On the quadrilateral circuminscribed to two circles.

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Figure 1.

Let ABCD be a quadrilateral inscribed in a circle (centre O, radius ρ) whose diagonals AC, BD intersect at right angles in S. From S draw SE, SF, SG, SH perpendiculars on AB, BC, CD, DA respectively.

Then EFGH is a quadrilateral circuminscribed to two circles. It is, moreover, the earliest and simplest form in which such a figure would ordinarily present itself to a student in Geometry.

[From the various cyclic quadrilaterals $S\widehat{E}F = S\widehat{B}C = S\widehat{A}D = S\widehat{E}H$ and SE bisects $F\widehat{E}H$. Again $F\widehat{E}H = 2S\widehat{A}D$ and $F\widehat{G}H = 2S\widehat{D}A$, so that $F\widehat{E}H$ and $F\widehat{G}H$ are supplementary].

S is the incentre of EFGH. Since $SE^2 = AE \,.\, BE = \rho^2 - OE^2$, therefore E (and similarly F, G, H) lies on the circular locus $SP^2 + OP^2 = \rho^2$ whose centre X is at the middle point of OS and whose radius R is given by the relation

$$2R^2 + 2d^2 = \rho^2 \qquad - \quad - \qquad \text{(i)}$$

where $d = SX$.

Again, if r be the radius of the circle inscribed in EFGH,

$$r = SE\sin SAD = SA\sin SAB \,.\, \sin SAD = SA \,.\, BC \,.\, CD/4\rho^2 = SA \,.\, SC/2\rho,$$

$$\text{thus} \quad 2\rho r = \rho^2 - 4d^2 \qquad - \quad - \qquad \text{(ii)}.$$

If we eliminate ρ between (i) and (ii) we get

$$(R + d)^{-2} + (R - d)^{-2} = r^{-2}$$

which is the known poristic relation.

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