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APPROXIMATE DUAL AND APPROXIMATE VECTOR VARIATIONAL INEQUALITY FOR MULTIOBJECTIVE OPTIMIZATION

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Abstract

An approximate dual is proposed for a multiobjective optimization problem. The approximate dual has a finite feasible set, and is constructed without using a perturbation. An approximate weak duality theorem and an approximate strong duality theorem are obtained, and also an approximate variational inequality condition for efficient multiobjective solutions.

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I. Introduction

Consider the finite dimensional linear space $\mathbf{R}^p$, equipped with a partial ordering $\geq_S$, defined by a closed pointed convex cone S, with interior $\operatorname{int} S \neq \varnothing$.

DEFINITION 1 (see [3]). For a given $e \in S$, a point $c \in A \subset \mathbf{R}^p$ is said to be an e-[*weak*] *minimum* of A if there exists no $x \in A$ satisfying $0 \neq c-x-e \in S$ [$\in \operatorname{int} S$]. A point $c \in A$ is said to be an e-[*weak*] *maximum* if there exists no $x \in A$ such that $0 \neq x-c-d \in S$ [$\in \operatorname{int} S$]. When $e = 0$, an e-[weak] minimum (e-[weak] maximum) is said to be a [*weak*] *minimum* ([*weak*] *maximum*). A set W is said to *e-upper dominate* a set V if

$$(\forall v \in V)(\exists w \in W)\ w + e - v \in S.$$

A set W is said to *e-strongly upper dominate* a set V if

$$(\forall v \in V)(\exists w \in W)\ w + e - v \in \operatorname{int} S.$$

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The *order-interval* $[-e,e] := \{x \in \mathbf{R}^p : e \geq_S x \geq_S -e\}$. Denote its interior by $(-e,e)$. Denote by $\mathbf{R}^{p\times m}$ the space of all $p \times m$ real matrices Λ, and by $\|\Lambda\|$ the norm of Λ in this space. Let $D \subset \mathbf{R}^m$ be a closed convex cone. Let $M := \{\Lambda \in \mathbf{R}^{p\times m} : \|\Lambda\| = 1, \Lambda(D) \subset S\}$. Then M is compact.

A function $h: C \to \mathbf{R}^p$ is *S-convex* if $C \subset \mathbf{R}^n$ is convex, and

$$(\forall x, y \in C,\ \forall \alpha \in (0,1)) \qquad h(\alpha x + (1-\alpha)y) \leq_S \alpha h(x) + (1-\alpha)h(y).$$

Consider a nonlinear multiobjective optimization problem:

$$\text{(P)} \qquad \operatorname{WMin} f(x) \text{ subject to } x \in X := \{x \in E, -g(x) \in D\},$$

where $f: \mathbf{R}^n \to \mathbf{R}^p$, $g: \mathbf{R}^n \to \mathbf{R}^m$ are vector functions, $E \subset \mathbf{R}^n$, and weak minimum (WMin; and WMax later) are as in Definition 1.

Definition 2. A vector valued *Lagrangian* for (P) is $L(x,\Lambda) := f(x) + \Lambda g(x)$.

Following Sawaragi, Nakayama and Tanino [1], it will be assumed that

(i) E is nonempty compact,

(ii) f is continuous, and

(iii) g is continuous.

Under these assumptions, it is readily shown that X and $f(X)$ are compact.

Definition 3. The *dual map* $\Phi: M \to \mathbf{R}^p$ is defined by

$$\Phi(\Lambda) := \operatorname{WMin} L(E,\Lambda),$$

for each $\Lambda \in M$. A dual problem [1] to (P) is

$$(\mathrm{D_{TS}}) \qquad \operatorname{WMax} \bigcup_{\Lambda \in M} \Phi(\Lambda).$$

Lemma 1. *Under the above assumptions, for each $\Lambda \in M$, the sets $L(E,\Lambda)$ and $\Phi(\Lambda)$ are compact.*

Proof. Since E is compact, and f and g are continuous, $L(E,\Lambda)$ is compact, for each $\Lambda \in M$. Hence $\Phi(\Lambda)$ is nonempty and bounded. Consider a sequence $\{y_k\} \subset \Phi(\Lambda)$, $\{y_k\} \to y_0$. If $y_0 \notin \Phi(\Lambda)$ then there exists $v \in L(E,\Lambda)$ such that $y_0 - v \in \operatorname{int} S$. Since $\{y_k\} \to y_0$, there exists $k_0 \in \mathbf{N}$ such that $y_k - v \in \operatorname{int} S$ when $k \geq k_0$, contradicting the weak minimum. So $\Phi(\Lambda)$ is closed, and therefore compact.

Lemma 2. *The map $\Phi: M \to \mathbf{R}^p$ is upper semicontinuous, and $\Phi(M)$ is compact.*

Proof. Let $\{\Lambda_k\} \to \Lambda_0$ in M, $\{v_k\} \to v_0$, $(\forall k) v_k \in L(E,\Lambda_k)$. For each k, there exists $x_k \in E$ such that $v_k = L(x_k,\Lambda_k)$. Since E is compact, there is a subsequence $\{x_{k_j}\} \subset \{x_k\}$ such that $\{x_{k_j}\} \to x_0 \in E$. Since f and g are continuous, $\{v_{k_j}\} \to L(x_0, \Lambda_0) = v_0 \in L(E, \Lambda_0)$. Thus the mapping $L(E, \cdot)$ is upper semicontinuous on M. To show also lower semicontinuity, let $\{\Lambda_k\} \to \Lambda_0$ in M, and let $v_0 \in L(E, \Lambda_0)$; thus $v_0 = L(x_0, \Lambda_0)$ for some $x_0 \in E$; then $\{v_k\} := \{L(x_0, \Lambda_k)\} \to v_0$, and $(\forall k)\ v_k \in L(E, \Lambda_k)$.

Let $\{\Lambda_k\} \to \Lambda_0$, $\varphi_k \in \Phi(\Lambda_k)$ for each k, $\{\varphi_k\} \to \varphi_0$. If $\varphi_0 \notin \Phi(\Lambda_0)$ then $\varphi_0 - v_0 \in \operatorname{int} S$ for some $v_0 \in L(E, \Lambda_0)$. Since $\{\varphi_k\} \to \varphi_0$, there exists a $k_0 > 0$ such that for each $k \geq k_0$ $\varphi_k - v_0 \in \operatorname{int} S$. Since $L(E, \cdot)$ is continuous at Λ_0, for each k there exists $v_k \in L(E_k)$ such that $\{v_k\} \to v_0$, thus $\varphi_k - v_k \in \operatorname{int} S$, and thus $\varphi_k \notin \operatorname{WMin} L(E, \Lambda_k)$. The contradiction shows that Φ is upper semicontinuous. Since M is compact, $\Phi(M)$ is compact [6].

REMARK. The conclusions of Lemma 1 and Lemma 2 still hold if $\Phi(\Lambda)$ is changed to $e - \operatorname{WMin} L(E, \Lambda)$.

2. Approximate dual

We now introduce an approximate dual for the multiobjective problem (P). No perturbation map is required.

LEMMA 3. *Let $A \subset \mathbf{R}^p$ be compact, and $e \in \operatorname{int} S$. Then there exists a finite subset $K \subset A$ such that K e-strongly upper dominates A.*

PROOF. The family of open order intervals $\{(a-e, a+e) : a \in A\}$ covers A, so there exists a finite subfamily $\{(a_i - e, a_i + e) : i = 1, 2, \ldots, k\}$ which covers A. Let $K = \{a_1, a_2, \ldots, a_k\}$. If $a \in A$, then $a \in (a_i - e, a_i + e)$ for some i, and $a_i + e - a \in \operatorname{int} S$; thus K ε-strongly upper dominates A.

REMARK. Similar results hold with upper replaced by lower.

Since f is continuous and X is compact, $f(X)$ is compact. By Lemma 3, there exists a finite subset $U \subset f(X)$, such that U e-strongly lower dominates $f(X)$, and thus $(\forall v \in f(X))\ (\exists u \in U)\ u - e - v \in -\operatorname{int} S$. So a *primal approximate problem* may be defined as

$$(\mathrm{P}^\#) \qquad \operatorname{WMin} U.$$

For each $\Lambda \in M$, define $\Phi^\#(\Lambda)$ to be the set of e-weak minima of $L(E, \Lambda)$. Then $\Phi^\#(\Lambda) \neq \emptyset$. Set $Q := \bigcup_{\Lambda \in M} \Phi^\#(\Lambda)$. By Lemma 2, the set Q is compact. By Lemma 3, there exists a finite subset $W \subset Q$ such that $(\forall q \in Q)\ (\exists w \in W)$ $w - e - q \in -S$. So an *approximate dual problem* may be defined as

$$(\mathrm{D}^\#) \qquad \operatorname{WMax} W.$$

Two obvious corollaries follow.

COROLLARY 1. *If u_0 is a weak minimum of* ($\mathrm{P}^{\#}$), *then u_0 is an e-weak minimum of* (P).

PROOF. If not, then there exists $v \in f(X)$ such that $u_0 - v - e \in \operatorname{int} S$. But, given $v \in f(X)$, there exists $u' \in U$ with $u' - v - e \in -\operatorname{int} S$. Combining these, $u_0 - u' \in \operatorname{int} S$, contradicting the weak minimum.

COROLLARY 2. *If w_0 is a weak maximum of* ($\mathrm{D}^{\#}$) *then w_0 is an e-weak maximum of* ($\mathrm{D_{TS}}$).

PROOF. If not, then there exists $v \in \Phi^{\#}(M)$ such that $v - (w_0 + e) = \operatorname{int} S$. Given this v, there exists $w' \in W$ such that $w' + e - v \in \operatorname{int} S$. Combining these, we obtain $w' - w_0 \in \operatorname{int} S$, contradicting the weak maximum.

THEOREM 1 (APPROXIMATE WEAK DUALITY). *For each $u \in U$ and each $w \in W$, $w - u - e \notin S \backslash \{0\}$.*

PROOF. If $w \in W$, then $w \in \Phi^{\#}(\Lambda)$ for some $\Lambda \in \mathbf{M}$. Then
$$(\forall x \in X)\ w - e - [f(x) + \Lambda g(x)] \in H := \mathbf{R}^p \backslash \operatorname{int} S.$$

THEOREM 2 (APPROXIMATE STRONG DUALITY). *Let $x^* \in X$ and $\Lambda^* \in \mathbf{M}$ satisfying $w^* := f(x^*) \in \Phi^{\#}(\Lambda^*) \cap U \cap W$. Then x^* is an e-weak minimum of $f(X)$, and w^* is an e-weak maximum of $\Phi(M)$.*

PROOF. If x^* is not an e-weak minimum of $f(X)$, then there exists $x \in X$ such that $f(x^*) - e - f(x) \in \operatorname{int} S$. Since $f(x^*) \in W$ and $\Lambda^* g(x^*) \in -S$, it follows that $f(x^*) - e - [f(x) + \Lambda^* g(x)] \in \operatorname{int} S$, contradicting $f(x^*) \in \Phi^{\#}(\Lambda^*)$.

If $w^* = f(x^*)$ is not an e-weak maximum of $\Phi(M)$, there is $w \in \bigcup_{\Lambda \in \mathbf{M}} \Phi(\Lambda)$ such that $w - f(x^*) - e \in \operatorname{int} S$. Now $w = \Phi(\Lambda)$ for some $\Lambda \in \mathbf{M}$. Since $\Lambda g(x^*) \in -S$, $w - [f(x^*) + \Lambda g(x^*) + e] \in \operatorname{int} S$, and hence w is not an e-weak maximum of $L(E, \Lambda)$.

3. Approximate vector variational inequality

Let $C \in \mathbf{R}^n$ be a nonempty convex set, $S \subset \mathbf{R}^p$ a closed convex cone with $\operatorname{int} S \neq \emptyset$, and $f: C \to \mathbf{R}^p$ a vector valued function. Denote by $\mathbf{L}(\mathbf{R}^n, \mathbf{R}^p)$ the space of all linear mappings from $\mathbf{R}^n$ into $\mathbf{R}^n$. Let $G: C \to \mathbf{L}(\mathbf{R}^n, \mathbf{R}^p)$ be a mapping. A *generalized vector variational inequality* is the problem of finding $x_0 \in C$ and $A \in G(x_0)$ such that
$$(\forall x \in C)\ A(x - x_0) \notin -\operatorname{int} S.$$

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Theorem 3. *Let $C \subset \mathbf{R}^n$ be compact convex; let $S \subset \mathbf{R}^p$ be a closed convex pointed cone with* $\operatorname{int} S \neq \emptyset$; *let $f: C \to \mathbf{R}^p$ be S-convex, (Fréchet) continuously differentiable at $x_0 \in C$, and differentiable on C; let x_0 be a weak minimum of $f(x)$ subject to $x \in C$. Then, for each $e \in \operatorname{int} S$, there exists a finite subset $W \subset C$ such that* $(\forall w \in W)\ f'(x_0)(w - x_0) + e \notin -\operatorname{int} S$.

Proof. Since C is a compact convex, and $f'(x_0)$ is a continuous linear mapping, $K := f'(x_0)(C - x_0)$ is compact convex. By Lemma 3, for each $e \in \operatorname{int} S$, there exists a finite subset $W' \subset C$ such that, for each $x \in C$, there is $w_x \in W'$ such that $[f'(x_0)(w_x - x_0) + e] - f'(x_0)(x - x_0) \in S$. Denote by W the finite subset $\{w_x \in W' : x \in C\}$. From [2, Theorem 4], since f is S-convex, $(\forall x \in C)\ f'(x_0)(x - x_0) \notin -\operatorname{int} S$. Let $Q := \mathbf{R}^p \backslash (-\operatorname{int} S)$. Then $f'(x_0)(w_x - x) + e \in S + Q \subset Q$. Hence $(\forall w \in W)\ f'(x_0)(w - x_0) + e \notin -\operatorname{int} S$.

Since $f'(x)(C - x)$ is compact convex, for each $x \in C$, Lemma 3 shows that, for each $x \in C$ and each $e \in \operatorname{int} S$, there exists a finite subset $M(x) \subset f'(x)(C - x)$ such that $(\forall m \in f'(x)(C - x))\ (\exists m' \in M(x))\ m - [m' - e] \in S$. For each $x \in C$ and each $m' \in M(x)$, let $u \equiv u(x, m')$ be an element of C such that $f'(x)(u - x) \in M(x)$; denote by $U(x)$ the finite set $\{u(x, m') : m' \in M(x)\}$.

Consider now the generalized inequality system, for $x \in C$:

$$\text{(GI)} \qquad (\forall u \in U(x)) f'(x)(u - x) - e \notin -\operatorname{int} S.$$

Theorem 4. *Let $f: C \to \mathbf{R}^p$ be S-convex and continuously differentiable on C, where $C \subset \mathbf{R}^n$ is a compact convex set; let $e \in \operatorname{int} S$. If x_0 is a solution* (GI), *then x_0 is a weak minimum of $f(x)$ subject to $x \in C$.*

Proof. Since f is S-convex and differentiable, $(\forall x \in C)\ f(x) - f(x_0) - f'(x_0)(x - x_0) \in S$. Since $f'(x)(C - x)$ is compact, Lemma 3 shows that, for each $x \in C$, there exists $u \in U(x_0)$ such that

$$f'(x_0)(x - x_0) - [f'(x_0)(u - x_0) - e] \in S.$$

Since x_0 is a solution of (GI), $f'(x_0)(u - x_0) \in Q$, where $Q := \mathbf{R}^p \backslash (-\operatorname{int} S)$. Adding these inclusions, $f(x) - f(x_0) \in S + S + Q \subset Q$. Thus x_0 is a weak minimum of $f(x)$ subject to $x \in C$.

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