On the quadrilateral circuminscribed to two circles.

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Figure 1.

Let ABCD be a quadrilateral inscribed in a circle (centre O, radius \( \rho \)) whose diagonals AC, BD intersect at right angles in S. From S draw SE, SF, SG, SH perpendiculars on AB, BC, CD, DA respectively.

Then EFGH is a quadrilateral circuminscribed to two circles. It is, moreover, the earliest and simplest form in which such a figure would ordinarily present itself to a student in Geometry.

[From the various cyclic quadrilaterals \( \overline{SEF} = \overline{SBC} = \overline{SAD} = \overline{SEH} \) and \( SE \) bisects \( \overline{FEH} \). Again \( \overline{FEH} = 2\overline{SAD} \) and \( \overline{FGH} = 2\overline{SDA} \), so that \( \overline{FEH} \) and \( \overline{FGH} \) are supplementary].

S is the incentre of EFGH. Since \( SE^2 = AE \cdot BE = \rho^2 - OE^2 \), therefore E (and similarly F, G, H) lies on the circular locus \( SP^2 + OP^2 = \rho^2 \) whose centre X is at the middle point of OS and whose radius R is given by the relation

\[
2R^2 + 2d^2 = \rho^2 \tag{i}
\]

where \( d = SX \).

Again, if \( r \) be the radius of the circle inscribed in EFGH,

\[
r = SE \sin SAD = SA \sin SAB \cdot \sin SAD = SA \cdot BC \cdot CD / 4\rho^2 = SA \cdot SC / 2\rho,
\]

thus

\[
2\rho r = \rho^2 - 4d^2 \tag{ii}
\]

If we eliminate \( \rho \) between (i) and (ii) we get

\[
(R + d)^{-2} + (R - d)^{-2} = r^{-2}
\]

which is the known poristic relation.