


RIGID STABLE VECTOR BUNDLES ON HYPERKÄHLER VARIETIES OF TYPE $K3^{[n]}$

KIERAN G. O'GRADY 

*Dipartimento di Matematica, Sapienza Università di Roma, P.le A. Moro 5,
00185 Roma - ITALIA
(ogrady@mat.uniroma1.it)*

(Received 2 December 2022; revised 17 October 2023; accepted 18 October 2023)

Abstract We prove existence and unicity of slope-stable vector bundles on a general polarized hyperkähler (HK) variety of type $K3^{[n]}$ with certain discrete invariants, provided the rank and the first two Chern classes of the vector bundle satisfy certain equalities. The latter hypotheses at first glance appear to be quite restrictive, but, in fact, we might have listed almost all slope-stable rigid projectively hyperholomorphic vector bundles on polarized HK varieties of type $K3^{[n]}$ with 20 moduli.

1. Introduction

1.1. Background

A prominent rôle in the theory of $K3$ surfaces is played by spherical (i.e. rigid and simple) vector bundles. In [O'G22], we have proved existence and uniqueness results for stable vector bundles on general polarized hyperkähler (HK) variety of type $K3^{[2]}$ with certain discrete invariants (of the polarization and of the vector bundle). In the present paper, we show that the main result in [O'G22] extends to HK varieties of type $K3^{[n]}$ of arbitrary (even) dimension. More precisely, we prove that for certain choices of rank and first two Chern classes on a polarized HK variety (X, h) of type $K3^{[n]}$, there exists one and only one stable vector bundle with the assigned rank and first two Chern classes provided the moduli point of (X, h) is a general point of a certain irreducible component of the relevant moduli space of polarized HK varieties.

We like to think of this result as evidence in favour of the following slogan: *stable vector bundles on higher dimensional HK manifolds behave as well as stable sheaves on $K3$ surfaces, provided one restricts to (stable) vector bundles whose projectivization extends to all small deformations of the base HK manifold (i.e. projectively hyperholomorphic vector bundles).*

Keywords: stable vector bundles; hyperkähler manifolds; Lagrangian fibrations; modular sheaves

© The Author(s), 2023. Published by Cambridge University Press. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (<https://creativecommons.org/licenses/by/4.0>), which permits unrestricted re-use, distribution and reproduction, provided the original article is properly cited.

1.2. The main result

Let $\mathcal{K}_e^d(2n)$ be the moduli space of polarized HK varieties of type $K3^{[n]}$ of degree e and divisibility d . Thus, $\mathcal{K}_e^d(2n)$ parametrizes isomorphism classes of couples (X, h) , where X is an HK manifold of type $K3^{[n]}$, $h \in \text{NS}(X)$ is the class of an ample divisor class (we assume that h is primitive), such that $q_X(h) = e$ and $\text{div}(h) = d$, where q_X is the Beauville-Bogomolov-Fujiki (BBF) quadratic form of X and $\text{div}(h)$ is the divisibility of h , that is the positive generator of $q_X(h, H^2(X; \mathbb{Z}))$. Note that $\text{div}(h)$ divides $2(n-1)$. It is known under which hypotheses $\mathcal{K}_e^d(2n)$ is not empty. If that is the case, then it is a quasi-projective variety (not necessarily irreducible) of pure dimension 20.

We recall that if \mathcal{F} is a (coherent) sheaf on a complex smooth variety, the *discriminant* of \mathcal{F} is defined to be the Betti cohomology class

$$\Delta(\mathcal{F}) := 2rc_2(\mathcal{F}) - (r-1)c_1(\mathcal{F})^2 = -2r \text{ch}_2(\mathcal{F}) + \text{ch}_1(\mathcal{F})^2. \tag{1.2.1}$$

Theorem 1.1. *Let $n, r_0, g, l, e \in \mathbb{N}_+$, with $n \geq 2$, and let*

$$\bar{e} := \begin{cases} e & \text{if } r_0 \text{ is even,} \\ 4e & \text{if } r_0 \text{ is odd.} \end{cases}$$

Assume that

$$g \text{ divides } \begin{cases} (r_0 - 1) & \text{if } r_0 \text{ is even,} \\ (r_0 - 1)/2 & \text{if } r_0 \text{ is odd,} \end{cases} \tag{1.2.2}$$

that

$$l|(n-1), \quad \text{gcd}\{l, r_0\} = 1, \quad \text{gcd}\left\{l, \frac{r_0-1}{g}\right\} = 1, \tag{1.2.3}$$

that

$$g^2 \cdot \bar{e} + 2(n-1)(r_0-1)^2 + 8 \equiv 0 \pmod{8r_0}, \tag{1.2.4}$$

and that

$$\bar{e} + \frac{2(n-1)(r_0-1)^2}{g^2} \equiv 0 \pmod{8l^2}. \tag{1.2.5}$$

Let $i \in \{1, 2\}$ be such that $i \equiv r_0 \pmod{2}$. Then there exists an irreducible component of $\mathcal{K}_e^{il}(2n)$, denoted by $\mathcal{K}_e^{il}(2n)^{\text{good}}$ (see Definition 5.3), such that for a general $[(X, h)] \in \mathcal{K}_e^{il}(2n)^{\text{good}}$, there exists one and only one (up to isomorphism) h slope-stable vector bundle \mathcal{E} on X , such that

$$r(\mathcal{E}) = r_0^n, \quad c_1(\mathcal{E}) = \frac{g \cdot r_0^{n-1}}{i} h, \quad \Delta(\mathcal{E}) = \frac{r_0^{2n-2}(r_0^2-1)}{12} c_2(X). \tag{1.2.6}$$

Moreover, for such a vector bundle, $H^p(X, \text{End}^0(\mathcal{E})) = 0$ for all p .

1.3. Comments

1.3.1. Special cases. Let $g = l = 1$ in Theorem 1.1. Then the hypotheses reduce to the following congruences:

$$e \equiv \begin{cases} 4(n-1)r_0 - 2n - 6 \pmod{8r_0} & \text{if } r_0 \equiv 0 \pmod{4}, \\ \frac{1}{2}((n-1)r_0 - n - 3) \pmod{2r_0} & \text{if } r_0 \equiv 1 \pmod{4}, \\ -2n - 6 \pmod{8r_0} & \text{if } r_0 \equiv 2 \pmod{4}, \\ -\frac{1}{2}((n-1)r_0 + n + 3) \pmod{2r_0} & \text{if } r_0 \equiv 3 \pmod{4}. \end{cases} \tag{1.3.1}$$

In particular, for $n = 2$ (and $g = l = 1$), Theorem 1.1 reduces to the main result in [O’G22]. Note also that in this case, $\mathcal{X}_e^{il}(2n)$ is irreducible by [Deb22, Theorem 3.5], and hence $\mathcal{X}_e^{il}(2n)^{\text{good}} = \mathcal{X}_e^{il}(2n)$.

1.3.2. Rank and Chern classes. The choice of rank and first two Chern classes in Theorem 1.1 is not as special as one would think. Let us first consider a rigid stable vector bundle \mathcal{E} on a polarized $K3$ surface (S, h) of rank r with $c_1(\mathcal{E}) = ah$. Since $\chi(S, \text{End } \mathcal{E}) = 2$, we have

$$2rc_2(\mathcal{E}) - (r - 1)a^2h^2 = \Delta(\mathcal{E}) = 2(r^2 - 1). \tag{1.3.2}$$

It follows that $\text{gcd}\{r, a\} = 1$. The following result is a (weak) extension to higher dimensions.

Proposition 1.2. *Let (X, h) be a polarized HK variety of type $K3^{[n]}$, and let \mathcal{F} be a slope-stable vector bundle on X . Suppose that $c_1(\mathcal{F}) = ah$ and that the natural morphism $\text{Def}(X, \mathcal{F}) \rightarrow \text{Def}(X, h)$ is surjective. Then*

$$r(\mathcal{F}) = r_0^n m, \quad a \cdot \text{div}(h) = r_0^{n-1} m b'_0, \tag{1.3.3}$$

where r_0, m, b'_0 are integers, and $\text{gcd}\{r_0, b'_0\} = 1$.

Note that in Proposition 1.2, we do not assume that \mathcal{F} is rigid. There are examples of slope-stable projectively hyperholomorphic vector bundles for which $m > 1$ which are not rigid (see [Mar21, Bot22] and [Fat23]). The tangent vector bundle of an HK manifold of type $K3^{[2]}$ is an example of a rigid slope-stable projectively hyperholomorphic vector bundle with $m > 1$ (in fact, $m = 4$) (see [Gav21]). There are no other examples of the latter type that I am aware of.

The stable vector bundles in Theorem 1.1, together with those obtained by tensoring with powers of the polarization, cover many of the choices of rank and first Chern class with $m = 1$ which are a priori possible according to Proposition 1.2.

Regarding the discriminant of the vector bundle(s) \mathcal{E} in Theorem 1.1, we note the following. First, the formula for the discriminant in (1.2.6) for $n = 1$ is exactly the formula in (1.3.2). Next, since $-\Delta(\mathcal{E})$ is the second Chern class of the pushforward to X of the relative tangent bundle of $\mathbb{P}(\mathcal{E}) \rightarrow X$, and $\mathbb{P}(\mathcal{E})$ extends to all small deformations of X (because $H^2(X, \text{End}^0(\mathcal{E})) = 0$), the cohomology class $\Delta(\mathcal{E})$ remains of type (2,2) for all deformations of X . It follows that $\Delta(\mathcal{E})$ is a linear combination of $c_2(X)$ and q_X^\vee (see the main result in [Zha15]). If $n \in \{2, 3\}$, then $c_2(X)$ and q_X^\vee are linearly dependent, and

hence, it follows (without doing any computation) that $\Delta(\mathcal{E})$ is a multiple of $c_2(X)$. If $n > 3$, then $c_2(X)$ and q_X^\vee are linearly independent, hence, there is no 'a priori' reason why $\Delta(\mathcal{E})$ should be a multiple of $c_2(X)$. In fact, I know of no examples of stable projectively hyperholomorphic vector bundles on HK varieties of type $K3^{[n]}$ whose discriminant is not a multiple of the second Chern class.

The vector bundles \mathcal{E} in Theorem 1.1 are atomic, in fact, they are in the \mathcal{O}_X -orbit (see [Mar21]), and hence, the Beckmann-Markman extended Mukai vector $\tilde{v}(\mathcal{E})$ (see [Mar21, Theorem 6.13], [Bec23, Definition 4.16], [Bec22, Definition 1.1]) is determined uniquely (if we require that its first entry equals $r(\mathcal{E})$), in particular, $\tilde{q}(\tilde{v}(\mathcal{E})) = -(n+3)r_0^{2n-2}/2$ by [Bec23, Lemma 4.8]. By the formula relating the projection of the discriminant $\Delta(\mathcal{E})$ on the Verbitsky subalgebra and the square $\tilde{q}(\tilde{v}(\mathcal{E}))$ in [Bot22, Proposition 3.11], we get that if $\Delta(\mathcal{E})$ is a multiple of $c_2(X)$, then it is given by the formula in (1.2.6).

The natural question to ask is the following: Are we close to having listed all slope-stable rigid vector bundles on a polarized HK variety of type $K3^{[n]}$ with 20 moduli?

1.3.3. Projective bundles. Let X and \mathcal{E} be as in Theorem 1.1. The projectivization $\mathbb{P}(\mathcal{E})$ extends (uniquely) to a projective bundle on all (small) deformations of X because $H^2(X, \text{End}^0(\mathcal{E})) = 0$. Actually, Markman [Mar21, Theorem 1.4] shows that $\mathbb{P}(\mathcal{E})$ extends to a projective bundle on all deformations of X (it is projectively hyperholomorphic). In fact, the (possibly twisted) locally free sheaf E on $S^{[n]}$ appearing in Markman loc. cit. is obtained by deforming the vector bundle $\mathcal{F}[n]^+$ associated to a spherical vector bundle \mathcal{F} on S , see Definition 3.1 (or Definition 5.1 in [O'G22]), and likewise the vector bundles in Theorem 1.1 are obtained by deforming $\mathcal{F}[n]^+$. Theorem 1.1 should provide a uniqueness result for stable projective bundles \mathbf{P} of dimension $(r_0^n - 1)$ with characteristic class given by the third equality in (1.2.6) (i.e. $-c_2(\Theta_{\mathbf{P}/X})$, where $\Theta_{\mathbf{P}/X}$ is the relative tangent bundle of $\mathbf{P} \rightarrow X$) on a general HK manifold of type $K3^{[n]}$. In order to turn this into a precise statement, one would need to specify with respect to which Kähler classes the projective bundle is supposed to be stable. The zoo of conditions in Theorem 1.1 would then correspond to the cases in which the projective bundle is the projectivization of a vector bundle, that is to the vanishing of the relevant Brauer class.

1.3.4. Franchetta property. Let $\mathcal{U}_e^{il}(2n) \subset \mathcal{H}_e^{il}(2n)$ be an open nonempty subset with the property that there exists one and only one stable vector bundle \mathcal{E} on $[(X, h)] \in \mathcal{U}_e^{il}(2n)$, such that the equations in (1.2.6) hold, and let $\mathcal{X} \rightarrow \mathcal{U}_e^{il}(2n)$ be the tautological family of HK (polarized) varieties (we might need to pass to the moduli stack). By Theorem A.5 in [Muk87], there exists a quasitautological vector bundle E on \mathcal{X} , that is a vector bundle whose restriction to a fibre (X, h) of $\mathcal{X} \rightarrow \mathcal{U}_e^{il}(2n)$ is isomorphic to $\mathcal{E}^{\oplus d}$ for some $d > 0$, where \mathcal{E} is the vector bundle of Theorem 1.1. If $[(X, h)] \in \mathcal{U}_e^{il}(2n)$, the generalized Franchetta conjecture, see [FLV19], predicts that the restriction to $\text{CH}^2(X)_{\mathbb{Q}}$ of $\text{ch}_2(E) \in \text{CH}(\mathcal{X})_{\mathbb{Q}}$ is equal to $-d \frac{r_0^{2n-2}(r_0^2-1)}{12} c_2(X)$. In other words, it predicts that the third equality in (1.2.6) holds at the level of (rational) Chow groups. In general, it is not easy to give a rationally defined algebraic cycle class on a nonempty open subset of the moduli stack of polarized HK varieties. Theorem 1.1 produces such a cycle, and hence, it provides a good test for the generalized Franchetta conjecture.

1.4. Basic ideas

The key elements in the proof of the main result are the following. First, there is the extension to modular sheaves (defined in [O'G22]) on higher dimensional HK manifolds of the decomposition of the (real) ample cone of a smooth projective surface into open chambers for which slope stability of sheaves with fixed numerical characters does not change (see [O'G22]).

The second element is the behaviour of modular vector bundles on a Lagrangian HK manifold. If the polarization is very close to the pullback of an ample line bundle from the base, then the restriction of a slope-stable vector bundle to a general Lagrangian fibre is slope semistable, and if it is slope-stable, then it is a semihomogeneous vector bundle, in particular, it has no nontrivial infinitesimal deformations keeping the determinant fixed. In the reverse direction, if the restriction of a vector bundle to a general Lagrangian fibre is slope-stable, then the vector bundle is slope-stable (provided the polarization is very close to the pullback of an ample line bundle from the base). The key element in the proof of existence is a construction discussed in [O'G22] (and in [Mar21]) which associates to a vector bundle \mathcal{F} on a $K3$ surface S a sheaf $\mathcal{F}[n]^+$ on $S^{[n]}$. The sheaf $\mathcal{F}[n]^+$ is locally free by Haiman's highly nontrivial results in [Hai01]. If \mathcal{F} is a spherical vector bundle, then $\text{End}^0(\mathcal{F}[n]^+)$ has no nonzero cohomology by Bridgeland-King-Reid's derived version of the McKay correspondence. This gives that $\mathcal{F}[n]^+$ extends to all (small) deformations of $(S^{[n]}, \det \mathcal{F}[n]^+)$, and that the projectivization $\mathbb{P}(\mathcal{F}[n]^+)$ extends to all (small) deformations of $S^{[n]}$ (the last result follows from a classical result of Horikawa). We prove slope stability of $\mathcal{F}[n]^+$ in the case of an elliptic $K3$ surface S by using our results on vector bundles on Lagrangian HK manifolds. In fact, if S is an elliptic $K3$ surface, then there is a Lagrangian fibration $S^{[n]} \rightarrow (\mathbb{P}^1)^{(n)} \cong \mathbb{P}^n$, whose general fibre is the product of n fibres of the elliptic fibration. If \mathcal{F} is a slope-stable rigid vector bundle on S , then the restriction to an elliptic fibre is slope-stable. It follows that the restriction of $\mathcal{F}[n]^+$ to a general fibre of the Lagrangian fibration $S^{[n]} \rightarrow \mathbb{P}^n$ is slope-stable. From this, one gets that the (unique) extension of $\mathcal{F}[n]^+$ to a general Lagrangian deformation of $(S^{[n]}, \det \mathcal{F}[n]^+)$ is slope-stable with respect to $\det \mathcal{F}[n]^+$ (provided we move in a Noether-Lefschetz locus with high enough discriminant). Uniqueness of a general slope-stable vector bundle with the given numerical invariants is obtained by proving uniqueness for vector bundles on (polarized) HK varieties with Lagrangian fibrations (with discriminant high enough and almost coprime to the rank). The main points in the proof of the latter result are the following. Let \mathcal{F} be a spherical vector bundle on an elliptic $K3$ surface S : the vector bundle \mathcal{E}_X on a (small) Lagrangian deformation X of $S^{[n]}$ obtained by extension of $\mathcal{F}[n]^+$ restricts to slope-stable semihomogeneous vector bundles on Lagrangian fibres parametrized by a large open subset of the base (the complement has codimension at least 2). Any slope-stable vector bundle \mathcal{E} on X with the same rank c_1 and c_2 as \mathcal{E}_X restricts to a slope-stable semihomogeneous vector bundle on a general Lagrangian fibre. Any two simple semihomogeneous vector bundles on an Abelian variety with the same rank and determinant are obtained one from the other via tensorization with a (torsion) line bundle. This, together with a monodromy argument, gives that \mathcal{E}_X and \mathcal{E} restrict to isomorphic vector bundles on a general Lagrangian fibre. Since the set of Lagrangian

fibres for which the restriction of \mathcal{E}_X is slope-stable has the complement of codimension at least 2, one concludes that \mathcal{E}_X and \mathcal{E} are isomorphic.

1.5. Outline of the paper

Sections 2 and 3 are devoted to the computation of the discriminant of the vector bundle $\mathcal{F}[n]^+$ on $S^{[n]}$, provided \mathcal{F} is a spherical vector bundle on the $K3$ surface S . Since $\mathbb{P}(\mathcal{F}[n]^+)$ extends to all (small) deformations of $S^{[n]}$, one knows a priori that the discriminant is a linear combination of $c_2(S^{[n]})$ and the inverse $q_{S^{[n]}}^\vee$ of the BBF quadratic form. From this, it follows that one can work on the open subset of $S^{[n]}$ parametrizing subschemes whose support has cardinality at least $n - 1$, and then a straightforward computation gives that the discriminant is as in (1.2.6).

In Section 4, we show that by starting from slope-stable spherical vector bundles \mathcal{F} on an elliptic surface $S \rightarrow \mathbb{P}^1$, we can produce vector bundles $\mathcal{F}[n]^+$ on $S^{[n]}$ with rank and first two Chern classes covering all the cases in Theorem 1.1. Moreover, we study the restriction of such an $\mathcal{F}[n]^+$ to fibres of the Lagrangian fibration $S^{[n]} \rightarrow (\mathbb{P}^1)^{(n)} \cong \mathbb{P}^n$.

Section 5 is the most demanding part of the paper. The key ideas, outlined in Section 1.4, are combined together in order to give the proof of Theorem 1.1 (and of Proposition 1.2).

2. The isospectral Hilbert scheme

2.1. Summary of results

We start by introducing notation and recalling known results. Let S be a $K3$ surface. The isospectral Hilbert scheme of n points on S , denoted by $X_n = X_n(S)$, was introduced and studied by Haiman (see Definition 3.2.4 in [Hai01]). We have a commutative diagram

$$\begin{array}{ccc}
 X_n(S) & \xrightarrow{\tau} & S^n \\
 \rho \downarrow & & \downarrow \pi \\
 S^{[n]} & \xrightarrow{\gamma} & S^{(n)}
 \end{array} \tag{2.1.1}$$

In fact, $X_n(S)$ is the reduced scheme associated to the fibre product of S^n and $S^{[n]}$ over $S^{(n)}$. Moreover, the map τ is identified with the blow up of S^n with centre the big diagonal (see Corollary 3.8.3 in [Hai01]). Let $\text{pr}_i : S^n \rightarrow S$ be the i -th projection, and let $\tau_i : X_n(S) \rightarrow S$ be the composition $\tau_i := \text{pr}_i \circ \tau$. Let

$$(S^n)_* := \{x = (x_1, \dots, x_n) \in S^n \mid \text{at most two entries of } x \text{ are equal}\}, \tag{2.1.2}$$

and let $X_n(S)_* := \tau^{-1}((S^n)_*)$. Let $E_n \subset X_n(S)_*$ be the exceptional divisor of $X_n(S)_* \rightarrow (S^n)_*$. Then E_n is smooth because the restriction of the big diagonal to $(S^n)_*$ is smooth. We let

$$e_n := \text{cl}(E_n) \in H^2(X_n(S)_*; \mathbb{Q}). \tag{2.1.3}$$

Let $\eta \in H^4(S; \mathbb{Q})$ be the fundamental class.

If X is an HK manifold, the nondegenerate BBF symmetric bilinear form $H^2(X) \times H^2(X) \rightarrow \mathbb{C}$ defines a symmetric bilinear form $H^2(X)^\vee \times H^2(X)^\vee \rightarrow \mathbb{C}$, that is a symmetric element of $H^2(X) \otimes H^2(X)$, whose image in $H^4(X)$ via the cup product map is a rational Hodge class q_X^\vee .

Below are the results obtained in the present section.

Proposition 2.1. *Let $n \geq 2$. We have the following equalities in the rational cohomology of $X_n(S)_*$:*

$$\rho^* \left(\text{ch}_2(S^{[n]}) \right)_{|X_n(S)_*} = -24 \sum_{l=1}^n \tau_l^*(\eta)_{|X_n(S)_*} + 3e_n^2, \tag{2.1.4}$$

$$\rho^*(q^\vee)_{|X_n(S)_*} = -(2n - 24) \sum_{l=1}^n \tau_l^*(\eta) - \frac{4n - 3}{2n - 2} e_n^2. \tag{2.1.5}$$

Before stating the next result, we note that while q_X^\vee is a rational cohomology class, $(2n - 2)q_X^\vee$ lifts to an integral class (uniquely because the group $H^*(S^{[n]}; \mathbb{Z})$ is torsion-free by the main result in [Mar07]).

Proposition 2.2. *Let $n \geq 4$, and let $T_n \subset H^4(S^{[n]}; \mathbb{Z})$ be the saturation of the subgroup spanned by $c_2(S^{[n]})$ and $(2n - 2)q_X^\vee$. The map*

$$\begin{aligned} T_n &\longrightarrow H^4(X_n(S)_*; \mathbb{Z}) \\ a &\longmapsto \rho^*(a)_{|X_n(S)_*} \end{aligned} \tag{2.1.6}$$

is injective.

The proof of Propositions 2.1 and 2.2 are, respectively, in Sections 2.2 and 2.3.

2.2. Proof of Proposition 2.1

We start by recalling a couple of formulae. First, suppose that $j: D \hookrightarrow W$ is the embedding of a smooth divisor in a smooth ambient variety, and that \mathcal{F} is a sheaf on D . Then, by Grothendieck-Riemann-Roch and the push-pull formula, we have

$$\text{ch}(j_*(\mathcal{F})) = j_*(\text{ch}(\mathcal{F})) \cdot \text{Td}(\mathcal{O}_W(D))^{-1} = j_*(\text{ch}(\mathcal{F})) \cdot \left(1 - \frac{\text{cl}(D)}{2} + \frac{\text{cl}(D)^2}{6} + \dots \right). \tag{2.2.1}$$

Next, we recall how one computes the Chern classes of a blow up. Let Z be a smooth variety, and let $Y \subset Z$ be a smooth subvariety of pure codimension c . Let $f: \tilde{Z} \rightarrow Z$ be the blow up of Y . Let $j: E \hookrightarrow \tilde{Z}$ be the inclusion of the exceptional divisor of f , and let $e \in H^2(\tilde{Z}; \mathbb{Q})$ be the class of E . If $\mathcal{N}_{Y/Z}$ is the normal bundle of Y in Z , then $E \cong \mathbb{P}(\mathcal{N}_{Y/Z})$, and the restriction of $\mathcal{O}_{\tilde{Z}}(E)$ to E is isomorphic to the tautological subline bundle $\mathcal{O}_E(-1)$. Let \mathcal{Q} be the quotient bundle on E , that is the vector bundle fitting into the exact sequence

$$0 \longrightarrow \mathcal{O}_E(-1) \longrightarrow f_E^* \mathcal{N}_{Y/Z} \longrightarrow \mathcal{Q} \longrightarrow 0, \tag{2.2.2}$$

where $f_E: E \rightarrow Y$ is the restriction of f . The differential of f gives the exact sequence

$$0 \longrightarrow \Theta_{\tilde{Z}} \xrightarrow{df} f^*(\Theta_Z) \longrightarrow j_*(\mathcal{Q}) \longrightarrow 0. \tag{2.2.3}$$

Taking Chern characters, and applying the formula in (2.2.1) to the inclusion $j: E \hookrightarrow \tilde{Z}$ and the sheaf \mathcal{Q} , we get the formula

$$\begin{aligned} \text{ch}(\tilde{Z}) &= f^*(\text{ch}(Z)) - \text{ch}(j_*(\mathcal{Q})) = f^*(\text{ch}(Z)) - j_*(\text{ch}(\mathcal{Q})) \cdot \left(1 - \frac{e}{2} + \dots\right) = \\ &= f^*(\text{ch}(Z)) - j_*\left((c-1) + f_E^*(c_1(\mathcal{N}_{Y/Z}) - j^*(e))\right) \cdot \left(1 - \frac{e}{2} + \dots\right) \equiv \\ &\equiv f^*(\text{ch}(Z)) - j_*\left(f_E^*(c_1(\mathcal{N}_{Y/Z})) - (c-1)e + \frac{c+1}{2}e^2\right) \pmod{H^6(\tilde{Z};\mathbb{Q})}. \end{aligned} \tag{2.2.4}$$

Proof of the equality in (2.1.4). Since $X_n(S)_*$ is the blow up of $(S^n)_*$ with centre the smooth locus of the big diagonal, we can relate the Chern characters of $X_n(S)_*$ and $(S^n)_*$ via the equality in (2.2.4). Since $\text{ch}_2(S^n) = -24 \sum_{i=1}^n \tau_i^*(\eta)$, and the normal bundle of the big diagonal in S^n has trivial first Chern class, the equation in (2.2.4) gives that

$$\text{ch}_2(X_n(S)_*) = \frac{3}{2}e_n^2 - 24 \sum_{l=1}^n \tau_l^*(\eta)|_{X_n(S)_*}. \tag{2.2.5}$$

The differential of the map $\rho: X_n(S) \rightarrow S^{[n]}$ gives the exact sequence

$$0 \longrightarrow \rho^*(\Omega_{(S^{[n]})_*}^1) \xrightarrow{(d\rho)^t} \Omega_{X_n(S)_*}^1 \longrightarrow \iota_*(\iota^*(\mathcal{O}_{X_n(S)_*}(-E_n))) \longrightarrow 0, \tag{2.2.6}$$

where $\iota: E_n \hookrightarrow X_n(S)_*$ is the inclusion map. Taking Chern characters, we get that

$$\text{ch}_2(X_n(S)_*) = \rho^* \text{ch}_2((S^{[n]})_*) - \frac{3}{2}e_n^2. \tag{2.2.7}$$

The equality in (2.1.4) follows from the equalities in (2.2.5) and (2.2.7). □

Proof of the equality in (2.1.5). Let $\{\alpha_1, \dots, \alpha_{22}\}$ be an orthonormal basis of $H^2(S; \mathbb{C})$. Then

$$q^\vee = \sum_{i=1}^{22} \mu(\alpha_i)^2 - \frac{1}{2n-2} \delta_n^2, \tag{2.2.1}$$

and hence

$$\begin{aligned} \rho^*(q^\vee)|_{X_n(S)_*} &= \sum_{i=1}^{22} \left(\sum_{l=1}^n \tau_l^*(\alpha_i)\right)^2 - \frac{1}{2n-2} e_n^2 = \\ &= 22 \sum_{l=1}^n \tau_l^*(\eta) + 2 \sum_{1 \leq l < m \leq n} \left(\sum_{i=1}^{22} \tau_l^*(\alpha_i) \cdot \tau_m^*(\alpha_i)\right) - \frac{1}{2n-2} e^2. \end{aligned} \tag{2.2.2}$$

Let $D_n \subset S^n$ be the big diagonal. Then

$$\text{cl}(D_n) = (n-1) \sum_{l=1}^n \tau_l^*(\eta) + \sum_{1 \leq l < m \leq n} \left(\sum_{i=1}^{22} \tau_l^*(\alpha_i) \cdot \tau_m^*(\alpha_i)\right). \tag{2.2.3}$$

TABLE 1. Integrals of α_n, e_n^2 over Γ, Ω .

| | α_n | e_n^2 |
|----------|------------|----------|
| Γ | 1 | $-(n-1)$ |
| Ω | 0 | $-d$ |

Moreover, it follows from the “Key Formula” (see, for example, Proposition 6.7 in [Ful84]) that we have the relation

$$\tau^*(\text{cl}(D_n))|_{X_n(S)_*} = -e_n^2. \tag{2.2.4}$$

The equality in (2.1.5) follows at once from the equalities in (2.2.3) and (2.2.4). \square

2.3. Proof of Proposition 2.2.

Lemma 2.3. *Let $n \geq 2$. Then the classes $\alpha_n := \sum_{l=1}^n \tau_l^*(\eta)|_{X_n(S)_*}$ and e_n^2 are linearly independent in $H^4(X_n(S)_*; \mathbb{Z})$.*

Proof. We prove the lemma by integrating α_n and e_n^2 over algebraic 2 cycles on $X_n(S)_*$ defined as follows. Let $p_1, \dots, p_{n-1} \subset S$ be $n-1$ distinct points, and let

$$\Gamma := \rho^{-1}(\{p_1, \dots, p_{n-1}, x \mid x \in S\}). \tag{2.3.1}$$

Clearly, $\Gamma \subset X_n(S)_*$, and it is isomorphic to the blow up of S at p_1, \dots, p_{n-1} . In order to define the second 2 cycle, we assume (as we may) that S contains two smooth curves C_1, C_2 intersecting with transverse intersection of cardinality $d > 0$. Let $q_1, \dots, q_{n-2} \subset (S \setminus C_1 \setminus C_2)$ be $n-2$ distinct points, and let

$$\Omega := \rho^{-1}(\{q_1, \dots, q_{n-2}, x_1, x_2 \mid x_i \in C_i\}). \tag{2.3.2}$$

Clearly, $\Omega \subset X_n(S)_*$, and it is isomorphic to the blow up of $C_1 \times C_2$ at the d points (x, x) for $x \in C_1 \cap C_2$.

It makes sense to integrate α_n and e_n^2 over Γ, Ω because the latter are compact (complex) surfaces contained in $X_n(S)_*$. One checks easily that the 2×2 “Gram matrix” of the integrals of α_n and e_n^2 over Γ and Ω is given by Table 1. It follows that α_n and e_n^2 are linearly independent. \square

Now we can prove Proposition 2.2. Proposition 2.1 expresses the restriction to $X_n(S)_*$ of $\rho^*(c_2(S^{[n]}))$ and q^\vee as linear combinations of α_n and e_n . The determinant of the 2×2 matrix with entries the corresponding coefficients is nonsingular if and only if $n \notin \{2, 3\}$, hence, Proposition 2.2 follows from Lemma 2.3.

Remark 2.4. Let $n \in \{2, 3\}$. By Proposition 2.1, the classes $\rho^*(\text{ch}_2(S^{[n]}))|_{X_n(S)_*}$ and $\rho^*(q^\vee)|_{X_n(S)_*}$ are linearly dependent. This agrees with known results. In fact, if X is an HK of type $K3^{[2]}$, then $c_2(X)$ and q_X^\vee are linearly dependent because $\text{Sym}^2 H^2(X; \mathbb{Q}) = H^4(X; \mathbb{Q})$, and if X is an HK of type $K3^{[3]}$, then $c_2(X)$ and q_X^\vee are linearly dependent although $\text{Sym}^2 H^2(X; \mathbb{Q})$ is strictly contained in $H^4(X; \mathbb{Q})$ (see Example 14 in [Mar02], or Remark 3.3 in [GKLR22]).

3. Basic modular vector bundles on $S^{[n]}$

3.1. Summary of results

Let S be a K3 surface. We maintain the notation introduced in Section 2.1. Let \mathcal{F} be a locally free sheaf on S . Then

$$X_n(\mathcal{F}) := \tau_1^*(\mathcal{F}) \otimes \dots \otimes \tau_n^*(\mathcal{F}) \tag{3.1.1}$$

is a locally free sheaf on $X_n(S)$. The map ρ in (2.1.1) is finite, and moreover, it is flat because $X_n(S)$ is Cohen–Macaulay (CM) by Theorem 3.1 in [Hai01]. It follows that the pushforward $\rho_*(X_n(\mathcal{F}))$ is also locally free. The symmetric group \mathcal{S}_n acts on $X_n(S)$ compatibly with its permutation action on S^n , and hence, the action lifts to an action μ_n^+ on $X_n(\mathcal{F})$. Since μ_n maps to itself any fibre of $\rho: X_n(S) \rightarrow S^{[n]}$, we get an action $\bar{\mu}_n^+ : \mathcal{S}_n \rightarrow \text{Aut}(\rho_*X_n(\mathcal{F}))$.

Definition 3.1. Let $\mathcal{F}[n]^+ \subset \rho_*X_n(\mathcal{F})$ be the sheaf of \mathcal{S}_n -invariants for $\bar{\mu}_n^+$.

Since $\rho_*X_n(\mathcal{F})$ is locally free, so is $\mathcal{F}[n]^+$.

Let r_0 be the rank of \mathcal{F} . Below is the main result of the present section.

Proposition 3.2. *Suppose that \mathcal{F} is spherical, that is $h^p(S, \text{End}^0(\mathcal{F})) = 0$ for all p . Let $n \geq 2$. Then*

$$\text{rk}(\mathcal{F}[n]^+) = r_0^n, \tag{3.1.2}$$

$$c_1(\mathcal{F}[n]^+) = r_0^{n-1} \left(\mu(c_1(\mathcal{F})) - \frac{r_0 - 1}{2} \delta_n \right), \tag{3.1.3}$$

$$\Delta(\mathcal{F}[n]^+) = \frac{r_0^{2n-2}(r_0^2 - 1)}{12} c_2(S^{[n]}). \tag{3.1.4}$$

Remark 3.3. The notation in (3.1.3) and (3.1.4) is unambiguous because the group $H^*(S^{[n]}; \mathbb{Z})$ is torsion-free by the main result in [Mar07] (see also [Tot20]). Notice that Proposition 3.2 holds also (trivially) for $n = 1$, provided we set $\delta_1 = 0$.

If \mathcal{F} is spherical, then the vector bundle $\mathcal{F}[n]^+$ is modular by Proposition 3.2, and we refer to it as a basic modular vector bundle. The proof of Proposition 3.2 is in Section 3.4.

Remark 3.4. The equalities in Proposition 3.2 should hold with the weaker hypothesis $\chi(S, \text{End}^0(\mathcal{F})) = 0$. To prove this, it would suffice to show that such a vector bundle is the limit of spherical vector bundles.

3.2. Chern classes of $\rho^*\mathcal{F}[n]^+$ restricted to $X_n(S)_*$

Let $h_+ \in H_{\mathbb{Q}}^{1,1}(S^{[n]})$ be given by

$$h_+ := \mu(c_1(\mathcal{F})) - \frac{r_0 - 1}{2} \delta_n. \tag{3.2.1}$$

In the present subsection, we prove the following result.

Proposition 3.5. *Let S be a $K3$ surface, and let \mathcal{F} be a vector bundle on S , such that $\chi(S, \text{End}^0(\mathcal{F})) = 2$. Let r_0 be the rank of S , and let $h^+ \in H_{\mathbb{Q}}^{1,1}(S^{[2]})$ be as in (3.2.1). Then the following equalities hold:*

$$\text{ch}_0(\rho^* \mathcal{F}[n]^+) |_{X_n(S)_*} = r_0^n \tag{3.2.2}$$

$$\text{ch}_1(\rho^* \mathcal{F}[n]^+) |_{X_n(S)_*} = r_0^{n-1} \rho^*(h^+) |_{X_n(S)_*} \tag{3.2.3}$$

$$\text{ch}_2(\rho^* \mathcal{F}[n]^+) |_{X_n(S)_*} = r_0^{n-2} \rho^* \left(\frac{(r_0^2 - 1)}{24} \text{ch}_2(S^{[n]}) + \frac{1}{2} h_+^2 \right) |_{X_n(S)_*}. \tag{3.2.4}$$

Proof. Let $D_n \subset (S^n)_*$ be the (intersection of $(S^n)_*$ with the) big diagonal. For $1 \leq j < k \leq n$, let $D_n(j, k) \subset D_n$ be the set of points (x_1, \dots, x_n) , such that $x_j = x_k$. We have the open embedding

$$\begin{array}{ccc} D_n(j, k) & \xrightarrow{\bar{\epsilon}_{j,k}} & S^{n-1} \\ (x_1, \dots, x_n) & \mapsto & (x_j, x_1, \dots, x_{j-1}, \widehat{x}_j, x_{j+1}, \dots, x_{k-1}, \widehat{x}_k, x_{k+1}, \dots, x_n). \end{array} \tag{3.2.5}$$

Let $\tau_{E_n} : E_n \rightarrow D_n$ be the restriction of τ to E_n , and let $E_n(j, k) := \tau_{E_n}^{-1}(D_n(j, k))$. Then $E_n = \coprod E_n(j, k)$. Let $\tau_{j,k} : E_n(j, k) \rightarrow D_n(j, k)$ be defined by the restriction of τ_{E_n} , and let

$$\begin{array}{ccc} E_n(j, k) & \xrightarrow{\epsilon_{j,k}} & S^{n-1} \\ y & \mapsto & \bar{\epsilon}_{j,k}(\tau_{j,k}(y)). \end{array} \tag{3.2.6}$$

Let $\mathcal{Q}_{j,k}$ be the locally free sheaf on $E_n(j, k)$ defined by

$$\mathcal{Q}_{j,k} := \epsilon_{j,k}^* \left(\bigwedge^2 \mathcal{F} \boxtimes \mathcal{F} \boxtimes \dots \boxtimes \mathcal{F} \right). \tag{3.2.7}$$

Let $\iota_{j,k} : E_n(j, k) \hookrightarrow X_n(S)_*$ be the inclusion map. We have an exact sequence

$$0 \longrightarrow \rho^* \mathcal{F}[n]^+ \longrightarrow \tau_1^*(\mathcal{F}) \otimes \dots \otimes \tau_n^*(\mathcal{F}) \longrightarrow \bigoplus_{1 \leq j < k \leq n} \iota_{j,k,*}(\mathcal{Q}_{j,k}) \longrightarrow 0. \tag{3.2.8}$$

It follows that

$$\rho^* \text{ch}(\mathcal{F}[n]^+) = \tau_1^* \text{ch}(\mathcal{F}) \cdot \dots \cdot \tau_n^* \text{ch}(\mathcal{F}) - \sum_{1 \leq j < k \leq n} \text{ch}(\iota_{j,k,*}(\mathcal{Q}_{j,k})). \tag{3.2.9}$$

Since $\chi(S, \text{End}^0(\mathcal{F})) = 2$, the Hirzebruch-Riemann-Roch theorem gives that

$$2r_0 \text{ch}_2(\mathcal{F}) = \text{ch}_1(\mathcal{F})^2 - 2(r_0^2 - 1)\eta. \tag{3.2.10}$$

Using the above equality, one gets that modulo $H^6(X_n(S)_*; \mathbb{Q})$, we have

$$\begin{aligned} \tau_1^*(\text{ch}(\mathcal{F})) \cdot \dots \cdot \tau_n^*(\text{ch}(\mathcal{F})) &= r_0^n + r_0^{n-1} \sum_{l=1}^n \tau_l^*(c_1(\mathcal{F})) + \\ &+ \frac{1}{2} r_0^{n-2} \sum_{l=1}^n \tau_l^*(c_1(\mathcal{F})^2 - 2(r_0^2 - 1)\eta) + \\ &+ r_0^{n-2} \sum_{1 \leq l < m \leq n} \tau_l^*(c_1(\mathcal{F})) \cdot \tau_m^*(c_1(\mathcal{F})). \end{aligned} \tag{3.2.11}$$

Let $e_n(j, k) = \text{cl}(E_n(j, k))$. Using the equality in (2.2.1), one gets that modulo $H^6(X_n(S)_*; \mathbb{Q})$, we have

$$\begin{aligned} \text{ch}(\iota_{j,k,*}(\mathcal{Q}_{j,k})) &= \iota_{j,k,*}(\text{ch}(\mathcal{Q}_{j,k})) \cdot \left(1 - \frac{e_n(j,k)}{2}\right) = \\ &= \iota_{j,k,*} \left(\iota_{j,k}^* \left(\binom{r_0}{2} r_0^{n-2} + \frac{1}{2}(r_0 - 1)r_0^{n-2} \sum_{l=1}^n \tau_l^* c_1(\mathcal{F}) \right) \right) \cdot \left(1 - \frac{e_n(j,k)}{2}\right) = \\ &= \binom{r_0}{2} r_0^{n-2} e_n(j,k) + \frac{1}{2}(r_0 - 1)r_0^{n-2} e_n(j,k) \cdot \sum_{l=1}^n \tau_l^* c_1(\mathcal{F}) - \frac{1}{2} \binom{r_0}{2} r_0^{n-2} e_n(j,k)^2. \end{aligned}$$

The equalities in (3.2.2), (3.2.3) and (3.2.4) follow at once from the equalities in (3.2.9), (3.2.11) and the above equality. \square

3.3. Deformations of $(S^{[n]}, \mathbb{P}(\mathcal{F}[n]^+))$

Let \mathcal{F} be a vector bundle on S , and let $f: \mathbb{P}(\mathcal{F}[n]^+) \rightarrow S^{[n]}$ be the structure map. We let $\text{Def}(\mathbb{P}(\mathcal{F}[n]^+), f, S^{[n]})$ be the deformation functor of the map f (see Definition 8.2.7 in [Man22]).

Proposition 3.6. *Suppose that the K3 surface S is projective and that \mathcal{F} is a spherical vector bundle on S . Then the natural map $\text{Def}(\mathbb{P}(\mathcal{F}[n]^+), f, S^{[n]}) \rightarrow \text{Def}(S^{[n]})$ is smooth.*

Proof. The result follows from a theorem of Horikawa. In fact, let $X := \mathbb{P}(\mathcal{F}[n]^+)$, $Y := S^{[n]}$, and consider the exact sequence of locally free sheaves on X

$$0 \longrightarrow \Theta_{X/Y} \longrightarrow \Theta_X \xrightarrow{df} f^* \Theta_Y \longrightarrow 0. \tag{3.3.1}$$

By [Hor74, Theorem 6.1] (see also Corollary 8.2.14 in [Man22]), it suffices to prove that the map $H^1(X, \Theta_X) \rightarrow H^1(X, f^* \Theta_Y)$ is surjective and the map $H^2(X, \Theta_X) \rightarrow H^2(X, f^* \Theta_Y)$ is injective. By the exact sequence in (3.3.1), it suffices to show that $H^2(X, \Theta_{X/Y}) = 0$. By the local-to-global spectral sequence abutting to $H^2(X, \Theta_{X/Y})$, we are done if we prove that

$$H^p(X, R^q f_*(\Theta_{X/Y})) = 0 \quad p + q = 2. \tag{3.3.2}$$

We have

$$R^q f_*(\Theta_{X/Y}) \cong \begin{cases} \text{End}^0 \mathcal{F}[n]^+ & \text{if } q = 0, \\ 0 & \text{if } q > 0. \end{cases} \tag{3.3.3}$$

It follows from the McKay correspondence (see Proposition 5.4 in [O’G22]) that

$$H^p(S^{[n]}, \text{End}^0 \mathcal{F}[n]^+) = 0 \quad \forall p, \tag{3.3.4}$$

and this finishes the proof. \square

Corollary 3.7. *Suppose that the K3 surface S is projective and that \mathcal{F} is a spherical vector bundle on S . If $n \leq 3$, then $\Delta(\mathcal{F}[n]^+)$ belongs to the saturation of $c_2(S^{[n]})$, if $n \geq 4$, then $\Delta(\mathcal{F}[n]^+)$ belongs to the saturation of the span of $c_2(S^{[n]})$ and q^\vee .*

Proof. Let $\mathcal{X} \xrightarrow{F} \mathcal{Y} \xrightarrow{G} T$ be representative of $\text{Def}(\mathbb{P}(\mathcal{F}[n]^+), f, S^{[n]})$. Thus both F and G are proper holomorphic maps of analytic spaces, there exists $0 \in T$, such that $F^{-1}(G^{-1}(0)) \rightarrow G^{-1}(0)$ is identified with $f: \mathbb{P}(\mathcal{F}[n]^+) \rightarrow S^{[n]}$, and every (small) deformation of f is identified with $F^{-1}(G^{-1}(t)) \rightarrow G^{-1}(t)$ for some $t \in T$ (close to 0). For $t \in T$ (close to 0), the map $F^{-1}(G^{-1}(t)) \rightarrow G^{-1}(t)$ is a \mathbb{P}^{r-1} fibration, where $r = \text{rk}(\mathcal{F}[n]^+)$, and hence, the pushforward $F_*(\Theta_{\mathcal{X}/\mathcal{Y}})$ is a vector bundle on \mathcal{Y} (of rank $r^2 - 1$). By Proposition 3.6, the family $G: \mathcal{Y} \rightarrow T$ is versal at $t = 0$, and hence, the characteristic class $c_2(F_*(\Theta_{X_0/Y_0}))$ (here, $X_0 = F^{-1}(G^{-1}(0))$ and $Y_0 = G^{-1}(0)$) remains of type (2,2) for all small deformation of $Y_0 = S^{[n]}$. If $n \leq 3$, it follows that $c_2(F_*(\Theta_{X_0/Y_0}))$ belongs to the saturation of $c_2(S^{[n]})$, and if $n \geq 4$, it follows that $c_2(F_*(\Theta_{X_0/Y_0}))$ belongs to the saturation of the span of $c_2(S^{[n]})$ and q^\vee (see [Zha15]). We are done because

$$c_2(F_*(\Theta_{X_0/Y_0})) = c_2(\text{End}^0 \mathcal{F}[n]^+) = -\Delta(\mathcal{F}[n]^+). \quad \square$$

Remark 3.8. Let $\text{Def}(S^{[n]}, \det \mathcal{F}[n]^+)$ be the deformation functor of the couple $(S^{[n]}, \det \mathcal{F}[n]^+)$. The natural map $\text{Def}(\mathcal{F}[n]^+) \rightarrow \text{Def}(S^{[n]}, c_1(\mathcal{F}[n]^+))$ is an isomorphism, by the Artamkin-Mukai theorem [Muk84, Art88] (see also [IM19]) and by the vanishing in (3.3.4). Hence, $\mathcal{F}[n]^+$ extends (uniquely) to a vector bundle on any small deformation of $S^{[n]}$ keeping $c_1(\mathcal{F}[n]^+)$ of type (1,1).

3.4. Proof of Proposition 3.2

The equality in (3.1.2) follows at once from the equality in (3.2.2). Similarly, the equality in (3.1.3) follows at once from the equality in (3.2.3), because the restriction map $H^2(S^{[n]}; \mathbb{Z}) \rightarrow H^2(\rho(X_n(S)_*))$ is an isomorphism (the complement of $\rho(X_n(S)_*)$ in $S^{[n]}$ has codimension greater than one). Lastly, we prove the equality in (3.1.4). Proposition 3.5 and a straightforward computation give that

$$\rho^* \Delta(\mathcal{F}[n]^+)_{|X_n(S)_*} = \rho^* \left(\frac{r_0^{2n-2}(r_0^2 - 1)}{12} c_2(S^{[n]}) \right)_{|X_n(S)_*}.$$

If $n \leq 3$, then by Proposition 3.7 (note: we may assume that S is projective), $\Delta(\mathcal{F}[n]^+)$ is a (possibly rational) multiple of $c_2(S^{[n]})$. Since the restriction of $\rho^* c_2(S^{[n]})$ to $X_n(S)_*$ is nonzero (by the equality in (2.1.4) and Lemma 2.3), the equality in (3.1.4) follows. If $n \geq 4$, then by Proposition 3.7 $\Delta(\mathcal{F}[n]^+)$ is a linear combination (possibly with rational coefficients) of $c_2(S^{[n]})$ and q^\vee , and the equality follows from Proposition 2.2.

4. Basic modular vector bundles on $S^{[n]}$ for S an elliptic $K3$ surface

4.1. Contents of the section

We show that by starting from slope-stable spherical vector bundles \mathcal{F} on an elliptic surface S , we can produce vector bundles $\mathcal{F}[n]^+$ on $S^{[n]}$ with rank and first two Chern classes covering all the cases in Theorem 1.1. We also study the restriction of such an $\mathcal{F}[n]^+$ to fibres of the Lagrangian fibration $S^{[n]} \rightarrow \mathbb{P}^n$.

4.2. Basic modular sheaves with the required topology

The present section contains analogues of the results in Sections 6.2 and 6.3 of [O’G22]. Let S be a $K3$ surface with an elliptic fibration $S \rightarrow \mathbb{P}^1$; we let C be a fibre of the elliptic fibration. The claim below follows from surjectivity of the period map for $K3$ surfaces.

Claim 4.1. *Let m_0, d_0 be positive natural numbers. There exist $K3$ surfaces S with an elliptic fibration $S \rightarrow \mathbb{P}^1$, such that*

$$\text{NS}(S) = \mathbb{Z}[D] \oplus \mathbb{Z}[C], \quad D \cdot D = 2m_0, \quad D \cdot C = d_0. \tag{4.2.1}$$

The following result is a (slight) extension of Proposition 6.2 in [O’G22] (and is more or less well-known by experts).

Proposition 4.2. *Let $m_0, r_0, s_0 \in \mathbb{N}_+$, and let $t, d_0 \in \mathbb{Z}$. Suppose that*

- (a) $t^2 m_0 = r_0 s_0 - 1$,
- (b) d_0 is coprime to r_0 ,
- (c) we have

$$d_0 > \frac{(2m_0 + 1)r_0^2(r_0^2 - 1)}{4}. \tag{4.2.2}$$

Let S be an elliptic $K3$ surface as in Claim 4.1. Then there exists a vector bundle \mathcal{F} on S , such that the following hold:

- (1) $v(\mathcal{F}) = (r_0, tD, s_0)$,
- (2) $\chi(S, \text{End}(\mathcal{F})) = 2$,
- (3) \mathcal{F} is L slope-stable for any polarization L of S ,
- (4) and the restriction of \mathcal{F} to every fibre of the elliptic fibration $S \rightarrow \mathbb{P}^1$ is slope-stable.

(Notice that every fibre is irreducible by our assumptions on $\text{NS}(S)$, hence, slope-stability of a sheaf on a fibre is well-defined, i.e. independent of the choice of a polarization.)

Proof. One proceeds, literally, as in the proof of Proposition 6.2 in [O’G22]. □

Assume that $n, r_0, g, l, e \in \mathbb{N}_+$, and that the equalities in (1.2.3), (1.2.4) and (1.2.5) (in Theorem 1.1) hold. Let

$$s_0 := \frac{g^2 \bar{e} + (2n - 2)(r_0 - 1)^2 + 8}{8r_0}, \quad m_0 := \frac{g^2 \bar{e} + 2(n - 1)(r_0 - 1)^2}{8g^2 l^2}. \tag{4.2.3}$$

Then s_0, m_0 are integers by the equalities in (1.2.4) and in (1.2.5). A straightforward computation gives that

$$(gl)^2 m_0 = r_0 s_0 - 1. \tag{4.2.4}$$

Let S be a $K3$ surface as in Claim 4.1, where m_0 is as in (4.2.3), and d_0 is an integer coprime to r_0 , such that the inequality in (4.2.2) holds. By Proposition 4.2, there exists a vector bundle \mathcal{F} on S , such that

$$v(\mathcal{F}) = (r_0, glD, s_0) \tag{4.2.5}$$

and Items (2)–(4) of that same proposition hold.

Claim 4.3. *Keep notation and hypotheses as above, in particular, \mathcal{F} is the vector bundle on S , such that the equation in (4.2.5) and Items (2)–(4) of Proposition 4.2 hold. Let $\mathcal{E} := \mathcal{F}[n]^+$. Then we have*

$$r(\mathcal{E}) = r_0^n, \quad c_1(\mathcal{E}) = \frac{g \cdot r_0^{n-1}}{i} h, \quad \Delta(\mathcal{E}) = \frac{r_0^{2n-2}(r_0^2 - 1)}{12} c_2(S^{[n]}), \tag{4.2.6}$$

where $h \in \text{NS}(S^{[n]})$ is primitive, $q(h) = e$ and $\text{div}(h) = il$.

Proof. Let

$$h := il\mu(\text{cl}(D) - i\frac{r_0 - 1}{2g}\delta_n). \tag{4.2.7}$$

Then h is integral by the hypothesis in (1.2.2), primitive by the third equality in (1.2.3), $q(h) = e$ by the second equality in (4.2.3) and $\text{div}(h) = il$. The equalities in (4.2.6) hold by Proposition 3.2. \square

4.3. Restriction of $\mathcal{F}[n]^+$ to Lagrangian fibres

Definition 4.4. Let $S \rightarrow \mathbb{P}^1$ be an elliptically fibred $K3$ surface. If $x \in \mathbb{P}^1$, we let C_x be the (scheme theoretic) elliptic fibre over x . Let $B = \{b_1, \dots, b_m\} \subset \mathbb{P}^1$ be the set of x , such that C_x is singular. Then B is not empty (generically, $m = 24$). The *Lagrangian fibration associated to $S \rightarrow \mathbb{P}^1$* is the map $\pi: S^{[n]} \rightarrow \mathbb{P}^n$ given by the composition

$$S^{[n]} \rightarrow S^{(n)} \rightarrow (\mathbb{P}^1)^{(n)} \cong \mathbb{P}^n. \tag{4.3.1}$$

We record a few facts regarding the (scheme theoretic) fibres of π . Let $x_1, \dots, x_n \in \mathbb{P}^1$ be pairwise distinct: then

$$\pi^{-1}(x_1 + \dots + x_n) \cong C_{x_1} \times \dots \times C_{x_n}. \tag{4.3.2}$$

Next, we describe the discriminant locus of $\pi: S^{[n]} \rightarrow (\mathbb{P}^1)^{(n)}$, that is the subset $\mathcal{D} \subset (\mathbb{P}^1)^{(n)}$ parametrizing cycles $x_1 + \dots + x_n$, such that $\pi^{-1}(x_1 + \dots + x_n)$ is singular. For $b_j \in B$, let $\mathcal{D}(b_j) \subset (\mathbb{P}^1)^{(n)}$ be the irreducible divisor parametrizing cycles $x_1 + \dots + x_n$, such that $x_i = b_j$ for some $i \in \{1, \dots, n\}$. Let

$$\mathcal{T} := \left\{ \sum_i m_i x_i \in (\mathbb{P}^1)^{(n)} \mid m_i \geq 2 \text{ for some } i \in \{1, \dots, n\} \right\}. \tag{4.3.3}$$

Note that the fibres of π over points of \mathcal{T} are reducible and nonreduced.

Proposition 4.5. *The irreducible decomposition of the discriminant locus \mathcal{D} of $\pi: S^{[n]} \rightarrow \mathbb{P}^n$ is given by*

$$\mathcal{D} = \mathcal{T} \cup \bigcup_{j=1}^m \mathcal{D}(b_j). \tag{4.3.4}$$

Below is the main result of the present subsection.

Proposition 4.6. *Let S be a $K3$ surface with an elliptic fibration $S \rightarrow \mathbb{P}^1$ as in Claim 4.1, and let $\pi: S^{[n]} \rightarrow \mathbb{P}^n$ be the associated Lagrangian fibration. Let \mathcal{F} be a vector bundle on S as in Proposition 4.2. Then the following hold:*

- (a) If $x_1, \dots, x_n \in \mathbb{P}^1$ are pairwise distinct, then the restriction of $\mathcal{F}[n]^+$ to $\pi^{-1}(x_1 + \dots + x_n)$ is slope-stable for any product polarization (this makes sense by the isomorphism in (4.3.2)).
- (b) Let $U \subset (\mathbb{P}^1)^{(n)}$ be the open subset parametrizing cycles $x_1 + \dots + x_n$, such that the restriction of $\mathcal{F}[n]^+$ to $\pi^{-1}(x_1 + \dots + x_n)$ is a simple sheaf. The complement of U has codimension at least 2.

Before proving Proposition 4.6, we notice that Proposition 6.10 in [O’G22] holds for products of projective varieties of arbitrary dimension.

Lemma 4.7. For $i \in \{1, 2\}$, let (X_i, L_i) be an irreducible polarized projective variety of dimension d_i , and let \mathcal{V}_i be a slope-stable vector bundle on X_i . Then $\mathcal{V}_1 \boxtimes \mathcal{V}_2$ is slope-stable for any product polarization $\mathcal{L} := L_1^{m_1} \boxtimes L_2^{m_2}$ (of course, $m_1, m_2 \in \mathbb{N}_+$).

Proof. Suppose that there exists an injection $\alpha: \mathcal{E} \rightarrow \mathcal{V}_1 \boxtimes \mathcal{V}_2$ with torsion-free cokernel, such that $0 < r(\mathcal{E}) < r(\mathcal{V}_1 \boxtimes \mathcal{V}_2)$ and

$$\mu_{\mathcal{L}}(\mathcal{E}) \geq \mu_{\mathcal{L}}(\mathcal{V}_1 \boxtimes \mathcal{V}_2). \tag{4.3.5}$$

The open subset $U \subset X_1 \times X_2$ of points p at which α is an injection of vector bundles (i.e. the stalk of \mathcal{E} at p is free and α defines an injection of the fibre of \mathcal{E} at p to the fibre of $\mathcal{V}_1 \boxtimes \mathcal{V}_2$ at p) has the complement of codimension at least 2. Let $p = (x_1, x_2) \in U$. The restrictions of α to $\{x_1\} \times X_2$ and to $X_1 \times \{x_2\}$ are generically injective maps of vector bundles. Let

$$A_1 := m_1^{d_1-1} m_2^{d_2} \deg X_2 \binom{d_1 + d_2 - 1}{d_2}, \quad A_2 := m_1^{d_1} m_2^{d_2-1} \deg X_1 \binom{d_1 + d_2 - 1}{d_1}.$$

We have

$$\mu_{\mathcal{L}}(\mathcal{E}) = A_1 \mu_{L_1}(\mathcal{E}|_{X_1 \times \{x_2\}}) + A_2 \mu_{L_2}(\mathcal{E}|_{\{x_1\} \times X_2}), \tag{4.3.6}$$

and

$$\mu_{\mathcal{L}}(\mathcal{V}_1 \boxtimes \mathcal{V}_2) = A_1 \mu_{L_1}(\mathcal{V}_1) + A_2 \mu_{L_2}(\mathcal{V}_2). \tag{4.3.7}$$

Since the restrictions of $\mathcal{V}_1 \boxtimes \mathcal{V}_2$ to $X_1 \times \{x_2\}$ and to $\{x_1\} \times X_2$ are isomorphic to the polystable vector bundles $\mathcal{V}_1 \otimes_{\mathbb{C}} \mathbb{C}^{r(\mathcal{V}_2)}$ and $\mathcal{V}_1 \otimes_{\mathbb{C}} \mathbb{C}^{r(\mathcal{V}_1)}$, respectively, it follows from (4.3.5), (4.3.6) and (4.3.7) that $\mu(\mathcal{E}|_{X_1 \times \{x_2\}}) = \mu(\mathcal{V}_1)$ and $\mu(\mathcal{E}|_{\{x_1\} \times X_2}) = \mu(\mathcal{V}_2)$. In turn, these equalities give that there exist vector subspaces $0 \neq W_i \subset \mathbb{C}^{r(\mathcal{V}_i)}$, such that $\mathcal{E}|_{X_1 \times \{x_2\}} = \mathcal{V}_1 \otimes_{\mathbb{C}} W_2$ on $U \cap (X_1 \times \{x_2\})$ and $\mathcal{E}|_{\{x_1\} \times X_2} = W_1 \otimes_{\mathbb{C}} \mathcal{V}_2$ on $U \cap (\{x_1\} \times X_2)$. It follows that $\text{Im } \alpha|_U = (\mathcal{V}_1 \boxtimes \mathcal{V}_2)|_U$. This is a contradiction. \square

Proof of Proposition 4.6. (a): Follows from the stability of the restriction of \mathcal{F} to any elliptic fibre of $S \rightarrow \mathbb{P}^1$ (Proposition 4.2), and Lemma 4.7.

(b): Let $V \subset (\mathbb{P}^1)^{(n)}$ be the open subset parametrizing cycles $\Gamma := d_1 x_1 + \dots + d_m x_m$, such that $d_j \leq 2$ for all $j \in \{1, \dots, m\}$, and C_{x_j} is smooth if $d_j = 2$. If Γ is such a cycle, then the restriction of $\mathcal{F}[n]^+$ to $\pi^{-1}(\Gamma)$ is a simple sheaf. In fact, $\pi^{-1}(\Gamma)$ is a product of schemes $\mathbf{C}_1 \times \dots \times \mathbf{C}_m$, where $\mathbf{C}_j = C_{x_j}$ if $d_j = 1$, while if $d_j = 2$, then \mathbf{C}_j is identified with

the scheme theoretic fibre over $2x_j \in (\mathbb{P}^1)^{(2)}$ of the Lagrangian fibration $S^{[2]} \rightarrow (\mathbb{P}^1)^{(2)}$. Moreover

$$\mathcal{F}[n]_{|\pi^{-1}(\Gamma)}^+ \cong (\mathcal{F}|_{C_1}) \boxtimes \dots \boxtimes (\mathcal{F}|_{C_m}). \tag{4.3.8}$$

If $d_j = 1$, then $C_j = C_{x_j}$ and $\mathcal{F}|_{C_j}$ is simple by Proposition 4.2. If $d_j = 2$, then C_j is the scheme considered in [O’G22, Section 6.4], and $\mathcal{F}|_{C_j}$ is simple by [O’G22, Proposition 6.13]. By the isomorphism in (4.3.8), it follows that $\mathcal{F}[n]_{|\pi^{-1}(\Gamma)}^+$ is simple. Since the complement of V in $(\mathbb{P}^1)^{(n)}$ is a (closed) subset of codimension at least 2, this proves Item (b). \square

Remark 4.8. By Item (a) of Proposition 4.6, the restriction of $\mathcal{F}[n]^+$ to a (singular) Lagrangian fibre X_t parametrized by a general point $t \in \mathcal{D}(b_j)$ (notation as in (4.3.4)) is slope-stable for any product polarization.

The following remarks place Item (a) of Proposition 4.6 in the context of known results.

Remark 4.9. Let $X \rightarrow \mathbb{P}^n$ be a Lagrangian fibration of an HK manifold. For $t \in \mathbb{P}^n$, let $X_t := \pi^{-1}(t)$ be the schematic fibre of $X \rightarrow \mathbb{P}^n$ over t . If X_t is smooth, there exists an ample primitive class $\theta_t \in H_{\mathbb{Z}}^{1,1}(X_t)$, such that the image of the restriction map $H^2(X; \mathbb{Z}) \rightarrow H^2(X_t; \mathbb{Z})$ is contained in $\mathbb{Z}\theta_t$ (see [Wie16]). If \mathcal{F} is a sheaf on X_t , slope-(semi)stability of \mathcal{F} will always mean θ_t slope-(semi)stability.

Remark 4.10. Let $X \rightarrow \mathbb{P}^n$ be a Lagrangian fibration of an HK manifold of type $K3^{[n]}$, and let X_t be a smooth Lagrangian fibre. Then the primitive ample class $\theta_t \in H_{\mathbb{Z}}^{1,1}(X_t)$ is a principal polarization of X_t (see [Wie16]). If $S^{[n]} \rightarrow \mathbb{P}^n$ is the Lagrangian fibration in (4.3.1), and $\pi^{-1}(x_1 + \dots + x_n) \cong C_{x_1} \times \dots \times C_{x_n}$ is a smooth Lagrangian fibre, then $\theta_{x_1 + \dots + x_n}$ is the product principal polarization.

5. Proof of Theorem 1.1 and Proposition 1.2

5.1. Contents of the section

In the present section, we prove the following two statements.

Proposition 5.1. *Let n, r_0, g, l, e, i be as in Theorem 1.1. There exists an irreducible component $\mathcal{X}_e^{il}(2n)^{\text{good}}$ of $\mathcal{X}_e^{il}(2n)$, such that for a general polarized HK variety (X, h) parametrized by $\mathcal{X}_e^{il}(2n)^{\text{good}}$, there exists an h slope-stable vector bundle \mathcal{E} on X , such that the equalities in (1.2.6) hold, and moreover, $H^p(X, \text{End}^0(\mathcal{E})) = 0$ for all p .*

Proposition 5.2. *Let n, r_0, g, l, e, i be as in Theorem 1.1. If $[(X, h)] \in \mathcal{X}_e^{il}(2n)^{\text{good}}$ is a general point, then there exists a unique h slope-stable vector bundle \mathcal{E} on X , such that the equalities in (1.2.6) hold.*

The same exact argument given in the ‘‘Proof of Theorem 1.4’’ on p. 30 of [O’G22] shows that Theorem 1.1 follows from Propositions 5.1 and 5.2.

In Sections 5.2 and 5.3, we prove results that are used in the proof of Propositions 5.1 and 5.2. Proposition 5.1 is proved in Section 5.4. Sections 5.5 and 5.6 contain results that

are used in the proof of Proposition 5.2. Propositions 5.2 and 1.2 are proved in Sections 5.7 and 5.8, respectively.

5.2. The relevant component of $\mathcal{K}_e^{il}(2n)$, and Noether-Lefschetz divisors

Let S be an elliptic $K3$ surface as in Claim 4.3, and let C, D be divisor classes generating $NS(S)$ as in loc. cit. Let $X_0 = S^{[n]}$, and let

$$X_0 \xrightarrow{\pi_0} (\mathbb{P}^1)^{(n)} = \mathbb{P}^n \tag{5.2.1}$$

be the Lagrangian fibration associated to the elliptic fibration of S (see (4.3.1)). Let

$$h_0 := i\mu(\text{cl}(D)) - i\frac{r_0 - 1}{2g}\delta_n, \quad f_0 := \mu(\text{cl}(C)) = c_1(\pi_0^* \mathcal{O}_{\mathbb{P}^n}(1)). \tag{5.2.2}$$

Definition 5.3. Let $\mathcal{K}_e^{il}(2n)^{\text{good}} \subset \mathcal{K}_e^{il}(2n)$ be the irreducible component containing $[(S^{[n]}, h_0)]$.

Let $d_0 = C \cdot D$ (see (4.2.1)). The sublattice $\langle f_0, h_0 \rangle \subset H_{\mathbb{Z}}^{1,1}(X_0)$ is saturated and

$$q(f_0) = 0, \quad q(h_0, f_0) = i\text{ld}_0, \quad q(h_0) = e \tag{5.2.3}$$

(the last equality follows from (4.2.3)). Let L_0, F_0 be the line bundles on X_0 , such that $c_1(L_0) = h_0$ and $c_1(F_0) = f_0$.

Definition 5.4. Let $\varphi: \mathcal{X} \rightarrow B$ be an (analytic) contractible representative of the functor $\text{Def}(X_0, L_0, F_0)$. Let $0 \in B$ be the base point, so that $X_0 = \varphi^{-1}(0) \cong S^{[n]}$. For $b \in B$, we let $X_b := \varphi^{-1}(b)$, and we let L_b, F_b be the line bundles on X_b which are deformations of L_0, F_0 , respectively. We let $h_b := c_1(L_b)$ and $f_b := c_1(F_b)$.

Our first observation is that if d_0 is large enough, then h_b is ample for a general $b \in B$. Before proving this, we recall the following elementary result.

Lemma 5.5 (Lemma 4.3 in [O’G22]). *Let (Λ, q) be a nondegenerate rank 2 lattice which represents 0, and hence, $\text{disc}(\Lambda) = -d^2$, where d is a strictly positive integer. Let $\alpha \in \Lambda$ be primitive isotropic, and complete it to a basis $\{\alpha, \beta\}$, such that $q(\beta) \geq 0$. If $\gamma \in \Lambda$ has strictly negative square (i.e. $q(\gamma) < 0$), then*

$$q(\gamma) \leq -\frac{2d}{1+q(\beta)}. \tag{5.2.4}$$

Proposition 5.6. *Keep notation as above, and assume that*

$$i\text{ld}_0 > (n - 1)^2(n + 3)(e + 1). \tag{5.2.5}$$

Then L_b is ample on X_b for a general $b \in B$.

Proof. Let $b \in B$ be a very general point, in the sense that $NS(X_b) = \langle h_b, f_b \rangle$. By the inequality in (5.2.5) and Lemma 5.5, there are no $\xi \in NS(X_b)$, such that $-2(n - 1)^2(n + 3) \leq q(\xi) < 0$. By [Mon15, Corollary 2.7], it follows that the ample cone of X_b is equal to the intersection of $NS(X_b)$ and the positive cone (if R is the integral generator of an extremal ray, then, viewed by duality as an element of $H^2(K3^{[n]}, \mathbb{Q})$, the multiple

$2(n-1)R$ is integral because the divisibility of any element of $H^2(X_b, \mathbb{Z})$ is a divisor of $2n-2$). Hence, either h_b or $-h_b$ is ample. By considering the limit case $b=0$, we get that h_b is ample. This proves that h_b is ample for b very general. Since h_b is ample for b in the complement of an analytic subset of B , we are done. \square

Assume that the inequality in (5.2.5) holds. By Proposition 5.6, we have the moduli map

$$\begin{aligned} B &\longrightarrow \mathcal{H}_e^{il}(2n) \\ b &\longmapsto [(X_b, h_b)]. \end{aligned} \tag{5.2.6}$$

Note that the image of the above period map is contained in a unique (irreducible) Noether-Lefschetz divisor in $\mathcal{H}_e^{il}(2n)^{\text{good}}$.

Definition 5.7. If the inequality in (5.2.5) holds, we let $NL(d_0) \subset \mathcal{H}_e^{il}(2n)^{\text{good}}$ be the unique irreducible Noether-Lefschetz divisor containing the image of the moduli map in (5.2.6).

Remark 5.8. Let $[(X, h)] \in NL(d_0)$ be a general point. Then there exists a well-defined rank two subspace $V \subset \text{NS}(X)$, such that $V = \langle h, f \rangle$, where

$$q(h, f) = id_0, \quad q(f) = 0 \tag{5.2.7}$$

(for $[(X, h)]$ in a proper Zariski closed subset of $NL(d_0)$, there might be more than one such rank two subspace). For almost all choices (of n, r_0, g, l, e, i and d_0), there is a unique class $f \in V$, such that $V = \langle h, f \rangle$ and the equalities in (5.2.7) hold, while for special choices, there are two such classes. If monodromy exchanges these two isotropic classes, there is no intrinsic way of distinguishing them. Abusing language, we will speak of “the class f ”. If $(X, h) = (X_0, h_0)$ (recall that $X_0 = S^{[n]}$, where S is our elliptic $K3$ surface), then $f_0 = c_1(\pi_0^* \mathcal{O}_{\mathbb{P}^n}(1))$, where π_0 is the Lagrangian fibration given in (5.2.1). By [Mat17, Theorem 1.2], it follows that if $[(X, h)] \in NL(d_0)$ is a general point, then there exists a Lagrangian fibration $\pi_X: \mathcal{X} \rightarrow \mathbb{P}^n$, such that $f = c_1(\pi_X^* \mathcal{O}_{\mathbb{P}^n}(1))$.

Proposition 5.9. *Keep notation as above, and assume that the inequality in (5.2.5) holds, and that d_0 is coprime to r_0 . Let $NL(d_0) \subset \mathcal{H}_e^{il}(2n)^{\text{good}}$ be the Noether-Lefschetz divisor of Definition 5.7. There exist an open dense $NL(d_0)^* \subset NL(d_0)$, and for each $[(X, h)] \in NL(d_0)^*$, a vector bundle \mathcal{E}_X on X , such that*

$$r(\mathcal{E}_X) = r_0^n, \quad c_1(\mathcal{E}_X) = \frac{g \cdot r_0^{n-1}}{i} h, \quad \Delta(\mathcal{E}_X) = \frac{r_0^{2n-2}(r_0^2-1)}{12} c_2(X), \tag{5.2.8}$$

$H^p(X, \text{End}^0(\mathcal{E}_X)) = 0$ for all p , and the restriction of \mathcal{E}_X to a general smooth fibre of the Lagrangian fibration $X \rightarrow \mathbb{P}^n$ (see Remark 5.8) is slope-stable.

Proof. Let $\mathcal{E}_0 := \mathcal{F}[n]^+$ be the vector bundle on X_0 of Claim 4.3. Note that $c_1(\mathcal{E}_0) = h_0$ by loc.cit. If B is small enough, then by Remark 3.8, the vector bundle \mathcal{E}_0 on X_0 deforms uniquely to a vector bundle \mathcal{E}_b on X_b , and hence, we get a vector bundle \mathcal{E}_X on X for $[(X, h)]$ in a dense open subset $\mathcal{U} \subset NL(d_0)$. The equations in (5.2.8) hold by the equations in (4.2.6). Since $H^p(X, \text{End}^0(\mathcal{E}_0)) = 0$ for all p (see (3.3.4)), it follows from upper

semicontinuity of the dimension of cohomology sheaves that for $[(X, h)]$ in a smaller dense open subset $\mathcal{U}' \subset \mathcal{U}$, we have $H^p(X, \text{End}^0(\mathcal{E}_X)) = 0$ for all p . Lastly, it follows from Item (a) of Proposition 4.6 and Remark 4.10 that for $[(X, h)]$ in a smaller dense open subset $\mathcal{U}'' \subset \mathcal{U}'$, the restriction of \mathcal{E}_X to a general fibre of the Lagrangian fibration $X \rightarrow \mathbb{P}^n$ (see Remark 5.8) is slope-stable. We set $NL(d_0)^* := \mathcal{U}''$. \square

5.3. Suitable polarizations

We recall that if h is a -suitable (see [O’G22, Definition 3.5]) and \mathcal{E} is a vector bundle on X with $a(\mathcal{E}) \leq a$ (see (3.1.1) loc.cit. for the definition of $a(\mathcal{E})$), then slope stability of the restriction of \mathcal{E} to a general Lagrangian fibre (there is a canonical choice of polarization of any smooth Lagrangian fibre, see Remark 4.9) implies slope stability of \mathcal{E} , and the following weak converse holds: slope stability of \mathcal{E} implies that the restriction of \mathcal{E} to a general Lagrangian fibre is slope semistable.

Lemma 5.10. *Keep assumptions and notation as above, and let $a > 0$. Suppose that*

$$ild_0 > a(e + 1). \tag{5.3.1}$$

Let $[(X, h)] \in NL(d_0)$ be a general point, and let $f \in V \subset \text{NS}(X)$ be as in Remark 5.8. Then there does not exist $\xi \in \text{NS}(X)$, such that

$$-a \leq q_X(\xi) < 0, \quad q_X(\xi, h) > 0, \quad q_X(\xi, f) > 0. \tag{5.3.2}$$

Proof. Let $\langle h, f \rangle = V \subset \text{NS}(X)$ be as in Remark 5.8. Applying Lemma 5.5 to $\Lambda := V$, $\alpha = f$ and $\beta = h$, one gets that there are no $\xi \in V$, such that $-a \leq q_X(\xi) < 0$. In particular, if $\text{NS}(X) = \langle h, f \rangle$ (as is the case for very general $[(X, h)] \in NL(d_0)$), then there is no $\xi \in V$, such that the inequalities in (5.3.2) hold.

It follows that the set of $[(X, h)] \in NL(d_0)$ for which *there exists* $\xi \in \text{NS}(X)$, such that the inequalities in (5.3.2) hold, is the intersection of $NL(d_0)$ with a finite union of Noether-Lefschetz divisors in $\mathcal{K}_e^{il}(2n)$, each of which does not contain $NL(d_0)$. In fact, suppose that the inequalities in (5.3.2) hold. Let D be the (finite) index of $\langle h, f \rangle \oplus (\langle h, f \rangle^\perp \cap \text{NS}(X))$ in $\text{NS}(X)$. It is crucial to note that D has an upper bound which only depends on the discriminant of the restrictions of q_X to V (i.e. $-(ild_0)^2$) and to V^\perp , and the latter has an upper bound depending only on the discriminant of V and the discriminant of q_X (i.e. $2n - 2$). Then

$$\xi = \frac{\xi_1}{D} + \frac{\xi_2}{D}, \quad \xi_1 \in \langle h, f \rangle, \quad \xi_2 \in \langle h, f \rangle^\perp. \tag{5.3.3}$$

Moreover, we have just proved that ξ_2 is nonzero. Since the restriction of q_X to $h^\perp \cap \text{NS}(X)$ is negative definite, we get that $q(\xi_2) < 0$. We also have $q(\xi_1) < 0$ by the last two inequalities in (5.3.2).

Hence, by the first inequality in (5.3.2), there exists a positive M independent of (X, h) , such that $-M \leq q(\xi_2) < 0$ (here, it is crucial that D has an upper bound which only depends on $(ild_0)^2$ and $2n - 2$). Hence, the moduli point of (X, h) belongs to the intersection of $NL(d_0)$ with a finite union of Noether-Lefschetz divisors in $\mathcal{K}_e^{il}(2n)$, and

none of them contains $NL(d_0)$ because if $\rho(X) = 2$, then $[(X, h)]$ is not contained in any of these Noether-Lefschetz divisors. \square

Proposition 5.11. *Keep notation as above, and assume that*

$$ild_0 > \frac{r_0^{4n-2}(r_0^2 - 1)(n + 3)(e + 1)}{8}. \tag{5.3.4}$$

Let $[(X, h)] \in NL(d_0)$ (the inequality in (5.3.4) implies that the inequality in (5.2.5) holds, and hence, the Noether-Lefschetz divisor $NL(d_0) \subset \mathcal{K}_e^{il}(2n)^{\text{good}}$ is defined). If \mathcal{E} is a vector bundle on X , such that the equalities in (1.2.6) hold, then h is $a(\mathcal{E})$ -suitable (relative to the associated Lagrangian fibration $\pi: \mathcal{X} \rightarrow \mathbb{P}^n$).

Proof. If $\alpha \in H^2(X)$, we have

$$\int_X c_2(X) \cdot \alpha^{2n-2} = 6(n + 3)(2n - 3)!! q_X(a)^{n-1},$$

(the equality above can be obtained from the known Hirzebruch-Huybrechts-Riemann-Roch formula for HK manifolds of Type $K3^{[n]}$) gives that $d(\mathcal{E}_X) = r_0^{2n-2}(r_0^2 - 1)(n + 3)/2$ (the definition of $d(\mathcal{E}_X)$ is in [O’G22, (1.2.2)]), and hence

$$a(\mathcal{E}_X) = \frac{r_0^{4n-2}(r_0^2 - 1)(n + 3)}{8}.$$

The inequality in (5.3.4) and Lemma 5.10 give that h is $a(\mathcal{E})$ -suitable. \square

5.4. Proof of Proposition 5.1

Keep notation and assumptions as above, and assume, in addition, that d_0 is coprime to r_0 and that the inequality in (5.3.4) holds (note that the set of such d_0 is infinite). Let $[(X, h)] \in NL(d_0)^*$, and let \mathcal{E}_X be a vector bundle on X as in Proposition 5.9. By Proposition 5.11, the polarization h is $a(\mathcal{E}_X)$ -suitable relative to the Lagrangian fibration $\pi_X: \mathcal{X} \rightarrow \mathbb{P}^n$. By Proposition 5.9, the restriction of \mathcal{E}_X to a general fibre of $\pi_X: \mathcal{X} \rightarrow \mathbb{P}^n$ is slope-stable. Since h is $a(\mathcal{E}_X)$ -suitable, the vector bundle \mathcal{E}_X is h slope-stable by [O’G22, Proposition 3.6]. We have $H^p(X, \text{End}^0(\mathcal{E}_X)) = 0$ for all p , and hence, \mathcal{E}_X extends (uniquely) to all small deformations of $(X, \det \mathcal{E}_X)$ by Remark 3.8. Since $c_1(\mathcal{E}_X)$ is a multiple of h , we get that \mathcal{E}_X extends to a vector bundle \mathcal{E}' on a general deformation (X', h') of (X, h) . By openness of slope stability, \mathcal{E}' is slope-stable, and by upper semicontinuity of cohomology dimension, $H^p(X, \text{End}^0(\mathcal{E}')) = 0$ for all p .

5.5. Tate-Shafarevich twists

A basic example of Lagrangian fibration is obtained as follows. Let (S, h_S) be a polarized $K3$ surface of genus n . Let $\mathcal{J}(S)$ be the moduli space of rank 0 pure $\mathcal{O}_S(1)$ semistable sheaves ξ with $\chi(\xi) = 1 - n$, that is sheaves with Mukai vector $(0, h_S, 1 - n)$. The generic point of $\mathcal{J}(S)$ is represented by $i_*\mathcal{L}$, where $i: C \hookrightarrow S$ is the inclusion of a smooth $C \in \mathcal{O}_S(1)$, and \mathcal{L} is a line bundle of degree 0. Suppose that all divisors in the complete linear system $|\mathcal{O}_S(1)|$ are irreducible and reduced. Then every semistable sheaf parametrized by

$\mathcal{J}(S)$ is stable, and $\mathcal{J}(S)$ is an HK projective variety of Type $K3^{[n]}$. Moreover, the support map $\mathcal{J}(S) \rightarrow |\mathcal{O}_S(1)| \cong \mathbb{P}^n$ is a Lagrangian fibration. Let $NL(d_0) \subset \mathcal{H}_e^{il}(2n)^{\text{good}}$ be the Noether-Lefschetz divisor of Definition 5.7, and let $[(X, h)] \in NL(d_0)$ be a general point. Then the associated Lagrangian fibration $\pi: X \rightarrow \mathbb{P}^n$ is related to a (general) moduli space $\mathcal{J}(S)$ via a Tate-Shafarevich twist. In order to be more precise, we recall a result of Markman. First, if $[(X, h)] \in NL(d_0)$ is a general point, then there is an associated polarized $K3$ surface (S, D) of genus n , and moreover, (S, D) is a general such polarized surface (see [Mar14, Section 4.1]).

Proposition 5.12. *Keep notation as above, and assume that the inequality in (5.2.5) holds. Let $[(X, h)] \in NL(d_0)$ be a general point, let $X \rightarrow \mathbb{P}^n$ be the associated Lagrangian fibration and let (S, D) be the associated polarized $K3$ surface (which is a general polarized $K3$ surface of genus n). Then $X \rightarrow \mathbb{P}^n$ is isomorphic to a Tate-Shafarevich twist of $\mathcal{J}(S) \rightarrow |D|$ via an identification $\mathbb{P}^n \xrightarrow{\sim} |D|$.*

Proof. Suppose first that $\rho(X) = 2$. Then, as shown in the proof of Proposition 5.6, the ample cone of X is equal to the positive cone (because of the inequality in (5.2.5)), and hence, every bimeromorphic map $X \dashrightarrow X'$, where X' is an HK , is actually an isomorphism. It follows that X is isomorphic to a Tate-Shafarevich twist of $\mathcal{J}(S) \rightarrow |D|$ by Theorem 7.13 in [Mar14]. The result follows from this because the locus in $NL(d_0)$ parametrizing (X, h) , such that $\rho(X) = 2$ is dense. \square

Let $\text{Pic}^0(X/\mathbb{P}^n)$ be the relative Picard scheme of the Lagrangian fibration $X \rightarrow \mathbb{P}^n$ (notice that all fibres of $X \rightarrow \mathbb{P}^n$ are irreducible by Proposition 5.12). Let $U \subset \mathbb{P}^n$ be the open dense set of regular values of $X \rightarrow \mathbb{P}^n$. If $t \in U$, the fibre of $\text{Pic}^0(X/\mathbb{P}^n) \rightarrow \mathbb{P}^n$ over t is an Abelian variety A_t (of dimension n) and the fundamental group $\pi_1(U, t)$ acts by monodromy on the subgroup $A_{t, \text{tors}}$ of torsion points.

Corollary 5.13. *Keep hypotheses and notation as above, and suppose that $V \subset A_t[r_0^n]$ is a coset (of a subgroup of $A_t[r_0^n]$) of cardinality r_0^{2n} invariant under the action of monodromy. Then $V = A_t[r_0]$.*

Proof. Let (S, D) be the polarized $K3$ surface of genus n associated to X following Markman. Let $\mathcal{J}(S)_0 \subset \mathcal{J}(S)$ be the open dense subset of smooth points (i.e. smooth points of $\mathcal{J}(S)$ with surjective differential) of the map $\mathcal{J}(S) \rightarrow |D|$. By Proposition 5.12, $\text{Pic}^0(X/\mathbb{P}^n) \rightarrow \mathbb{P}^n$ is isomorphic to $\mathcal{J}(S)_0 \rightarrow |D|$, for a certain identification $\mathbb{P}^n \xrightarrow{\sim} |D|$. Under this identification, $t \in \mathbb{P}^n$ corresponds to a smooth curve $C \in |D|$, and the corresponding Lagrangian fibre A_t is the Jacobian of C . Hence, we have a natural isomorphism

$$H_1(C; \mathbb{Q})/H_1(C; \mathbb{Z}) \xrightarrow{\sim} A_{t, \text{tors}}, \tag{5.5.1}$$

and the identification is compatible with the monodromy actions.

First, we prove the result under the assumption that V is a subgroup G . By the structure theorem for finite Abelian groups, $G \cong \mathbb{Z}/(d_1) \oplus \dots \oplus \mathbb{Z}/(d_r)$, where $r \leq 2n$ (because $A_t[r_0^n] \cong \mathbb{Z}/(r_0^n)^{\oplus 2n}$) and $d_i | r_0^n$ for all i . The monodromy action on $H_1(C; \mathbb{Z})$ is transitive on nonzero elements by [FMOS22, Proposition 4.4] (if $n \geq 3$, if $n = 2$ transitivity is proved

by hand). It follows from the isomorphism in (5.5.1) that $r = 2n$ and $d_1 = \dots = d_r$. Thus, $d_i = r_0$ for all $i \in \{1, \dots, 2n\}$ because $|G| = r_0^{2n}$. This proves the result under the assumption that V is a subgroup. Now let V be a translate of a group G . Then $G = \{a - b \mid a, b \in V\}$, and hence, G is also invariant for the monodromy action. Thus, $G = A_t[r_0]$ by what we have just proved, and the coset V gives a point of the quotient $A_t[r_0^{2n}]/A_t[r_0^n] \cong A_t[r_0^n]$ which is invariant for the monodromy action. There is a unique invariant element, namely 0, because of the isomorphism in (5.5.1) and the transitivity of the monodromy action on nonzero elements of $H_1(C; \mathbb{Z})$. Hence, $V = A_t[r_0]$. \square

5.6. More properties of \mathcal{E}_X for $[(X, h)] \in NL(d_0)^*$ a general point

The main result of the present subsection is the following improved version of Proposition 5.9.

Proposition 5.14. *Keep notation as in Section 5.2, and assume that d_0 is coprime to r_0 , and that the inequality in (5.3.4) holds. There exist an open dense subset $NL(d_0)^{**} \subset NL(d_0)^*$, and for each $[(X, h)] \in NL(d_0)^{**}$, an h slope-stable vector bundle \mathcal{E}_X on X , such that the following hold:*

- (1) *The equalities in (5.2.8) hold.*
- (2) *$H^p(X, \text{End}^0(\mathcal{E}_X)) = 0$ for all p .*
- (3) *There exists an open $\mathcal{U}_X \subset \mathbb{P}^n$, whose complement has codimension at least 2, such that for every $t \in \mathcal{U}_X$, the restriction of \mathcal{E}_X to the fibre X_t over t of the Lagrangian fibration $X \rightarrow \mathbb{P}^n$ is slope-stable for the restriction of h to X_t .*

Before proving Proposition 5.14, we need to go through a couple of results. The result below for Abelian surfaces is [O’G22, Proposition 4.4]. Part of the proof below is literally taken from the proof of Proposition 4.4 in [O’G22], but there is a crucial extra input (not needed in dimension 2), namely [Sim92, Theorem 2].

Proposition 5.15. *Let (A, θ) be a principally polarized Abelian variety of dimension n , and let \mathcal{F} be a slope semistable vector bundle on A , such that $c_1(\mathcal{F})$ is a multiple of θ and $\Delta(\mathcal{F}) \cdot \theta^{n-2} = 0$. Then there exist integers r_0, m, b_0 with $r_0, m \in \mathbb{N}_+$ and $\text{gcd}\{r_0, b_0\} = 1$, such that*

$$r(\mathcal{F}) = r_0^n m, \quad c_1(\mathcal{F}) = r_0^{n-1} b_0 m \theta. \tag{5.6.1}$$

If \mathcal{F} is not θ slope-stable, then we may assume that $m > 1$.

Proof. If \mathcal{F} is slope-stable, then it is simple semihomogeneous by [O’G22, Proposition A.2], and hence, we may write (5.6.1) with $m = 1$ by [O’G22, Proposition A.3].

Suppose that \mathcal{F} is strictly θ slope-semistable, that is there exists a destabilizing exact sequence of torsion-free sheaves

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{H} \longrightarrow 0 \tag{5.6.2}$$

with \mathcal{G} slope-stable. Arguing as in the proof of [O’G22, Proposition 4.4], one proves that

$$c_1(\mathcal{G}) = a\theta, \quad c_1(\mathcal{H}) = b\theta, \quad \Delta(\mathcal{G}) \cdot \theta^{n-2} = \Delta(\mathcal{H}) \cdot \theta^{n-2} = 0. \tag{5.6.3}$$

Let us prove that \mathcal{G} and \mathcal{H} are locally free. Let $r := r(\mathcal{F})$, and let $m_r: A \rightarrow A$ be the multiplication by r map. Let \mathcal{L} be a line bundle on A , such that $c_1(\mathcal{L}) = -ra\theta$, and let $\mathcal{E} := m_r^*(\mathcal{F}) \otimes \mathcal{L}$. Then \mathcal{E} is slope-semistable (because \mathcal{F} is) and

$$c_1(\mathcal{E}) = 0, \quad \Delta(\mathcal{E}) \cdot \theta^{n-2} = 0.$$

By [Sim92, Theorem 2], it follows that every quotient of the (slope) Jordan-Hölder filtration of \mathcal{E} is locally free. Since $m_r^*(\mathcal{G}) \otimes \mathcal{L}$ is a polystable subsheaf of \mathcal{E} , we get that it is locally free. Thus, $m_r^*(\mathcal{G})$ is locally free, and hence, also \mathcal{G} is locally free. By the equalities in (5.6.3), we may iterate this argument to show that also \mathcal{H} is locally free.

It follows that if

$$0 = \mathcal{G}_0 \subsetneq \mathcal{G}_1 \subsetneq \dots \subsetneq \mathcal{G}_m = \mathcal{F}$$

is a (slope) Jordan-Hölder filtration of \mathcal{F} , then each quotient $\mathcal{Q}_i := \mathcal{G}_i/\mathcal{G}_{i-1}$ is a slope-stable locally free sheaf with

$$\frac{c_1(\mathcal{Q}_i)}{\text{rk}(\mathcal{Q}_i)} = \frac{c_1(\mathcal{F})}{r(\mathcal{F})}, \quad \Delta(\mathcal{Q}_i) \cdot \theta^{n-2} = 0. \tag{5.6.4}$$

Hence, each \mathcal{Q}_i is simple semihomogeneous by [O’G22, Proposition A.2], and therefore, by [O’G22, Proposition A.3] (see also [Muk78, Remark 7.13]), there exist coprime integers r_i, b_i , with $r_i > 0$, such that $r(\mathcal{Q}_i) = r_i^n$ and $c_1(\mathcal{Q}_i) = r_i^{n-1}b_i\theta$. Let $i, j \in \{1, \dots, m\}$; since the slopes of \mathcal{Q}_i and \mathcal{Q}_j are equal, we get that $b_i r_j = b_j r_i$. It follows that $r_i = r_j$ and $b_i = b_j$ because $\text{gcd}\{r_i, b_i\} = \text{gcd}\{r_j, b_j\} = 1$. Thus, $r(\mathcal{F}) = mr_0^n$ and $c_1(\mathcal{F}) = mr_0^{n-1}b_0\theta$, where $r_0 = r_i$ and $b_0 = b_i$ for all $i \in \{1, \dots, m\}$. We have $m \geq 2$ because we assumed that \mathcal{F} is strictly slope-semistable. \square

Corollary 5.16. *Let (A, θ) be a principally polarized Abelian variety of dimension n , and let \mathcal{F} be a θ slope-semistable vector bundle on A , such that $\Delta(\mathcal{F}) \cdot \theta^{n-2} = 0$. If $r(\mathcal{F}) = r_0^n$ and $c_1(\mathcal{F}) = r_0^{n-1}b_0\theta$, where r_0, b_0 are coprime integers, then \mathcal{F} is θ slope-stable.*

Proof. By contradiction, suppose that \mathcal{F} is not θ slope-stable. By Proposition 5.15, we may write $r(\mathcal{F}) = s_0^n m$, $c_1(\mathcal{F}) = s_0^{n-1} c_0 m \theta$, where s_0, m, c_0 are integers (with $s_0, m > 0$), s_0, c_0 are coprime and $m > 1$. It follows that $s_0 b_0 = c_0 r_0$. Since $\text{gcd}\{r_0, b_0\} = 1$ and $\text{gcd}\{s_0, c_0\} = 1$, we get that $r_0 = s_0$, and hence, $m = 1$. This is a contradiction. \square

Proof of Proposition 5.14. Let $\varphi: \mathcal{X} \rightarrow B$ be as in Definition 5.4. Recall that $X_0 = \varphi^{-1}(0) \cong S^{[n]}$, where S is an elliptic K3 surface as in Claim 4.3. Let $\mathcal{E}_0 := \mathcal{F}[n]^+$ be the vector bundle on X_0 of Claim 4.3. If B is small enough, the vector bundle \mathcal{E}_0 on X_0 deforms uniquely to a vector bundle \mathcal{E}_b on X_b , hence, we get a vector bundle \mathcal{E}_X on X for $[(X, h)]$ in a dense open subset $\mathcal{U} \subset NL(d_0)$. Moreover, \mathcal{E}_X is h slope stable and Items (1) and (2) of Proposition 5.14 hold (see the proof of Proposition 5.1). For $[(X, h)] \in NL(d_0)^*$,

where $NL(d_0)^* \subset \mathcal{U}$ is an open dense subset, the restriction of \mathcal{E}_X to a general smooth fibre of the Lagrangian fibration $X \rightarrow \mathbb{P}^n$ is slope-stable (see Proposition 5.1).

Let $\mathcal{V}_0 \subset \mathbb{P}^n$ be the set of t , such that the restriction of \mathcal{E}_0 to the fibre over t of the Lagrangian fibration $S^{[n]} \rightarrow \mathbb{P}^n$ is simple. By Item (b) of Proposition 4.6, \mathcal{V}_0 is an open subset whose complement has codimension (in \mathbb{P}^n) at least 2. It follows that there exists an open dense subset $NL(d_0)_s^* \subset NL(d_0)^*$ with the following property: if $[(X, h)] \in NL(d_0)_s^*$, the subset $\mathcal{V}_X \subset \mathbb{P}^n$ has parametrizing fibres X_t of the Lagrangian fibration $X \rightarrow \mathbb{P}^n$, such that the restriction of \mathcal{E}_X to X_t is simple and has complement of codimension (in \mathbb{P}^n) at least 2.

Let $[(X, h)] \in NL(d_0)_s^*$. We claim that if $t \in \mathcal{V}_X$ and X_t is smooth, then the restriction $\mathcal{E}_{X|X_t}$ is slope-stable. In order to prove this, we start by noting that for any Lagrangian (scheme-theoretic) fibre X_t , we have

$$\int_{[X_t]} \Delta(\mathcal{E}_{X|X_t}) \cdot (h|_{X_t})^{n-2} = 0. \tag{5.6.5}$$

In fact, the above equality is an easy consequence of the modularity of \mathcal{E}_X (see [O'G22, Lemma 2.5]). Let X_t be a general smooth Lagrangian fibre. By Proposition 5.9, the restriction of \mathcal{E}_X to X_t is slope-stable, hence, $\mathcal{E}_{X|X_t}$ is semihomogeneous because of the equality in (5.6.5) (see [O'G22, Lemma 2.5]). It follows that if $t_0 \in \mathcal{V}_X$ and X_{t_0} is smooth, then $\mathcal{E}_{X|X_{t_0}}$ is (simple) semihomogeneous. To prove this, we introduce some notation. For $t \in \mathbb{P}^n$, such that X_t is smooth, let

$$\begin{aligned} \Phi^0(\mathcal{E}_{X|X_t}) &:= \{(x, [\xi]) \in X_t \times \widehat{X}_t \mid \exists T_x^*(\mathcal{E}_{X|X_t}) \xrightarrow{\sim} (\mathcal{E}_{X|X_t}) \otimes \xi\}, \\ \Psi^0(\mathcal{E}_{X|X_t}) &:= \{(x, [\xi]) \in X_t \times \widehat{X}_t \mid \text{Hom}(T_x^*(\mathcal{E}_{X|X_t}), (\mathcal{E}_{X|X_t}) \otimes \xi) \neq 0\}, \end{aligned}$$

where $T_x: X_t \rightarrow X_t$ is translated by $x \in X_t$ (locally, in t , we may assume that X_t is a family of Abelian varieties rather than torsors over Abelian varieties). Recall that $\mathcal{E}_{X|X_t}$ is semihomogeneous if and only if $\Phi^0(\mathcal{E}_{X|X_t})$ has dimension at least n , and that if that is the case, then the group $\Phi^0(\mathcal{E}_{X|X_t})$ has pure dimension n . Let X_t be a general smooth Lagrangian fibre, so that $\mathcal{E}_{X|X_t}$ is slope-stable and semihomogeneous. Then $\Phi^0(\mathcal{E}_{X|X_t})$ has pure dimension n , and moreover, (by slope stability) $\Phi^0(\mathcal{E}_{X|X_t}) = \Psi^0(\mathcal{E}_{X|X_t})$. By upper semicontinuity of cohomology dimension, it follows that every irreducible component of $\Psi^0(\mathcal{E}_{X|X_{t_0}})$ has dimension at least n . Now, $(0, [\mathcal{O}_{X_{t_0}}]) \in \Psi^0(\mathcal{E}_{X|X_{t_0}})$ and, since $\mathcal{E}_{X|X_{t_0}}$ is simple, every nonzero homomorphism $\mathcal{E}_{X|X_{t_0}} \rightarrow \mathcal{E}_{X|X_{t_0}}$ is an isomorphism. It follows that $\Phi^0(\mathcal{E}_{X|X_{t_0}})$ has dimension at least n , and hence, $\mathcal{E}_{X|X_{t_0}}$ is semihomogeneous. By [Muk78, Proposition 6.13], we get that $\mathcal{E}_{X|X_{t_0}}$ is slope-semistable (actually, it is Gieseker stable, see Proposition 6.16 loc.cit.). Lastly, we prove that $\mathcal{E}_{X|X_{t_0}}$ is slope-stable. Let θ_{t_0} be the principal polarization of X_{t_0} (see Remark 4.9). Since $q_{X_{t_0}}(h, f) = \text{ild}_0$ (see (5.2.7)), we have $h|_{X_{t_0}} = \text{ild}_0 \theta_{t_0}$. Hence

$$r(\mathcal{E}_{X|X_{t_0}}) = r_0^n, \quad c_1(\mathcal{E}_{X|X_{t_0}}) = r_0^{n-1} \text{gl}d_0 \theta_{t_0}. \tag{5.6.6}$$

By hypothesis g , l and d_0 are coprime to r_0 . By Corollary 5.16, we get that $\mathcal{E}_{X|X_{t_0}}$ is slope-stable.

We finish the proof by showing that if $[(X, h)] \in NL(d_0)_s^*$ is general, then the restriction of \mathcal{E}_X to a general singular Lagrangian fibre is slope-stable. Since $[(X, h)] \in NL(d_0)_s^*$ is general, the discriminant divisor $\mathcal{D}_X \subset \mathbb{P}^n$ parametrizing singular Lagrangian fibres of $X \rightarrow \mathbb{P}^n$ is the dual of an embedded K3 surface $S \subset (\mathbb{P}^n)^\vee$. In fact, this holds by Proposition 5.12. Hence, \mathcal{D}_X is an irreducible divisor. Thus, it suffices to prove that there exist $t \in \mathcal{D}_X$, such that $\mathcal{E}_{X|X_t}$ is slope-stable (for the restriction of h to X_t). This follows from Remark 4.8 and openness of slope stability. \square

5.7. Proof of Proposition 5.2

The key result is the following.

Proposition 5.17. *Keep notation as in Section 5.2. Assume, in addition, that d_0 is coprime to r_0 , and that the inequality in (5.3.4) holds. Let $[(X, h)] \in NL(d_0)^{**}$ be a general point. Then (up to isomorphism) there exists one and only one h slope-stable vector bundle \mathcal{E} on X , such that the equalities in (1.2.6) hold.*

Proof. Let \mathcal{E}_X be a vector bundle on X as in Proposition 5.14, and let \mathcal{E} be an h slope-stable vector bundle on X , such that the equalities in (1.2.6) hold. We prove that \mathcal{E}_X and \mathcal{E} are isomorphic.

Let $\pi: X \rightarrow \mathbb{P}^n$ be the associated Lagrangian fibration. By Proposition 5.11, the polarization h is $a(\mathcal{E})$ -suitable, and hence, the restriction of \mathcal{E} to a general Lagrangian fibre is slope-semistable. We claim that if X_t is a smooth Lagrangian fibre and $\mathcal{E}|_{X_t}$ is slope-semistable, then it is actually slope-stable. In fact, this follows from Corollary 5.16 — the computations showing that the hypotheses of Corollary 5.16 are satisfied have already been done (see (5.6.6)). The upshot is that there exists a dense open subset $\mathcal{U}_X^0 \subset \mathbb{P}^n$, contained in the set of regular values of the Lagrangian fibration, with the property that for all $t \in \mathcal{U}_X^0$, the restrictions $\mathcal{E}_{X|X_t}$ and $\mathcal{E}|_{X_t}$ are simple semihomogeneous vector bundles (since they are slope-stable vector bundles and $\Delta(\mathcal{E}_{X|X_t}) \cdot \theta_t^{n-2} = \Delta(\mathcal{E}|_{X_t}) \cdot \theta_t^{n-2} = 0$, they are semihomogeneous by [O’G22, Proposition A.2]).

We claim that if $t \in \mathcal{U}_X^0$, then $\mathcal{E}_{X|X_t}$ and $\mathcal{E}|_{X_t}$ are isomorphic. First, note that, since they are simple semihomogenous vector bundles with same rank and c_1 , the set

$$V_t := \{[\xi] \in X_t^\vee \mid \mathcal{E}_{X|X_t} \cong (\mathcal{E}|_{X_t}) \otimes \xi\}$$

is not empty by [Muk78, Theorem 7.11], and it has cardinality r_0^{2n} by Proposition 7.1 op. cit. Note that $V_t \subset X_t[r_0^n]$ because $\mathcal{E}_{X|X_t}$ and $\mathcal{E}|_{X_t}$ have rank r_0^n and isomorphic determinants. Next, we claim that V_t is invariant under the monodromy action of $\pi_1(\mathcal{U}_X^0, t)$. In fact, let

$$\mathcal{V} := \bigcup_{t \in \mathcal{U}_X^0} V_t. \tag{5.7.1}$$

We show that the forgetful map

$$\mathcal{V} \rightarrow \mathcal{U}_X^0 \tag{5.7.2}$$

is a topological covering. Let $t_1 \in \mathcal{U}_X^0$, and let $\xi_{t_1} \in V_{t_1}$. Let $B \subset \mathcal{U}_X^0$ be an open (in the classical topology) neighbourhood of t_1 which is contractible. For each $t \in B$, let $\xi_t \subset X_t[r_0^n]$ be obtained from ξ_{t_1} by parallel transport. We claim that

$$\mathcal{E}_{X|X_t} \cong (\mathcal{E}|_{X_t}) \otimes \xi_t \quad \forall t \in B. \tag{5.7.3}$$

In fact, the traceless endomorphism bundles of the right and left sides of (5.7.3) have vanishing cohomologies (see Theorem 5.8 in [Muk78]). Since their determinants remain of type (1,1) on X_t for all $t \in B$ (actually, for all $t \in \mathcal{U}_X^0$), it follows that each of them extends uniquely to all $X_{t'}$ for $t' \in B$ close enough to t . Since they are isomorphic for $t = t_1$, we get that they are isomorphic for all $t \in B$. This shows that the map in (5.7.2) is a topological covering. The proof shows also that monodromy takes V_t to itself.

Since V_t is invariant under the monodromy action of $\pi_1(\mathcal{U}_X^0, t)$, it follows from Corollary 5.13 that $V_t = A[r_0]$. Thus, $0 \in V_t$, and therefore, $\mathcal{E}_{X|X_t} \cong \mathcal{E}|_{X_t}$.

Now, we use the hypothesis that $[(X, h)] \in NL(d_0)^{**}$ is a general point. Then the polarized $K3$ surface (S, D) is a general surface of genus n (see Section 5.5), and by Proposition 5.12, the Lagrangian fibration $\pi: X \rightarrow \mathbb{P}^n$ is a Tate-Shafarevich twist of the relative Jacobian $\mathcal{J}(S) \rightarrow |D|$. It follows that the discriminant curve $B \subset \mathbb{P}^n$ is isomorphic to the dual of $S \subset |D|^\vee$, and hence is reduced. Let $\mathcal{U}_X^\dagger := \mathcal{U}_X \setminus \text{sing } B$, where \mathcal{U}_X is as in Proposition 5.14. Note that \mathcal{U}_X^\dagger is an open subset of \mathbb{P}^n whose complement has codimension at least 2. One proves that

$$\mathcal{E}_{X|X_t} \cong \mathcal{E}|_{X_t} \quad \forall t \in \mathcal{U}_X^\dagger \tag{5.7.4}$$

proceeding as in the proof of [O'G22, Proposition 7.4]. More precisely, there exists a smooth projective curve $T \subset \mathcal{U}_X^\dagger$ containing t and transverse to B (recall that the complement of \mathcal{U}_X^\dagger in \mathbb{P}^n has codimension at least 2). Then $Y := \pi^{-1}(T)$ is a smooth projective (integral) variety of dimension $n + 1$ and the sheaves $\mathcal{F} := \mathcal{E}_{X|Y}$ and $\mathcal{G} := \mathcal{E}|_Y$ satisfy the hypotheses of [O'G22, Lemma 7.5], and hence, the isomorphism in (5.7.4) holds by the quoted lemma.

Since $\mathcal{E}_{X|X_t}$ and $\mathcal{E}|_{X_t}$ are simple for all $t \in \mathcal{U}_X^\dagger$, and since $c_1(\mathcal{E}_X) = c_1(\mathcal{E})$, it follows that the restrictions of \mathcal{E}_X and \mathcal{E} to $\pi^{-1}(\mathcal{U}_X^\dagger)$ are isomorphic (see the proof of Proposition 7.4 in [O'G22], in particular, the beginning of the proof of Lemma 7.5). The complement of $\pi^{-1}(\mathcal{U}_X^\dagger)$ in X has codimension at least 2 because π is equidimensional, and hence, the isomorphism $\mathcal{E}_{X|\pi^{-1}(\mathcal{U}_X^\dagger)} \xrightarrow{\sim} \mathcal{E}|_{\pi^{-1}(\mathcal{U}_X^\dagger)}$ extends to an isomorphism $\mathcal{E}_X \xrightarrow{\sim} \mathcal{E}$. \square

We are ready to prove Proposition 5.2. Let n, r_0, g, l, e be as in Theorem 1.1. Since the result is trivially true for $r_0 = 1$, we assume that $r_0 \geq 2$. Let $\mathcal{X} \rightarrow T_e^{il}(2n)$ be a complete family of polarized HK varieties of Type $K3^{[n]}$ parametrized by $\mathcal{K}_e^{il}(2n)^{\text{good}}$. Since $\mathcal{K}_e^{il}(2n)^{\text{good}}$ is irreducible, we may, and will, assume that $T_e^{il}(2n)$ is irreducible. For $t \in T_e^{il}(2n)$, we let (X_t, h_t) be the corresponding polarized HK of Type $K3^{[n]}$. Let $m: T_e^{il}(2n) \rightarrow \mathcal{K}_e^{il}(2n)^{\text{good}}$ be the moduli map sending t to $[(X_t, h_t)]$.

By Gieseker and Maruyama, there exists a relative moduli space

$$\mathcal{M}(r_0, g) \xrightarrow{f} T_e^{il}(2n), \tag{5.7.5}$$

such that for every $t \in T_e^{il}(2n)$, the (scheme theoretic) fibre $f^{-1}(t)$ is isomorphic to the (coarse) moduli space of h_t slope-stable vector bundles \mathcal{E} on X_t , such that (1.2.6) holds. Moreover, the morphism f is of finite type by Maruyama [Mar81], and hence, $f(\mathcal{M}(r_0, g))$ is a constructible subset of $T_e^{il}(2n)$.

Let $d(r_0, e, l)$ be the right-hand side of the inequality in (5.3.4). For t in a dense subset of $\bigcup_{d>d(r_0, e, l)} m^{-1}(NL(d)^{\text{good}})$, the preimage $f^{-1}(t)$ is a singleton by Proposition 5.17.

Since $\bigcup_{d>d(r_0, e, l)} m^{-1}(NL(d)^{\text{good}})$ is Zariski dense in $T_e^{il}(2n)$ (it is the union of an infinite collection of pairwise distinct divisors), and since $f(\mathcal{M}(r_0, g))$ is a constructible subset of $T_e^{il}(2n)$, it follows that for general $t \in T_e^{il}(2n)$, the fibre $f^{-1}(t)$ is a singleton.

Let $[\mathcal{E}]$ be the unique point of $f^{-1}(t)$ for t a generic point of $m^{-1}(NL(d)^{\text{good}})$, where $d > d(r_0, e, l)$. Then $H^p(X_t, \text{End}^0(\mathcal{E})) = 0$ by Proposition 5.14. Hence, the last sentence of Theorem 1.1 follows from upper semicontinuity of cohomology.

5.8. Proof of Proposition 1.2

Since the natural morphism $\text{Def}(X, \mathcal{F}) \rightarrow \text{Def}(X, h)$ is surjective, for $d_0 \gg 0$, there exist extensions of \mathcal{F} to polarized HK varieties (Y, h) of type $K3^{[n]}$ with a Lagrangian fibration $\pi: Y \rightarrow \mathbb{P}^n$, such that

$$q_Y(h, f) = d_0 \cdot \text{div}(h) \tag{5.8.1}$$

(as usual, $f := c_1(\pi^* \mathcal{O}_{\mathbb{P}^n}(1))$). Let X_t be a smooth (Lagrangian) fibre of π , and let θ_t be the principal polarization of X_t induced by the Lagrangian fibration (see Remark 4.10). We claim that

$$h|_{Y_t} = d_0 \cdot \text{div}(h) \theta_t. \tag{5.8.2}$$

In fact, the above equality follows from the equalities

$$\int_{Y_t} (h|_{Y_t})^n = \int_X h^n \cdot f^n = n! q_X(h, f)^n = n! (d_0 \cdot \text{div}(h))^n.$$

By the equality in (5.8.2), we get that

$$c_1(\mathcal{E}|_{Y_t}) = a \cdot d_0 \cdot \text{div}(h) \theta_t.$$

By Proposition 5.15, we may write

$$r(\mathcal{E}) = r_0^n m, \quad a \cdot d_0 \cdot \text{div}(h) = r_0^{n-1} b_0 m, \tag{5.8.3}$$

where r_0, m, b_0 are integers, $r_0, m \in \mathbb{N}_+$ and $\text{gcd}\{r_0, b_0\} = 1$. Choose d_0 coprime to $r(\mathcal{E})$: then d_0 divides b_0 , and the equation in (1.3.3) holds with $b'_0 = b_0/d_0$.

Acknowledgments. Thanks to Marco Manetti for explaining Horikawa’s classical result [Hor74, Theorem 6.1], and to Emanuele Macrì for helping me out by introducing me to the powerful [Sim92, Theorem 2]. This work was partially supported by PRIN 2017YRA3LK.

Competing interests. The authors have no competing interest to declare.

References

- [Art88] I. V. ARTAMKIN, On the deformation of sheaves, *Izv. Akad. Nauk SSSR Ser. Mat.* **52**(3) (1988), 660–665, 672. MR 954302
- [Bec22] T. BECKMANN, Atomic objects on hyper-kähler manifolds, *Preprint*, 2022, <https://arxiv.org/abs/2202.01184v1>.
- [Bec23] T. BECKMANN, Derived categories of Hyper-Kähler manifolds: Extended Mukai vector and integral structure, *Compos. Math.* **159**(1) (2023), 109–152.
- [Bot22] A. BOTTINI, Towards a modular construction of OG10, *Preprint*, 2022, <https://arxiv.org/abs/2211.09033>.
- [Deb22] O. DEBARRE, Hyper-Kähler manifolds, *Milan J. Math.* **90** (2022), 305–387.
- [Fat23] E. FATIGHENTI, Examples of non-rigid, modular vector bundles on hyperkähler manifolds, *Preprint*, 2023, <https://doi.org/10.48550/arXiv:2302.09025v1> [math.AG].
- [FLV19] L. FU, R. LATERVEER AND C. VIAL, The generalized Franchetta conjecture for some hyper-Kähler varieties, *J. Math. Pures Appl.* **130**(9) (2019), 1–35. With an appendix by the authors and Mingmin Shen. MR 4001626
- [FMOS22] L. FLAPAN, E. MACRÌ, K. O’GRADY AND G. SACCÀ, The geometry of antisymplectic involutions, I, *Math. Z.* **300**(4) (2022), 3457–3495.
- [Ful84] W. FULTON, Intersection theory, in *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge.* (Springer-Verlag, Berlin, 1984), i–470.
- [Gav21] V. GAVRAN, Tangent bundle of a manifold of $K3^{[2]}$ -type is rigid, *Preprint*, 2021, <https://arxiv.org/abs/2103.13522>.
- [GKLR22] M. GREEN, Y.-J. KIM, R. LAZA AND C. ROBLES, The LLV decomposition of hyper-Kähler cohomology (the known cases and the general conjectural behaviour), *Math. Ann.* **382** (2022), 1517–1590.
- [Hai01] M. HAIMAN, Hilbert schemes, polygraphs and the Macdonald positivity conjecture, *J. Amer. Math. Soc.* **14**(4) (2001), 941–1006. MR 1839919
- [Hor74] E. HORIKAWA, On deformations of holomorphic maps II, *J. Math. Soc. Japan* **26** (1974), 647–667.
- [IM19] D. IACONO AND M. MANETTI, On deformations of pairs (manifold, coherent sheaf), *Canad. J. Math.* **71**(5) (2019), 1209–1241. MR 4010426
- [Man22] M. MANETTI, Lie methods in deformation theory, in *Springer Monographs in Mathematics* (Springer, Singapore, 2022), i–574
- [Mar81] M. MARUYAMA, On boundedness of families of torsion free sheaves, *J. Math. Kyoto Univ.* **21**(4) (1981), 673–701.
- [Mar02] E. MARKMAN, Generators of the cohomology ring of moduli spaces of sheaves on symplectic surfaces, *J. Reine Angew. Math.* **544** (2002), 61–82.
- [Mar07] E. MARKMAN, Integral generators for the cohomology ring of moduli spaces of sheaves over Poisson surfaces, *Adv. Math.* **208** (2007), 622–646.
- [Mar14] E. MARKMAN, Lagrangian fibrations of holomorphic-symplectic varieties of $K3^{[n]}$ -type, in Algebraic and complex geometry, in *Springer Proceedings in Mathematics & Statistics*, vol. 71 (Springer, Cham, 2014), 241–283. MR 3278577
- [Mar21] E. MARKMAN, Stable vector bundles on a hyper-Kähler manifold with a rank 1 obstruction map are modular, *Preprint*, 2021, <https://arxiv.org/abs/2107.13991> [math.AG].
- [Mat17] D. MATSUSHITA, On isotropic divisors on irreducible symplectic manifolds, in Higher dimensional algebraic geometry—in honour of Professor Yujiro Kawamata’s sixtieth

- birthday, *Advanced Studies in Pure Mathematics*, vol. 74 (Mathematical Society of Japan, Tokyo, 2017), 291–312. MR 3791219
- [Mon15] G. MONGARDI, A note on the Kähler and Mori cones of hyperkähler manifolds, *Asian J. Math.* **19**(4) (2015), 583–591.
- [Muk78] S. MUKAI, Semi-homogeneous vector bundles on an Abelian variety, *J. Math. Kyoto Univ.* **18**(2) (1978), 239–272. MR 0498572
- [Muk84] S. MUKAI, Symplectic structure of the moduli space of sheaves on an abelian or $K3$ surface, *Invent. Math.* **77**(1) (1984), 101–116. MR 751133
- [Muk87] S. MUKAI, On the moduli space of bundles on $K3$ surfaces. I, in *Vector bundles on algebraic varieties (Bombay, 1984)*, *Tata Institute of Fundamental Research Studies in Mathematics*, vol. 11 (Tata Institute of Fundamental Research, Bombay, 1987), 341–413. MR 893604
- [O'G22] K. G. O'GRADY, Modular sheaves on hyperkähler varieties, *Algebr. Geom.* **9** (2022), 1–38.
- [Sim92] C. T. SIMPSON, Higgs bundles and local systems, *Inst. Hautes Études Sci. Publ. Math.* **75** (1992), 5–95. MR 1179076
- [Tot20] B. TOTARO, The integral cohomology of the Hilbert scheme of points on a surface, *Forum Math. Sigma* **8** (2020), 1–6.
- [Wie16] B. WIENECK, On polarization types of Lagrangian fibrations, *Manuscripta Math.* **151**(3–4) (2016), 305–327. MR 3556822
- [Zha15] L. ZHANG, Character formulas on cohomology of deformations of Hilbert schemes of $K3$ surfaces, *J. Lond. Math. Soc. (2)* **92**(3) (2015), 675–688. MR 3431656