THE

MATHEMATICAL GAZETTE.

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FUROR ARITHMETICUS.

Though some astronomical observations require ten significant figures for their expression, few observational or experimental results are correct to the sixth figure. Hence in practical questions there is a definite limit, beyond which it is useless to extend arithmetical results.

Mathematical formulæ on the other hand may give results so large as to require expression by a great number of figures, or the result may be incommensurable and more nearly expressed the further the calculation is carried.

Thus, factorial forty is expressed by forty-eight figures

$$\underline{|40}=815915,283247,897734,344051,369596,115894,272000,000000,$$

and the sum of the infinite series e can be found to any number of decimal places.

$$e=1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+$$
$$=2{\cdot}718281,828459,045235,360287, \quad .$$

From Plato downwards mystical ideas have been connected with numbers, which have led to the laborious investigation of relations between them. Perfect numbers such as 6 and 28 were found, each of which is equal to the sum of its divisors, and amicable numbers, such as 220 and 284, each of which is equal to the sum of the divisors of the other.

Many curious facts and fancies are collected in the *Numerorum Mysteria* of Peter Bungus (1585).

Owing to the extremely cumbrous notations employed by all the nations of antiquity, except the Hindoos, such calculations could not be carried very far until the introduction of the so-called Arabian notation, which spread over Europe during the thirteenth century. Since that time, though most people have a strong repugnance to undertaking any long numerical calculation, some few seem to be seized with a divine afflatus which carries them through appalling series of figures. Thus, tables are calculated to a great number of places, long incommensurable roots are found, and constants are determined to many figures.

As de Morgan points out (*Budget of Paradoxes*, 290), "These tremendous stretches of calculation—at least we so call them in our day—are useful in several respects; they prove more than the capacity of this or that computer for laborious accuracy, they show that there is in the community an increase of skill and courage."

It is also impossible to know on which side of the line of utility to put any given result. What is useless to-day may in the progress of knowledge become invaluable to-morrow.

When Euler investigated, and Legendre calculated, the Γ function $\int_0^\infty e^{-x}x^{n-1}dx$ from 0 to 1 by thousandths to twelve places, they hardly anticipated that they were enabling Gilbert to obtain the value of Fresnel's integrals $\int_0^x \sin\frac{\pi x}{2}dx$ and $\int_0^x \cos\frac{\pi x}{2}dx$, indispensable in the theory of the diffraction of light.

The majority of mathematicians object to the drudgery of arithmetic. Lord Lytton remarks, even of Newton, "Qui genus humanum ingenio superavit," "That great master of calculations the most abstruse could not accurately cast up a sum in addition. Nothing brought him to the end of his majestic tether like dot and carry one."

A few even of the greatest mathematicians, such as James Bernoulli, Euler, Legendre, Gauss, de Morgan, Hamilton, and Adams have also been expert and laborious computers. Thus Hamilton and de Morgan congratulated one another on finding

$$\cos\frac{2\pi}{7} = 0{\cdot}62348,98018,58733,53052,50,$$

to obtain which by the ordinary method they must have summed $\left(x=\frac{2\pi}{7}\right)$

$$1-\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}- \quad +\frac{x^{24}}{\underline{|24}}.$$

In several noticeable instances the furor seems to have fallen upon military officers such as Wolfram, Vega, and Oakes, but not upon their naval brethren. It has fallen upon men in all ranks and conditions of life, even upon the great classic Porson, author of the equation

$$xy+zu=444 \qquad xz+yu=180$$
$$xu+yz=156 \qquad xyzu=5184.$$

It has increased the long list of circle-squarers and puzzle makers, and is not unknown among schoolboys. Though the miracles of calculation produced during last century surpassed all previous efforts, they were by no means so widely distributed. The victims of the furor were fewer.

This curious fact seems due to various causes. The commercial production of arithmometers has rendered many extensive tables unnecessary. The modern tendency is towards haste, hence more attention is paid to approximation in the methods of working and in the expression of results. Owing to the spread of educational facilities many, who in former times would have remained on the threshold, become mathematicians and devote themselves to the higher instead of to the lower branches. A still greater number now devote themselves to science and regard calculation merely as a means of expressing the results of experiments, which owing to errors require comparatively few figures.

Thus, in a modern physical research $x^4+44x^3-340x^2-6000x+18000=0$ required solution (*P. Phys. Soc.* xix. 625). The graphical method gave 2·7 as the root with quite sufficient accuracy; in former times a more elaborate and laborious solution would have given $x=2{\cdot}735419339$.

For these and possibly other reasons it seems very doubtful if more powerful elementary tables will be calculated. Such modern tables are merely shortened reprints of results obtained long ago, and attention is chiefly devoted to greater accuracy and convenience in arrangement.

In the higher and newer parts of mathematics, tables such as those of Dr. Meissel and the British Association frequently appear, but they are too often hidden in the pages of periodicals, and many more seem to be required to deal with recent advances in physics.

Notwithstanding this apparent trend towards approximation in recent times, long series of figures are occasionally required, and the means of ascertaining what results have been already obtained are very inadequate.

Many valuable results are hidden in the pages of old or out of the way periodicals, or in MS. in private libraries, or the archives of learned societies. A few tables and constants to many figures may occasionally be picked up in old book shops, but the lists of publishers will be searched for them in vain. A classified list of such tables and results, both in print and MS., with the places where they can be consulted, would save many an irritating and too often an unavailing search.

If the natural logarithm of π be required, how many remember that it is given by Callet to forty-eight places?

Nat. $\log \pi = 1{\cdot}14472,98858,49400,17414,34273,51353,05871,16472,94812,916.$

A brief notice of some of the larger elementary tables may recall what our predecessors have done for us and prove of use.

Crelle (1875) multiples to 1000×1000.

Oakes (1865) reciprocals 1-99999 to seven figures with differences.

Kulik (1848) squares and cubes to 100 000.

Blater (1887) quarter squares to 200 000.

Rheticus, the natural sines for every $1''$ for the first two degrees and for every $10''$ later to 15 places. Some values were added by Pitiscus (1613). Rheticus also calculated a complete $10''$ canon to ten places, published by Otho (1596). According to de Morgan this was "The most laborious work of calculation that any one man ever undertook." Probably no instrument in use at the time was accurate to $1'$.

The discovery of logarithms by Napier in 1614 opened up fresh fields to computers.

In 1624 Briggs published the logarithms from 1-20 000 and 90 000-100 000 to fourteen places. Four years later Vlacq calculated the missing 70 000 logarithms and published the complete table to ten places.

At his death in 1631 Briggs had nearly finished a complete canon to $0{\cdot}01°$, natural sines to fifteen places, logarithmic sines to fourteen places, natural and logarithmic tangents and secants to ten places. This table was published by Gellibrand in 1633.

In the same year Vlacq published a complete $10''$ canon to ten places. These two tables of Vlacq's are the basis of almost all modern tables. Had Briggs' table been adopted we should now be using a much more convenient division of the quadrant.

In 1717 Sharp published the logarithms to 100 and primes to 1100 to sixty-one places, and in 1742 Gardiner printed the logarithms to 1000 and of odd numbers to 1161 to twenty places. These tables were reprinted with additions by Hutton (1785) and Callet (1783), who added a twenty figure table of natural logarithms.

Dodson (1742) published antilogarithms to eleven places corresponding to five-figure logarithms.

Wolfram (1778) natural logarithms to 2200 and 1240 beyond to forty-eight places.

The enormous Tables de Cadastre are still in MS. at the Paris Observatory, but an abstract to eight places has been printed.

In 1871 Sang published a table containing the logarithms 100 000-200 000 from his own calculations. He also left MS. containing logarithms to 20 000 to twenty-eight places, and logarithms 100 000 to 370 000 to fifteen places; also a table of sines and tangents for grades, etc. The MSS. are in the care of the Royal Society of Edinburgh.

In 1876 Peter Gray published a table of radices, by the aid of which logarithms to twenty-four places can be dealt with.

Degen (1824) calculated the logarithms of factorials up to $|\underline{1200}$ to eighteen places.

In 1827 Legendre published a double entry table giving the values of elliptic integrals of the first and second kind for each degree of the modulus and amplitude to ten places.

It is hardly possible now-a-days to realize the enthusiasm and patience of the earlier tabulators, who plunged into an ocean of calculation often with very inefficient formulæ and no modern appliances. Even to use a table to twenty places taxes the industry and accuracy of a modern computer. In general twenty figures must be multiplied by two or three, fifteen figures by fifteen, ten by ten, and five by five. The tedium and risk of error in using such tables is shown by the fact that Callet and Hutton give identical examples. Few will take the further step of using Sharp's table, which has not been reprinted for many years.

Questions connected with prime numbers have taxed the power of mathematicians and the patience of computers.

The only known method of finding primes, or numbers which are exactly divisible only by one, is the tedious "sieve of Eratosthenes" of Cyrene (275-194 B.C.). But Fermat seems to have been in possession of a more rapid method. When asked by Mersenne, in 1643, if 100 895, 598 169 was prime, he replied, it is the product of $898\,423 \times 112\,303$. Rouse Ball remarks "Even with the aid of modern tables I do not think anyone would now undertake to answer such a question."

Tables of primes and factors up to 9 000 000 have been printed by Chernac, Burckhardt, Glaisher, and Dase, and that for the tenth million is said to exist in MS. at Berlin.

The highest prime now known is $2^{61}-1$ or 2,305843,009213,693951.

A convenient method for finding the factors of a comparatively small number consists in finding a square such that when the given number is subtracted from it a square is left. The sum and difference of the two square roots are the factors of the given number (*Nature*, Feb. 28, 1889).

Stanley Jevons asks (*Prin. of Sci.*, 123), "Can the reader say what two numbers multiplied together will produce 8 616 460 799? I think it unlikely that any one but myself will ever know, for they are two large prime numbers, and can only be rediscovered by trying in succession a long series of prime divisors until the right one is fallen upon. The work would probably occupy a good computer for many weeks."

It is proverbially unsafe to prophecy before the event. Subtracting from the next higher square,

$$\begin{array}{r} 8\,616\,480\,625 \text{ or } 92825^2 \\ \underline{8\,616\,460\,799} \\ 19\,826 \end{array}$$

Since $(a+n)^2=a^2+2an+n^2$, if any number of the form $n(2a+n)$ be added the sum remains a square. Hence, a number of the form $n(185\,650+n)$ must be added to 19826 such as to make the sum a square. The number of trials may be reduced by remembering the criteria of squared numbers, such as that they must terminate in 00, 1, 4, 5, 6 or 9. In the present case n is found to be 55. $92825+55=92880$.

$$\begin{array}{r} 55(185\,650+55)=10\,213\,775 \\ \underline{19\,826} \\ 10\,233\,601=3199^2. \end{array}$$

$$\begin{array}{rr} 92\,880 & 92\,880 \\ \underline{3\,199} & \underline{3\,199} \\ 96\,079 \times & 89\,681=8\,616\,460\,799. \end{array}$$

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It is obvious that for numbers so large as to be beyond the reach of an available table of squares the method may require much patience. It is easier the more nearly equal the factors are. Luckily the subject of primes, however important in the theory of numbers, does not enter much into practical calculations.

In solving a long or difficult equation it is generally worth while to draw the graph on squared paper, which gives the first figure or two of the roots, and then to consider if there is any way of dodging the difficulties.

It looks a little formidable at first to find the four roots of the equation

$$70x^4-140x^3+90x^2-20x+1=0$$

until it is noticed that the sum of the four roots is 2, and that if x be replaced by $t+\frac{1}{2}$, the equation reduces to

$$70t^4-15t^2+\tfrac{3}{8}=0,$$

a quadratic in t^2, and the roots are

$$\begin{aligned} x &= 0{\cdot}06943{,}18442{,}02975{,}4 \\ &= 0{\cdot}93056{,}81557{,}97024{,}6 \\ &= 0{\cdot}33000{,}94782{,}07567{,}7 \\ &= 0{\cdot}66999{,}05217{,}92432{,}3. \end{aligned}$$

For practical purposes it is, in general, only required to obtain one real root of an equation or number. The most generally accepted method is that of Horner, which is given in the text-books.

Find a number such that the sum of the first five powers is equal to 100.

$$x^5+x^4+x^3+x^2+x-100=0. \quad x=2{\cdot}239\,643.$$

Not only was de Morgan a victim of the furor, but he infected his pupils. Hicks found the real positive root of the classical equation

$$x^3-2x=5,$$

$x=2{\cdot}09455$ to 152 places.

Valuable as Horner's method is, in some cases the far less well-known method of Weddle seems to give the result with considerably less labour and risk of error.

Weddle gives the two following solutions which it would be very tedious to find by Horner:

$$(100\,000)^{\frac{1}{11}}=2{\cdot}848\,035\,869.$$

$$1379{\cdot}664x^{622}+2686034\times10^{432}x^{153}-17290224\times10^{518}x^{60}+2524156\times10^{574}=0.$$

$$x=8{\cdot}367\,975\,431.$$

Hutton's method of approximating to the root of a number seems not to be so generally known as it deserves to be. If a be nearly the n^{th} root of N, a still nearer root is

$$\frac{\overline{n+1}N+\overline{n-1}a^n}{\overline{n-1}N+\overline{n+1}a^n}\times a.$$

In the case of the cube root this reduces to

$$\frac{2N+a^3}{N+2a^3}\times a.$$

If $N=10$ and $a=2{\cdot}154$ the next approximation gives

$$\sqrt[3]{10}=2{\cdot}154\,434\,690\,02.$$

It may encourage the would-be computer to mention a few results which have been already obtained.

In 1657 Gaspar Schott found 2^{256}, which consists of 78 figures.

In 1863 Suffield and Lunn calculated the recurring period of 1/7699 consisting of 7698 figures, but this was beaten by Shanks, who found 1/17 389 to 17 388 figures.

Colville, a pupil of de Morgan, found $\sqrt{2}$ to 110 places. A preliminary attack may be made on $\sqrt[3]{19}=2{\cdot}668\,401\,648\,721\,944\,867\,339\,630$.

Probably no constant has caused the expenditure of so much labour as π. Illegitimate attempts to obtain an exact value are immortalized in the *Budget of Paradoxes.* Legitimate approximations have culminated in Shanks' value to 707 places. He also obtained e, $\log_e 2$, 3, 5, 10 and M to 205 places.

Adams, the discoverer of Neptune, was also an ardent computer. He found the numbers of Bernoulli from B_{32}–B_{62}, and more roughly to B_{100}. He also calculated Euler's constant E, $\log_e 2$, 3, 5, 7, 10 and M to 270 places, and the sums of the reciprocals of the first 500 and the first 1000 natural numbers to 260 figures.

The sum of the tenth powers of the first thousand natural numbers is

$$91{,}409{,}924{,}241{,}424{,}243{,}424{,}241{,}924{,}242{,}500.$$

James Bernoulli mentions that it took him rather less than seven and a half minutes to obtain this result.

Though, owing to unavoidable errors, it is useless to carry the arithmetical reduction of observational or experimental results beyond a comparatively small number of figures, the reduction, especially in astronomy, may severely tax the power and patience of the computer.

The value of the average result of equally good observations increases as the square root of their number. Hence, if a quantity is to be determined as accurately as possible, a large number of observations or experiments must be made, and if the method of least squares be rigorously applied the work becomes very tedious.

Theory shows that the length l of a pendulum in latitude ϕ is determined by the equation

$$l=l_0(1+k\sin^2\phi),$$

where k is a constant to be determined by observation at different places. Bowditch undertook the tremendous labour of combining 52 observational values in strict accordance with the method of least squares to obtain the equation

$$l=39{\cdot}01307+0{\cdot}20644\sin^2\phi.$$

The survey of a country entails an immense amount of arithmetical work in obtaining the most probable values from slightly discrepant observations and in solving a vast number of spherical triangles. A large staff of computers may be occupied for years.

A good many dodges by which long calculations may be lightened may be found in modern arithmetics. Special reference may be made to de Morgan's article on "Computation" in the *Supplement of the Penny Encyclopædia*, to Boccardi's *Guide du Calculateur*, and to Langley's *Computation.*

Many of these are of general utility, but too often a special dodge is required, which only suggests itself when the work is half done.

Some fruits of more or less bitter experience may be jotted down in the hopes of saving a neophyte from some of the numerous pitfalls which beset the way of the computer.

1. When long arithmetical results are required, endeavour to ascertain if they have been already found and lie buried in periodicals, old books, or MSS.

2. Draw a graph representing the data on squared paper, making free use of Prof. Perry's black thread. Or obtain an approximate result with the aid of a slide-rule or four-figure table of logarithms.

3. Obtain a formula as convenient as possible for calculation, having regard to the personal peculiarities of the computer, and to the tables and other aids available.

4. Consider to how many figures the data are accurate, or the answer required, and work to one or two more.

5. Use logarithmic or other tables of only the required accuracy, and correct them from the table of errata or otherwise. Errors are somewhat numerous in many of the older tables.

6. Obtain a quantity of "mark paper," ruled in small squares, and rule each fifth or sixth vertical line in red.

7. Write the nine multiples of numbers, which are frequently required, on slips of card; these slips can be arranged as required on a board by the aid of drawing pins. Blater's Table of Napier or Sawyer's Automatic Multiplier may be used instead of the slips.

8. A few wooden or metal slips are useful for ranging long rows of figures or covering up any not required.

9. It is a counsel of perfection to repeat a tedious calculation from a different formula with different tables.

It is to be remembered that in the value of π, published by Rutherford in 1841, to 208 places, only 152 figures are correct. Two errors crept into Shanks' result to 530 places in 1853. If such computers publish erroneous figures it may well behove their inferiors to be careful. SYDNEY LUPTON.

CORRESPONDENCE.

APPROXIMATION IN METHOD VERSUS APPROXIMATION IN ARITHMETIC.

TO THE EDITOR OF THE *Mathematical Gazette.*

DEAR SIR,—In the welcome Report on the Correlation of Mathematical and Science Teaching by the Joint Committee of the Mathematical Association and the Association of Public Schools' Science Masters, two examples are given (p. 6) on the method which should be followed in treating problems in Physics.

There can be no question of the main contention that great care should be taken that the pupil is not finding a numerical result simply by substitution in a given formula. But the first example as given raises another point also. The example is on the linear expansion of a brass rod, and is begun by directing the pupil's attention to the meaning of the coefficient of linear expansion as "the amount by which unit length (no temperature given) expands when heated through unit temperature." In working the example this unit length is tacitly assumed to be at 10° C.—or else it is tacitly assumed that there will be no appreciable difference in the result whether this unit length be taken to be at 0° C. or at 10° C.

This vagueness in method raises a point of considerable importance in the teaching of such questions when clothed with all the authority of occurring in a specially recommended example in a Report of such weight. But I venture to ask whether it is well to allow unnecessary inaccuracies in method simply for the sake of shortness and saving a little mathematics? I am not here speaking of approximations to what really occurs in Nature which must be assumed sufficiently to simplify Physical problems. But would it not be far better to work the theoretical parts of the problem clearly and logically from the accepted definitions for the Physical quantities (these definitions having probably been explained carefully and at length to the class), and then find the approximate numerical answer by accurate approximate arithmetic? By accurate approximate arithmetic is here meant such that the student knows to which significant figure he can trust. With logarithms or a slide rule this final arithmetic is short and easy, and will not withdraw the student's attention from the main principles of the problem.