM XIII. *Let α and β be positive irrational* (*rational*) *integers, and let N be any integer. Each of the following three statements is equivalent to* (1), (5) ((1), (11))*:*

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(i) *S_N' and T_N' are almost N-lower complementary;*
(ii) *S_N and T_N are almost N-upper complementary;*
(iii) *S and T are almost complementary (complementary).*

Proof. Suppose that the conditions are satisfied. There may exist integers, n and K, satisfying $n\beta + \delta = K$. If so, Lemma 3(i) implies that $m\alpha + \gamma = K$ for some integer m. Thus, K is repeated, and clearly, $K - 1 \notin S \cup T$. If (2) is satisfied, the first part of the proof of Theorem I shows that apart from these two irregularities, S and T are complementary. (By (11), $m\alpha + \gamma = n\beta + \delta = K$ cannot occur if α and β are rational.) If (2) is violated, a "shift" of the sequence ψ_n will make (2) hold, irrespective of whether α and β are rational or irrational. Thus, S_N and T_N are almost N-upper complementary.

The arguments used in establishing (5) and (11) in the proofs of Theorems I and III, respectively, show that the conditions are also necessary.

In addition to complementary sets, *disjoint* sets were also considered in (**11**; **12**). In (**12**, *Satz* 6), it is stated that S_1 and T_1 are 1-upper *disjoint* if and only if there exist positive integers, a and b, such that

$$\frac{a}{\alpha} + \frac{b}{\beta} = 1, \qquad a\frac{\gamma}{\alpha} + b\frac{\delta}{\beta} \equiv 0 \pmod 1. \tag{35}$$

That these conditions are not sufficient can be seen from Example 2, where conditions (35) are satisfied with $a = b = 1$, but $S_1 \cap T_1 = 2$. They become sufficient if a condition of the form (4) is adjoined. However, with or without (4), the conditions are not necessary. For $\alpha = \beta = \sqrt{5}$, $\gamma = 0$, $\delta = 1$, we clearly have that

$$[n\alpha + \gamma] < [n\beta + \delta] < [(n + 1)\alpha + \gamma],$$

which shows that $S \cap T = \emptyset$.

For the homogeneous case, Skolem (**11**) proved that $[n\alpha]$ and $[n\beta]$ are disjoint if and only if there exist positive integers, a and b, such that $a\alpha^{-1} + b\beta^{-1} = 1$. From this he easily deduced that there do not exist three irrationals, α, β, γ, such that $[n\alpha]$, $[n\beta]$, and $[n\gamma]$ are mutually disjoint. If (**12**, *Satz* 6) were correct, the same conclusion would apply to the non-homogeneous case. Actually, however, more than two mutually disjoint sets do exist in the latter case. For example, let $\alpha_i = \sqrt{10}$, $i = 1, 2, 3$, $\gamma_1 = 0$, $\gamma_2 = 1$, $\gamma_3 = 2$. Then the sets $[n\alpha_i + \gamma_i]$, $i = 1, 2, 3$, are mutually disjoint, since, clearly,

$$[n\alpha_1 + \gamma_1] < [n\alpha_2 + \gamma_2] < [n\alpha_3 + \gamma_3] < [(n + 1)\alpha_1 + \gamma_1].$$

For the rational case, we trivially have three complementary sets given by $3n$, $3n + 1$, $3n + 2$.

We remark finally that results on complementary sets of integers suggest natural generalizations of Wythoff's game. These may be pursued elsewhere in the future.

References

1. W. Ahrens, *Mathematische Unterhaltungen und Spiele*, zweite vermehrte und verbesserte Auflage, Bd. I (Teubner, Leipzig, 1910).
2. W. W. R. Ball, *Mathematical recreations and essays*, revised by H. S. M. Coxeter, p. 39 (Macmillan, New York, 1947).
3. Th. Bang, *On the sequence* $[n\alpha]$, $n = 1, 2, \ldots$, Math. Scand. *5* (1957), 69–76.
4. S. Beatty, Problem 3173, Amer. Math. Monthly *33* (1926), 159; *34* (1927), 159.
5. H. S. M. Coxeter, *The golden section, Phyllotaxis and Wythoff's game*, Scripta Math. *19* (1953), 135–143.
6. A. P. Domoryad, *Mathematical games and pastimes*, translated by H. Moss (Pergamon Press, Oxford, 1964).
7. Ky Fan, *The Dunkel memorial problem book*, Problem 4399, p. 57.
8. D. J. Newman, Problem 5252, Amer. Math. Monthly *71* (1964), 1138; *72* (1965), 1144.
9. I. Niven, *Diophantine approximations* (Interscience, New York, 1963).
10. T. H. O'Beirne, *Puzzles and paradoxes* (Oxford Univ. Press, London, 1965).
11. Th. Skolem, *On certain distributions of integers in pairs with given differences*, Math. Scand. *5* (1957), 57–68.
12. ——— *Über einige Eigenschaften der Zahlenmengen* $[\alpha n + \beta]$ *bei irrationalem* α *mit einleitenden Bemerkungen über einige kombinatorische Probleme*, Norske Vid. Selsk. Forh. (Trondheim) *30* (1957), 118–125.
13. B. M. Stewart, *Theory of numbers*, 2nd ed. (Macmillan, New York, 1964).
14. J. V. Uspensky and M. A. Heaslet, *Elementary number theory*, p. 98 (McGraw-Hill, New York, 1939).
15. W. A. Wythoff, *A modification of the game of Nim*, Nieuw Arch. Wisk. *7* (1907), 199–202.
16. A. M. Yaglom and I. M. Yaglom, *Challenging mathematical problems with elementary solutions*, translated by J. McCawley, Jr., revised and edited by B. Gordon, Vol. II (Holden-Day, San Francisco, 1967).

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