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The interrelations among various spaces of distributions

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In this paper we discuss the interrelations among various spaces of distributions and show that none of them can be linearly and differentiably homeomorphic to the space of Mikusiński operators. It is also shown that the distributions of Mikusiński-Sikorski can also be defined by the method described by Temple as the completion of the space of continuous functions after introducing a weaker notion of convergence in this space.

In this paper we develop the theory of infinite distributions and the interrelations among the various approaches to the theory of distributions. Our development of the infinite distribution is different from the usual methods of Schwartz or Mikusiński-Sikorski. In the various known theories of distributions every distribution is ultimately realised as an abstract finite derivative of a continuous function for each finite interval. This realisation apparently establishes a one-to-one correspondence among the various formulations of distributions - the only point to be checked up here being the reconstitution into a distribution in the respective defined sense starting with this realisation. As a matter of fact, in the course of this paper we prove that barring Lighthill's generalized functions, Avner Friedman's distributions, Mikusiński's operators, the spaces of distributions as developed in [*4*], [*5*], [*7*], [*8*], and our space of distributions are essentially of the same structure. However, we find that none of them can be linearly and differentiably homeomorphic to the

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space of Mikusiński operators. Though it is obvious that the space of Mikusiński operators, being a field, cannot be algebraically of the same structure as other spaces, it is not so obvious that as a topological translation vector space structure it is different from the other spaces of distributions. We have proved earlier that the field of Mikusiński operators is isomorphic topologically and algebraically to the quotient field of the integrity domain of the right distributions of Schwartz.

Considering the space of generalized functions as defined by Lighthill [3], we find that this is linearly and differentiably homeomorphic to the space of tempered distributions of Schwartz which forms a subspace of the space of distributions of Schwartz, and hence forms a subspace of our space of distributions. Again when we consider Avner Friedman's distributions, we observe that the space of Schwartz distributions is obtained as a particular space of Avner Friedman's distributions.

Temple, in his classical paper on the theories and applications of generalized functions makes the unproved assertion that the distributions obtained by various authors can be obtained essentially by one process, namely by completing the space of continuous functions, introducing a weaker notion of convergence. It is known that the class of distributions of Schwartz and the field of operators of Mikusiński can be obtained by this process. Here we establish that the class of distributions of Mikusiński-Sikorski can also be defined by this method.

Let I denote the collection of all finite open intervals in the real line, and F the collection of all pairs (f, k) where $f \in C(-\infty, \infty)$, $k \in Z_+$. Consider a map $\alpha : I \to F$ such that whenever $I_1, I_2 \in I$ and $I_1 \subset I_2$ we have $\alpha I_1 = (f_1, k_1)$ and $\alpha I_2 = (f_2, k_2)$ satisfying $\int_{k_2} f_1 - \int_{k_1} f_2$ is a polynomial $P_{k_1+k_2}$ of degree less than or equal to $k_1 + k_2$ in I_1. If β is another such map define $\alpha \sim \beta$ if for each $I \in I$, $\alpha I = (f, k)$ and $\beta I = (g, m)$ satisfy $\int_m f - \int_k g = P_{k+m}$ in I. It can easily be seen that this relation $\sim$ among the maps is an equivalence relation.

DEFINITION 1. An equivalence class $[\alpha]$ defined in this sense will

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be called a (infinite) distribution. A distribution $[\alpha]$ is said to be a finite distribution if the coset $[\alpha]$ contains at least one map which associates the same (f, k) to every $I \in \mathcal{I}$.

We denote the class of distributions by $\mathcal{D}$. In an obvious manner $\mathcal{D}$ can be made into a vector space.

REMARK 2. A Mikusiński-Sikorski distribution over (a, b) can also be thought of as a pair (f, k) where $f \in C(a, b)$ and $k \in Z_+$ in a canonical way. Then there is a natural map $t_n : V_{n+1} \to V_n$ where V_n is the space of all distributions over $(-n, n)$ for every $n \in Z_+$. The topology is taken to be the one in Theorem 1 of [2].

Then we may state that the vector space $\mathcal{D}$ is isomorphic to the projective limit of the vector spaces V_n of Mikusiński-Sikorski distributions over $(-n, n)$.

Note 3. If for each interval I , $\alpha I = (f, k)$, define $\alpha' I = (f, k+1)$. From this it can be easily seen that if $[\alpha] \in \mathcal{D}$, also $[\alpha'] \in \mathcal{D}$. Anticipating Theorem 4 below we write $[\alpha'] = [\alpha]'$ and call $[\alpha]'$ the derivative of $[\alpha]$.

In a similar manner, for each real number h , define $\tau_h . \alpha I = (f_h, k)$ where f_h denotes the translation of the function f through a distance h . We may check that if $[\alpha] \in \mathcal{D}$, then $[\tau_h \alpha] \in \mathcal{D}$. We write $[\tau_h, \alpha] = \tau_h[\alpha]$.

We state the following theorem whose proof is easy:

THEOREM 4. $\operatorname*{Lt}_{h \to 0} \dfrac{\tau_h[\alpha]-[\alpha]}{h} = [\alpha]'$ *where* $[\alpha] \in \mathcal{D}$.

Since it can be easily shown that whenever $[\alpha]_n$ converges to $[\alpha]$, $[\alpha]_n'$ also converges to $[\alpha]'$, the space $\mathcal{D}$ becomes a C-D space, [2].

REMARK 5. The same procedure can be followed in the case of half-line.

DEFINITION 6. A finite distribution $[\alpha]$ is said to have left

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bounded support if in the representation $\alpha I = (f, k)$, f is a polynomial of degree less than k in $(-\infty, a)$ for some a .

THEOREM 7. *There exists a one-one biocontinuous linear differentiable map from the space of finite distributions of* $\mathcal{D}$ *with left bounded support onto the space of distributional operators of* Mikusiński.

The proof follows easily from the definitions of distributions of $\mathcal{D}$ with left bounded support and distributional operators of Mikusiński [5] and from the defintions of convergence in both the spaces.

THEOREM 8. *There exists a one-one bicontinuous linear differentiable map from the space* $\mathcal{D}$ *onto the space of* Mikusiński *distributions* [10] *which maps every continuous function into itself.*

Proof. If $[\alpha] \in \mathcal{D}$, let α be a member of $[\alpha]$. Let us further assume that α is so chosen that if $\alpha(-n, n) = (f_n, k_n)$ and $\alpha(-n-1, n+1) = (f_{n+1}, k_{n+1})$ then $\int_{k_{n+1}} f_n - \int_{k_n} f_{n+1} = 0$ in $(-n, n)$ for all $n = 1, 2, 3, \ldots$. Consider the fundamental sequence f_n/l^{k_n} of distributional operators. Obviously, the fundamental sequence is changed into an equivalent such sequence by another similar choice of representative in $[\alpha]$. By the property of the map α , the distributional operators f_n/l^{k_n} and f_m/l^{k_m} are equivalent in $(-n, n)$ whenever $n < m$. Let θ be the map from $\mathcal{D}$ to the space of Mikusiński distributions defined by $\theta[\alpha] = f_n/l^{k_n}$ where f_n/l^{k_n} is the Mikusiński distribution represented by the fundamental sequence f_n/l^{k_n} . It is clear that the map is well-defined and linear. If $[\alpha] \neq [\beta]$ then $\alpha \not\approx \beta$. Therefore, for at least one interval $(-n, n)$, $\alpha(-n, n) \not\approx \beta(-n, n)$. Consequently, $\theta[\alpha] \neq \theta[\beta]$. The map θ is also onto. Consider a Mikusiński distribution $d = f_n/l^{k_n}$. For any interval I , consider the smallest integer interval $(-n, n)$ bigger than I . Let α be the map taking I to (g_n, k_n) where $g_n = f_n$. If J is any

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interval bigger than I , we can easily verify that αJ and αK are equivalent in I as elements of F . If we consider any other map β by choosing another fundamental sequence of the class defining g , then $\beta \approx \alpha$. Thus $[\alpha]$ will define a distribution and evidently $\theta[\alpha] = d$. It may be seen without any difficulty that θ is also bicontinuous and differentiable.

REMARK 9. Similar theorems can be established:

(i) between $\mathcal{D}$ and the space of Mikusiński-Sikorski distributions;

(ii) between $\mathcal{D}$ and the space of Schwartz distributions D' ; and hence

(iii) between the space of Mikusiński-Sikorski distributions and the space of Schwartz distributions.

THEOREM 10. *There does not exist any one-to-one linear onto bicontinuous map between the space* D' *and the operators of* Mikusiński [5] *which is considered as a linear space over the real field.*

Proof. The space of distributions of Schwartz does allow non-zero continuous linear maps but the space of Mikusiński operators does not allow any such continuous linear functional, [*1*]. Thus the result is clear.

By Remark 9 and Theorem 10, we have

THEOREM 11. *There does not exist any one-to-one linear onto bicontinuous map between the space of distributions of* Mikusiński-Sikorski *and the space of operators of* Mikusiński.

REMARK 12. We may identify each continuous function f in $(-\infty, \infty)$ with a distribution $[f]$ by defining a map $f : I \to$ F as fI to be the pair $(f, 0)$. Whenever a continuous function f has its ordinarily defined derivative f' , it may be seen that $[f]' = [f']$. Even if it happens that f is only a locally summable function, it may be identified with a distribution $[f]$ in $\mathcal{D}$ by defining $fI = \left(\int f, 1\right)$.

THEOREM 13. *There exists a one-to-one bicontinuous linear differentiable map from the space of generalized functions of* Lighthill *onto the space of tempered distributions of* Schwartz.

Proof. Let F be a generalized function of Lighthill. Then

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$F = [f_m]$ where f_m is a regular sequence of good functions defining F. To show that is can be viewed as a Schwartz distribution, we need show that it is a continuous linear functional over the space of smooth functions with compact support. That it is a linear functional is trivial. To show that it is continuous we have to show that whenever a sequence ϕ_n of smooth compact functions converges to zero, $\langle F, \phi_n \rangle$ also converges to zero. Now from [10] f_m is regular if and only if $f_m = \sum a_m G_m^{(k)}$ and $(1+x^2)^{\upsilon} G_m$ converges almost uniformly for some integer υ where the G_m are good functions.

$$\operatorname*{Lt}_{n} \langle F, \phi_n \rangle = \operatorname*{Lt}_{m} \operatorname*{Lt}_{n} \langle f_m, \phi_n \rangle$$
$$= \operatorname*{Lt}_{m} \operatorname*{Lt}_{n} \left\langle \sum a_m G_m^{(k)}, \phi_n \right\rangle$$
$$= \operatorname*{Lt}_{m} \operatorname*{Lt}_{n} (-1)^k \left\langle \sum a_m G_m, \phi_n^{(k)} \right\rangle ,$$

and this limit exists.

So every generalized function F can be thought of as a Schwartz distribution. That the correspondence is one-to-one is obvious. To show that the correspondence is also onto, consider a Schwartz distribution of T obtained as a continuous linear functional over the spaces of good functions. We know that S is dense in E. Therefore, by duality, we have that E' is dense in S'. So, every tempered distribution in S' is a limit of a sequence of compact distributions (that is, distributions with compact support).

Now define

$$f_{1/n}(x-t) = 0 \qquad \text{for} \quad |x-t| \geq 1/n$$
$$= e^{-\varepsilon^2/\varepsilon^2-(x-t)^2} \qquad \text{for} \quad |x-t| < 1/n .$$

Denote $f_{1/n}(x-t) = \alpha_n(x-t)$.

Now $T*\alpha_n = \operatorname*{Lt}_{m} \left(T_m*\alpha_n\right) = g_n$.

Now g_n is a sequence of infinitely differentiable functions with compact support hence, a sequence of good functions.

Consider

$$\langle g_n, G\rangle = \left\langle \operatorname*{Lt}_m \left(T_m * \alpha_n\right), G\right\rangle \quad \text{(where } G \text{ is a good function)}$$

$$= \left\langle \operatorname*{Lt}_m T_m, \hat{\alpha}_n * G\right\rangle \quad \left(\text{where } \hat{\alpha}_n(x) = \alpha_n(-x)\right) ;$$

$$\operatorname*{Lt}_n \langle g_n, G\rangle = \operatorname*{Lt}_m \operatorname*{Lt}_n \langle T_m, \hat{\alpha}_n * G\rangle$$

$$= \operatorname*{Lt}_m \langle T_m, G\rangle .$$

The limit on the right hand side exists, and therefore, g_n is a regular sequence. Hence, it defines a generalized function of Lighthill. That the map is bicontinuous is obvious since the topologies of the two spaces are identical. It is also easy to verify that the map is linear and differentiable.

THEOREM 14. *There exists a one-to-one bicontinuous linear differentiable map from the space of distributions of* Liverman *onto the space of distributions of* Schwartz *with right support which maps every smooth function vanishing in a neighbourhood of* $-\infty$ *into itself.*

Proof. Refer [4].

Temple in his paper [9] on the theories and applications of generalized functions makes the unproved assertion that the distributions obtained by various authors can be obtained essentially by one process, namely the following: Introduce a weaker notion of convergence and obtain the closure with respect to this convergence. Temple has stated that we could start from abstract spaces instead of starting from the space of continuous functions and introduce a notion of convergence with respect to which this is closed and obtain distributions over this space. He has taken up this idea from Mikusiński [6]. The idea is to consider three spaces F, Φ, C and C is assumed to have a notion of convergence. It is also assumed that there exists a definite mapping $\langle f, \phi\rangle$ of $f \in F$, $\phi \in \Phi$ into C. This mapping $\langle f, \phi\rangle$ is such that if $\langle f, \phi\rangle = \langle g, \phi\rangle$ for all $\phi \in \Phi$ then $f = g$. It can easily be seen that such a mapping introduces a notion of convergence in F. Completing the space F under this convergence by the well known method of Cauchy, gives the class of distributions obtained from F .

Keeping F, Φ, C to be the class of continuous functions on $[0, \infty)$ and defining $\langle f, \phi\rangle = \int_0^t f(t-x)\phi(x)dx$ we get the class of Mikusiński operators.

Keeping F, Φ to be the class of smooth compact functions and C to be the class of real numbers and defining $\langle f, \phi\rangle = \int_R f(x)\phi(x)dx$ we get the class of Schwartz distributions.

From Remark 9, we find that the Mikusiński-Sikorski distributions also can be viewed as continuous linear functionals over the class of smooth compact functions. Hence, the Mikusiński-Sikorski distributions could be obtained from the Temple process.

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