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SEMISIMPLE RADICAL CLASSES OF INVOLUTION ALGEBRAS

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J. Wichman has asked about semisimple radical classes of involution algebras. In the present paper we describe the semisimple radical classes of involution algebras over a field K^* with involution *. If K^* is infinite, then there are only trivial semisimple radical classes. If K^* is finite then these classes are subdirect closures of strongly hereditary finite sets of finite idempotent algebras. In proving this result we determine the structure of certain simple involution algebras. We prove that the variety of symmetric involution algebras over $Z^{(2)}$ does not have attainable identities, answering a problem posed by Gardner [2]. Most of the results are valid also for involution rings (over the integers).

1. Preliminaries

Let K be a field with involution *. K-algebra A is an *involution algebra* if in A there is defined a unary operation * such that $x^{**}=x$, $(x+y)^*=x^*+y^*$, $(xy)^*=y^*x^*$ and $(kx)^*=k^*x^*$ for all $x,y\in A$ and $k\in K$. Without the fear of ambiguity we shall denote by * both the involutions defined in K and A. We shall write K^* for the field K whenever we wish to emphasize that K is with involution *. In particular K^{id} will mean that the involution on K is the identical one. An involution subalgebra I of A is called an *ideal* of A if it is a ring-ideal of A^*. This fact will be indicated by $I\lhd^* A$.

A class $\mathbb{C}$ of involution algebras is called *extension-closed* if $I\lhd^* A$, $I\in\mathbb{C}$ and $A/I\in\mathbb{C}$ and $A/I\in\mathbb{C}$ implies $A\in\mathbb{C}$. As in [1, Theorem 1.5] we can show that if $\mathbb{C}$ is a variety of involution algebras, which is extension-closed, then $\mathbb{C}$ is *inductive* (that is, if an involution algebra A contains an ascending chain of ideals I_α such that $\cup I_\alpha=A$ and $I_\alpha\in\mathbb{C}$ for each α, then $A\in\mathbb{C}$). Thus the variety of $\mathbb{C}$ of involution algebras is a radical class (in the sense of Kurósh and Amitsur) if it is extension-closed.

Let $\mathbb{C}$ be any class of involution algebras. For each involution algebra A, let us define

$$A(\mathbb{C})=\cap\{I \,|\, I\lhd^* A \quad\text{and}\quad A/I\in\mathbb{C}\}.$$

Then $\mathbb{C}$ is said to have *attainable identities* if $A(\mathbb{C})(\mathbb{C})=A(\mathbb{C})$. Let us notice again that if $\mathbb{C}$ is a variety of involution algebra, and $\mathbb{C}$ has attainable identities, then $\mathbb{C}$ is extension-closed (see [1, Theorem 1.5]). In this case $\mathbb{C}$ is a semisimple class. Thus if $\mathbb{C}$ is a semisimple variety, then $\mathbb{C}$ is a semisimple radical class.

Let us recall that a class $\mathbb{C}$ of involution algebras is *hypernilpotent* (*hypoidempotent*) if $\mathbb{C}$ contains all nilpotent involution algebras (if $\mathbb{C}$ consists only of idempotent involution algebras, respectively).

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For further details of the basic fact of involution algebras and of radical theory we refer to [4] and to [11]. Radicals of involution albegras have been studied in the recent papers [2, 6, 7, 8 and 10]. The semisimple radical classes of algebras were investigated in several papers. The full list of references can be found in [3]. In this paper we work only with involution algebras over a field K^*, though many of the results can be extended to rings with involution as well as to some other varieties. These cases will be treated later.

2. Varieties and attainable identities

Proposition 1. *A free involution algebra is a subdirect sum of nilpotent involution algebras.*

Proof. Let A be any free involution algebra, then it is well known that $\bigcap_{n\in N} A^n = 0$. Thus A is a subdirect sum of A/A^n, which are clearly nilpotent involution algebras.

We will prove the analogous result of [11, Lemma 30.2] for involution algebras. Let us notice that this short and simple proof can be applied to the case of associative rings.

Proposition 2. *If $\mathbb{C}$ is a hypernilpotent class of involution algebra closed under homomorphisms and subdirect sums, then $\mathbb{C}$ is a class of all involution algebras.*

Proof. Since $\mathbb{C}$ is closed under subdirect sums, by Proposition 1 every free involution algebra is contained in $\mathbb{C}$. Since any involution algebra is a homomorphic image of a free involution algebra, the assertion is proved by the fact that $\mathbb{C}$ is closed under homomorphisms.

Now let P be any subset of an involution algebra A. As in [4] we shall make use of the following symbols:

$$S(P) = \{x \in P \mid x^* = x\},$$

$$K(P) = \{x \in P \mid x^* = -x\},$$

$$Z(P) = \{x \in P \mid xy = yx \text{ for all } y \in P\},$$

and

$$P^{(*)} = \{x \mid x = p^* \text{ for any } p \in P\}.$$

Let A_0 denote the additive group of an involution algebra A. Then A_0 can be considered as an algebra over K with zero-multiplication. Let us notice that in the case $K^* = K^{\mathrm{id}}$ the operators $\mathrm{id}(x^{\mathrm{id}} = x)$ and $-\mathrm{id}(x^{-\mathrm{id}} = -x)$ define involutions on the algebra A_0. These involution algebras will be denoted by A_0^{id} and $A_0^{-\mathrm{id}}$, respectively. The *characteristic* of a field K is denoted by $\operatorname{char} K$. In the following theorem we characterize certain varieties of involution algebras.

Theorem 3. *Let $\mathbb{V}_S$, $\mathbb{V}_K$ be the varieties of involution algebras defined by the identities $x^*=x$, $x^*=-x$ respectively. If $\mathbb{V}_S\neq 0$ and $\mathbb{V}_K\neq 0$, then they have not attainable identities. Moreover,*

(i) *$\mathbb{V}_S$, $\mathbb{V}_K$ are inductive,*
(ii) *if $K^*\neq K^{\mathrm{id}}$ then $\mathbb{V}_S=\mathbb{V}_K=\{0\}$,*
(iii) *if $\operatorname{char} K=2$ and $K^*=K^{\mathrm{id}}$ then $\mathbb{V}_S$ and $\mathbb{V}_K$ are not closed under extensions, and hence they are not radical classes,*
(iv) *if $\operatorname{char} K\neq 2$ and $K^*=K^{\mathrm{id}}$ then $\mathbb{V}_S$ and $\mathbb{V}_K$ are radical classes,*
(v) *if $\mathbb{W}$ is an extension-closed variety, which contains a non-zero nilpotent involution algebra in $\mathbb{V}_S$ (or in $\mathbb{V}_K$), then $\mathbb{V}_S\subseteq\mathbb{W}$ (or $\mathbb{V}_K\subseteq\mathbb{W}$).*

Proof. (i) is clear.

(ii) Suppose $0\neq a\in A\in\mathbb{V}_S$. Then for all $k\in K^*$ we have

$$k^*a=k^*a^*=(ka)^*=ka,$$

so $(k^*-k)a=0$ and therefore $k^*=k$. Thus $K^*=K^{\mathrm{id}}$ if $\mathbb{V}_S\neq\{0\}$. The proof for $\mathbb{V}_K$ is essentially the same.

(iii) Now, as is well known, $\mathbb{V}_S=\mathbb{V}_K\neq 0$. Using [**6**, Theorem 2], [**7**, Theorem 10] and [**8**, Theorem 1], it is straightforward to see that in this case any radical class is either hypernilpotent or hypoidempotent radical variety. Hence by Proposition 2 $\mathbb{V}_S$ contains all involution algebras, a contradiction.

(iv) Similar to [**2**, Theorem 1].

(v) As in [**1**, Theorem 1.4] $\mathbb{W}$ is a radical variety and by (ii), $K^*=K^{\mathrm{id}}$. Analogous to Proposition 1 we can show that any free involution algebra of $\mathbb{V}_S$ (or $\mathbb{V}_K$) is a subdirect sum of nilpotent involution algebras of $\mathbb{V}_S$ (or $\mathbb{V}_K$). It is easy to see that $K_0^{\mathrm{id}}\in\mathbb{W}$ (or $K_0^{-\mathrm{id}}\in\mathbb{W}$). Hence $\mathbb{W}$ contains all free involution algebras of $\mathbb{V}_S$ (or of $\mathbb{V}_K$), and consequently $\mathbb{V}_S\subseteq\mathbb{W}$ (or $\mathbb{V}_K\subseteq\mathbb{W}$).

Now we return to show that if $\mathbb{V}_S\neq 0$ and $\mathbb{V}_K\neq 0$, then $\mathbb{V}_S$ and $\mathbb{V}_K$ are not semisimple classes, and hence they have not attainable identities. Let us notice here that in this case $K^*=K^{\mathrm{id}}$. In the case $\operatorname{char} K=2$ the assertion is obvious by (ii). In the other case let F be the free involution algebra generated by a single element x. Denote by I and J the ideals of F generated by $\{(x-x^*)^2,(x-x^*)F(x-x^*)\}$ and by $(x-x^*)$, respectively. Let $H=F/I$, $P=J/I$. We have that $P\lhd^* H$ and $P^2=0$. Moreover, it is easy to verify by elementary verification that $K(P)\lhd^* P$ but $K(P)$ is not ideal of H, and H/P, $P/K(P)\in\mathbb{V}_S$. Thus for H we get

$$H(\mathbb{V}_S)(\mathbb{V}_S)=K(P)\neq P=H(\mathbb{V}_S),$$

that is $\mathbb{V}_S$ has not attainable identities. An analogous reasoning proves the assertion for $\mathbb{V}_K$.

Remark. The result of Theorem 3 is valid if K is the ring $Z^{(2)}=\{m/2^n \mid m, n \in Z\}$, that is

In the variety of all involution $Z^{(2)}$-algebra $\mathbb{V}_S$ has not attainable identities.

The proof follows from the above constructions. This is the answer for the question of Gardner in [2].

3. Algebras over an infinite field

Proposition 4. *If $\mathbb{C}$ is a semisimple radical class of involution algebras, then either $\mathbb{C}$ is the class of all involution algebras or $\mathbb{C}$ is a hypoidempotent radical.*

Proof. It is clear that in the case char $K=2$ and $K^*=K^{\mathrm{id}}$ (by [6, Theorem 2] and [7, Theorem 10]) and in the case $K^*\neq K^{\mathrm{id}}$ that $\mathbb{C}$ is hypoidempotent or $\mathbb{C}$ contains every nilpotent involution algebra. And hence by Proposition 2 for the latter case $\mathbb{C}$ is the class of all involution algebras. In the case char $K\neq 2$ and $K^*=K^{\mathrm{id}}$ we can show that $\mathbb{C}$ is either hypoidempotent or $K_0^{\mathrm{id}}\in\mathbb{C}$ or $K_0^{-\mathrm{id}}\in\mathbb{C}$. Hence, in the latter case since $\mathbb{C}$ is a variety, it follows that $\mathbb{C}$ contains nilpotent involution algebras, which are in $\mathbb{V}_S$ or in $\mathbb{V}_K$. By Theorem 3(v), either $\mathbb{V}_S$ or $\mathbb{V}_K$ is contained in $\mathbb{C}$. Assume that $\mathbb{V}_S\subseteq\mathbb{C}$. Since $\mathbb{C}$ has attainable identities, for the involution algebra H constructed above we have $H(\mathbb{C})=0$, that is $H\in\mathbb{C}$, and hence $K(P)\in\mathbb{C}\cap\mathbb{V}_K$. This implies that $\mathbb{V}_K\subseteq\mathbb{C}$. By [7, Theorem 10] every nilpotent involution algebra is contained in $\mathbb{C}$. Thus again by Proposition 2 $\mathbb{C}$ is the class of all involution algebras. Similar to the case $\mathbb{V}_K\subseteq\mathbb{C}$.

Corollary 5. *If $\mathbb{C}$ is a semisimple radical class of involution algebras, then $\mathbb{C}$ has the A–D–S property* (*i.e.*, $\mathbb{C}(I)\lhd^* A$ *for any ideal I of an involution algebra A*).

Proof. By [7, Theorem 10] and by Proposition 4 the assertion is obvious.

Proposition 6. *Let K be an involution field. If $S(K)$ is finite, then K is finite.*

Proof. Let f be any automorphism of the field K (without involution). The fixed subfield N of K over f is defined as the set of all elements of K, which are left fixed by f. If f is finite ordered, then by [5, paper 16, Theorem 11] K is finite-dimensional over N. Now we apply this fact to the automorphism $f=*$. Since $*$ is finite ordered ($*^2=1$) it follows that K is finite-dimensional over the fixed subfield of $*$. This field is exactly $S(K)$. Hence if $S(K)$ is finite, so is K.

Theorem 7. *If K is infinite, then there are no nontrivial semisimple radical classes in the variety of involution K^*-algebras.*

Proof. Suppose that $\mathbb{C}$ is a non-trivial semisimple radical class. Thus by Proposition 4 $\mathbb{C}$ is hypoidempotent. Let a be any non-zero symmetric element of an algebra $A\in\mathbb{C}$. Since $\mathbb{C}$ is a variety, every involution subalgebra, in particular, the involution subalgebra I generated by the element a, is in $\mathbb{C}$. Since $\mathbb{C}$ is hypoidempotent, it follows $I^2=I$. This means that $a=a^2p(a)$, where $p(a)$ is a polynomial of a. Hence $b=ap(a)$ is a non-zero idempotent element of A. Thus if $bb^*\neq 0$, then the involution subalgebra of A generated by the idempotent element b^{**} is isomorphic to K^*. In the other case the element $b+b^*$ is idempotent, and the involution subalgebra of A generated by $b+b^*$ is also isomorphic to K^*. It follows that $K^*\in\mathbb{C}$. By Proposition 6 it is enough to show that in K the set $S(K)$ of symmetric elements is finite. We can show this fact by a proof similar to that of [**1**, Theorem 2.3].

4. Algebras over a finite field

In this part we will determine the non-trivial semisimple radical classes of involution algebras.

Let us recall that the involution algebra A *is* $*$*-prime if* $I, J\lhd^* A$ and $IJ=0$ imply that either $I=0$ or $J=0$. Let us notice that if A is prime, then A is also $*$-prime but the converse is not true. The involution algebra $A\neq 0$ is called *simple* if A is semiprime and if $I\lhd^* A$ implies that $I=0$ or $I=A$. It is also clear that if A is simple as an algebra (without involution), then A is a simple involution algebra. The converse is not valid. For any algebra A let A^{op} denote the *opposite algebra* of A $(x\circ y=yx)$.

Proposition 8. *If A is a simple involution algebra, then either A is a simple algebra or there is a simple algebra I such that $A=I+I$ relative to the exchange involution $((x,y)^*=(y,x))$.*

Proof. Suppose that A is not a simple algebra. In this case A has a non-zero proper algebra-ideal I, so $I^{(*)}$ is also a proper algebra-ideal of A. Moreover, $I\cap I^{(*)}$ and $I+I^{(*)}$ are ideals of A. Since A is a simple involution algebra and $I\neq A$, therefore

$$A=I\oplus I^{(*)} \quad\text{and}\quad I\cap I^{(*)}=0.$$

This implies that $A=I\oplus I^{(*)}$ holds. Since $I^{(*)}=I^{op}$, it follows that $A\cong I\oplus I^{op}$ relative to the exchange involution.

Proposition 9. *Let F be a finite field and R be a ring of 2×2 matrices over F. If $*$ is any involution of R, then the following conditions are equivalent:*

(i) *if $x\in S(R)\cup K(R)$ and $x^n=0$, then $x=0$;*

(ii) *$x^*x=0$ if and only if $x=0$;*

(iii) *there exists a fixed element $\alpha\in F$ such that for any $r\in R$*

$$r^*=\begin{pmatrix} a & c\\ d & b\end{pmatrix}^*=\begin{pmatrix} a & \alpha^{-1}d\\ \alpha a & b\end{pmatrix}$$

and $\alpha\neq -t^2$ for every $t\in F$.

Proof. (i)⇒(ii). Since R is not commutative, we have $K(R)\neq 0$. Assume that $x^*x=0$ for some $x\in R$ $(x\neq 0)$. It is clear that $x^*K(R)x\subseteq K(R)$ and hence by the assumption

$(x^*K(R)x)^2=0$. This implies that $x^*K(R)x=0$. Therefore

$$x^*rx=x(r-r^*+r^*)x=x^*r^*x=(x^*rx)^*$$

holds for every $r\in R$. On the other hand R is a prime ring, thus there is an element $r\in R$ such that $s=x^*rx=0$. Since $s\in S(R)$ and $s^2=(x^*rx)(x^*rx)=0$ which is a contradiction.

(ii)⇒(iii). By [**4**, Theorem 2.5.1] and [**9**, Theorem 1] the involution $*$ of R is either symplectic or of transpose type. If $*$ is symplectic, then

$$s=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}=-\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}^*$$

and hence $s\in K(R)$. But $s^2=0$, a contradiction. Thus $*$ must be of transpose type. In this case as in the last part of the proof [**4**, Theorem 3.3.1] we can show that the involution induced by $*$ on F is identical and there is a fixed element $\alpha\in F$ $(\alpha\neq 0)$ such that

$$r^*=\begin{pmatrix}a & c\\ d & b\end{pmatrix}^*=\begin{pmatrix}a & \alpha^{-1}d\\ \alpha c & b\end{pmatrix}.$$

Now we show that $\alpha\neq -t^2$ for every $t\in F$. Suppose indirectly that $\alpha=-t^2$. Then for the matrix $x=\begin{pmatrix}t & 1\\ t & 1\end{pmatrix}$ we have that $x^*x=0$ contradicting condition (ii). Thus (iii) is valid.

(iii)⇒(i). If $a\in K(R)$, then a has the form $a=\begin{pmatrix}0 & x\\ -\alpha x & 0\end{pmatrix}$, and so it follows that in this case a is invertible, hence a is not nilpotent. If $a\in S(R)$, then a has the form $a=\begin{pmatrix}a & x\\ \alpha x & b\end{pmatrix}$. In this case $a^2=0$ if and only if either $a=0$ or $a=-b$ and $\alpha=-a^2/c^2$. Hence condition (iii) implies that $a=0$. Thus (i) holds.

Corollary 10. *Let R be an involution ring of 2×2 matrices over a finite field F. Suppose that $x^*x\neq 0$ for every $x\neq 0$, $x\in R$. Then every involution subring of R is idempotent, that is $A^2=A$.*

Proof. It is enough to show that if $a\in R$, then the involution subalgebra $[a]$ of R generated by a is idempotent. Since R is finite, also $[a]$ is finite. Moreover by Proposition 9 $[a]$ is semiprime. Hence it follows that $[a]$ is a semiprime Artin algebra. Thus it is clear that $[a]$ is idempotent.

Proposition 11. *Let $\mathbb{C}$ be any variety which consists of idempotent involution algebras. If $D\in\mathbb{C}$ is a simple involution algebra, then D is finite.*

Proof. First we will show that if $D\in\mathbb{C}$ is an involution field, then D is finite. Now let $A\in\mathbb{C}$ be any involution algebra and $a\in A$ be a symmetric element. Denote by $[a]$ the involution subalgebra of A generated by a. Since $\mathbb{C}$ is strongly hereditary, it follows that $[a]\in\mathbb{C}$. Hence $[a]$ is idempotent. This fact means that $a=a^2p(a)$ holds, where $p(x)$ is a polynomial over a field K. Thus every symmetric element is a root of some polynomial over K. Similarly we can show that this is satisfied for every skew-element of $A\in\mathbb{C}$. Applying this argument for the involution algebra $A=\prod_1^\infty D$, a direct product of infinite copies of an involution field D, we obtain that if $b=(\ldots,b_i,\ldots)\in A$ is a symmetric element (so is every component b_i), then b is a root of a polynomial $p(x)$. This implies that every component b_i is also a root of $p(x)$. On the other hand b_i can be chosen from the symmetric elements of D and $p(x)$ has only finitely many roots. It follows that $S(D)$ is a finite set and hence by Proposition 6 the involution field D is finite.

Now we return to show that if $D\in\mathbb{C}$ is simple, then D is finite. Since D is a simple involution algebra, by Proposition 8 D is either simple as an algebra (without involution) or D is $*$-prime but not prime. In the first case since $a=a^2p(a)$, for every $a\in S(D)\cap K(D)$, and by [**4**, Theorem 3.3.3] D is a 2×2 matrix ring over a field F. Thus it is enough to see that F is finite. Since $D\in\mathbb{C}$, so is every involution subalgebra of D, in particular, the involution algebra A of all matrices of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & a\end{smallmatrix}\right)$. Moreover, A is algebraically isomorphic to F. By the previous consideration, however, A is finite, and therefore so is F. In the second case there is a simple algebra I such that $D\cong I\oplus I^{op}$ relative to the exchange involution. Let

$$A=\{(x,x^{op})\mid x\in I\}.$$

Then A is an involution subalgebra of D and A is algebraically isomorphic to I. As in the first case, we can see that A is finite and consequently also $D\cong I\oplus I^{op}$ has to be finite. Thus the proof is complete.

Proposition 12. *Let $\mathbb{C}$ be a variety which consists of idempotent involution algebras. Then*

$$\mathscr{B}(\mathbb{C})=\{A\in\mathbb{C}\ A \text{ is a simple involution algebra}\}$$

is a finite set.

Proof. If $A\in\mathscr{B}(\mathbb{C})$, then by Proposition 11 A is finite. Moreover every involution subalgebra of A is idempotent. Thus if A is $*$-prime but not prime, then there is a finite field F such that $A=F\oplus F^{op}$ relative to the exchange involution. If A is prime, then either A is a finite involution field or A is a 2×2 matrix ring over a finite field. In each case there is a finite field F such that A is a vector space over F and $\mathrm{Dim}_F A\leqq 4$. Therefore for seeing that $\mathscr{B}(\mathbb{C})$ is finite it is enough to show that

$$\mathscr{D}(\mathbb{C})=\{F\in\mathbb{C},\ F \text{ is a field}\}$$

is a finite set. In this way we trace our proof back to the case of associative algebras (see [**3, 11**]). We will not enter into the details.

Before giving a necessary and sufficient condition for a class of idempotent involution algebras to be a variety, we need the following interesting results.

Let us recall that the involution algebra A is *locally finite* if every involution subalgebra of A which is generated by a finite set, is finite. Let A be any semiprime involution algebra. Denote by $\operatorname{soc} A$ the (two sided) *socle* of A (which is the sum of all minimal ideals of A).

Proposition 13. *Let A be any involution algebra. Suppose that $B=\operatorname{soc} B$ for all finitely generated involution subalgebras B of A. If I is an ideal of A such that A/I is finitely generated, then there is a finitely generated involution subalgebra B of A such that*

$$B\cap I=O \quad \text{and} \quad A=B+I.$$

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Proof. Since A/I is finitely generated, there are elements $x_1,\dots,x_n \in A$ such that

$$[x_1,\dots,x_n]+I\in A,$$

where $[x_1,\dots,x_n]$ is the involution subalgebra of A generated by $\{x_1,\dots,x_n\}$. Let $C=[x_1,\dots,x_n]$. By the supposition $C=\operatorname{soc} C$, every ideal of C is a direct summand of C, that is there exists an involution subalgebra B such that $C=B\oplus(C\cap I)$. Since $A=C+I$ it follows that

$$A=(B+(C\cap I))+I=B+I.$$

Clearly $B\cap I=0$. Furthermore B is a homomorphic image of C, thus B is finitely generated.

Theorem 14. *Let K be finite field and $\mathbb{C}$ be any class of involution algebras over K. The following conditions are equivalent:*

(i) *$\mathbb{C}$ is a semisimple class,*

(ii) *$\mathbb{C}$ is either the class of all involution algebras or there is a strongly hereditary finite set $\mathbb{F}$ of finite simple involution algebras such that $\mathbb{C}$ consists of all subdirect sums of elements in $\mathbb{F}$.*

Proof. (i)⇒(ii). By Proposition 4 we can suppose that the class $\mathbb{C}$ is hypoidempotent. If A is a subdirectly irreducible involution algebra in $\mathbb{C}$, then the heart H of A is a simple involution algebra. Since $\mathbb{C}$ is a variety, also $H\in\mathbb{C}$ holds. By Proposition 11 H is finite, and therefore H has an identity. Thus H is a direct summand of A, and hence $H=A$ as A is subdirectly irreducible. This fact shows that every subdirectly irreducible involution algebra in $\mathbb{C}$ is simple. Hence every involution algebra in $\mathbb{C}$ is a subdirect sum of simple involution algebras in $\mathbb{C}$. Let $\mathbb{F}=\mathscr{B}(\mathbb{C})$. Then $\mathbb{F}$ is finite by Proposition 12, and clearly $\mathbb{F}$ is strongly hereditary. Thus condition (ii) holds.

(ii)⇒(i). We may confine ourselves to the case when $\mathbb{C}$ is not the class of all involution algebras. Since every involution algebra of $\mathbb{F}$ is finite it follows that each of them has an identity. Thus we can show as in the case of rings that $\mathbb{C}$ is a semisimple class. It remains to see that $\mathbb{C}$ is also homomorphically closed. Since $\mathbb{F}$ is a strongly hereditary finite set of finite simple involution algebras, it is clear that every $A\in\mathbb{C}$ is locally finite. On the other hand if $C\in\mathbb{F}$ and $x\in S(C)\cup K(C)$, then the involution subalgebra $[x]$ of C generated by x is commutative and finite. Moreover, $[x]$ is idempotent. Hence $[x]$ is a finite direct sum of finite fields. From this it follows that there is a natural number n such that $x^n=x$ $(n>1)$. This is true for all elements $x\in S(C)\cup K(C)$. Furthermore, since $\mathbb{F}$ is finite, there exists a natural number $n_0>1$ such that for arbitrary $C\in\mathbb{F}$ and $x\in S(C)\cup K(C)$ $x^{n_0}=x$. Since C consists of all subdirect sums of elements of $\mathbb{F}$, this is satisfied for all $C\in\mathbb{C}$. Thus if $x=a+a^*$ or $x=a-a^*$, then $x^{n_0}=x$ holds. This property is invariant for a homomorphic image. Therefore if R is an image of $C\in\mathbb{C}$, then

$$(x+x^*)^{n_0}=x+x^* \quad\text{and}\quad (x-x^*)^{n_0}=x-x^*$$

for all $x\in R$. Using [**4**, Theorem 3.3.2] we obtain that R is a subdirect sum of fields and

2×2 matrix rings over fields. This implies that the homomorphic images of $\mathbb{C}$ are Brown–McCoy semisimple. Now we return to show that $\mathbb{C}$ is homomorphically closed. Since the homomorphic images of $\mathbb{C}$ are Brown–McCoy semisimple, it is enough to show that if $C\in\mathbb{C}$ and I is a maximal ideal of C, then $C/I\in\mathbb{C}$. Taking into account that C/I is a simple involution algebra and that $(x+x^*)^{n_0}=x+x^*$ and $(x-x^*)^{n_0}=x-x^*$ for all $x\in C/I$, by [4, Theorem 3.3.2] we know that C/I is a finite dimensional vector space over some finite field $\mathbb{F}$ and $\mathrm{Dim}_{\mathbb{F}}\, C/I\leqq 4$. Hence C/I is finite. On the other hand every finite involution subalgebra P of C is semiprime, so it follows that $P=\mathrm{soc}\, P$. Applying Proposition 13 for the involution algebra C, we get that there is an involution subalgebra B of C such that $C=B+I$ and $I\cap B=0$. Since $C\in\mathbb{C}$, we have $B\in\mathbb{C}$. Furthermore $B=C/I$, which implies $C/I\in\mathbb{C}$. Thus $\mathbb{C}$ is homomorphically closed, completing the proof.

It is known that in the case of associative rings, if $\mathbb{C}$ is not a trivial semisimple radical then $\mathbb{C}$ only contains commutative algebras. In the case of involution algebras, however, non-trivial semisimple radical classes may contain 2×2 matrix rings over finite fields which are certainly non-commutative involution algebras.

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