Let \( n \) be a positive integer or infinity (denoted \( \infty \)), \( k \) a positive integer. We denote by \( \Omega_k(n) \) the class of groups \( G \) such that, for every subset \( X \) of \( G \) of cardinality \( n + 1 \), there exist distinct elements \( x, y \in X \) and integers \( t_0, t_1, \ldots, t_k \) such that \([x_0, x_1, \ldots, x_k] = 1\), where \( x_i \in \{x, y\} \), \( i = 0, 1, \ldots, k \), \( x_0 \neq x_1 \). If the integers \( t_0, t_1, \ldots, t_k \) are the same for any subset \( X \) of \( G \), we say that \( G \) is in the class \( \Omega_k(n) \). The class \( \mathcal{U}_k(n) \) is defined exactly as \( \Omega_k(n) \) with the additional conditions \( x_i^2 \neq 1 \). Let \( t_2, t_3, \ldots, t_k \) be fixed integers. We denote by \( \mathcal{W}_k^* \) the class of all groups \( G \) such that for any infinite subsets \( X \) and \( Y \) there exist \( x \in X, y \in Y \) such that \([x_0, x_1, x_2^2, \ldots, x_k^t] = 1\), where \( x_i \in \{x, y\}, x_0 \neq x_1, i = 2, 3, \ldots, k \). Here we prove that

1. \( \mathcal{U}_k(2) \) is a finitely generated soluble group, then \( G \) is nilpotent.
2. If \( G \in \Omega_k(\infty) \) is a finitely generated soluble group, then \( G \) is nilpotent-by-finite.
3. If \( G \in \mathcal{W}_k(n), \) \( n \) a positive integer, is a finitely generated residually finite group, then \( G \) is nilpotent-by-finite.
4. If \( G \) is an infinite \( \mathcal{W}_k^* \)-group in which every nontrivial finitely generated subgroup has a nontrivial finite quotient, then \( G \) is nilpotent-by-finite.

1. INTRODUCTION AND RESULTS

In response to a question of Paul Erdős, B.H. Neumann proved in [19] that a group is centre-by-finite if and only if every infinite subset contains a commuting pair of distinct elements. The extension of the questions of Paul Erdős, firstly, is considered by Lennox and Wiegold [15]. Further questions of a similar nature, with different aspects, have been considered by many people (see for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 23, 24, 20, 25, 26, 27]).

Our notation and terminology are standard, and can be found in [22]. For a group \( G \), and elements \( x, y, x_1, \ldots, x_k \in G \) we write

\[
[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2, \quad [x_1, \ldots, x_k] = [[x_1, \ldots, x_{k-1}], x_k]
\]
A group is said to be \( k \)-Engel (Engel) group if for all \( x, y \in G \), \([x, ky] = 1\) (respectively, there exist a positive integer \( t \) depending on \( x \) and \( y \) such that \([x, ty] = 1\)). The class of \( k \)-Engel (Engel) groups will be denoted by \( \mathcal{E}_k \) (respectively, \( \mathcal{E} \)). For elements \( x, y, z \) of a group \( G \), the following commutative identities will be used frequently without special reference.

\[
[x, y, z] = [x, z]y[y, z], \quad [x, yz] = [x, z][x, y]z
\]

Let \( k \) be a positive integer, \( n \) a positive integer or infinity (denoted \( \infty \)). We denote by \( (\mathcal{N}, n) \) (respectively, \( (\mathcal{N}_k, n) \)) the class of all groups \( G \) such that, for every subset \( X \) of cardinality \( n + 1 \), there exist distinct elements \( x, y \in X \) such that \((x, y)\) is nilpotent (respectively, nilpotent of class at most \( k \)). We also denote by \( \mathcal{E}_k(n) \) (respectively, \( \mathcal{E}(n) \)) the class of all groups \( G \) such that, for every subset \( X \) of cardinality \( n + 1 \), there exist distinct elements \( x, y \in X \) such that \([x, ky] = 1\) (respectively, \([x, ty] = 1\) for some positive integer \( t \) depending on \( x, y \)). Lennox and Wiegold [15] proved that a finitely generated soluble group \( G \in (\mathcal{N}, \infty) \) if and only if \( G \) is finite-by-nilpotent. Abdollahi and Taeri [3] proved that a finitely generated soluble group \( G \in (\mathcal{N}_k, \infty) \) if and only if \( G \) is a finite extension by a group in which any two generator subgroup is nilpotent of class at most \( k \).

Let \( n \) be a positive integer or \( n = \infty \). We denote by \( \Omega(n) \) the class of groups \( G \) in which for every subset \( X \) of \( G \) of cardinality \( n + 1 \), there exist distinct elements \( x, y \in X \), such that the following condition holds.

There exist a positive integer \( k \) and elements \( x_0, x_1, \ldots, x_k \in \{x, y\} \) with \( x_0 \neq x_1 \), and integers \( t_0, t_1, \ldots, t_k \), such that \([x_0^{t_0}, x_1^{t_1}, \ldots, x_k^{t_k}] = 1\).

If the integer \( k \) is the same for any subset \( X \) of \( G \), we say that \( G \) is in the class \( \Omega_k(n) \). If \( G \in \Omega_k(n) \) and the integers \( t_0, t_1, \ldots, t_k \) are the same for any subset \( X \) of \( G \), we say that \( G \) is in the class \( \overline{\Omega}_k(n) \). Since all torsion groups belong to \( \Omega_k(n) \), we define another class of groups: the class \( \mathcal{U}_k(n) \) (respectively, \( \overline{\mathcal{U}}_k(n) \)) is defined exactly as \( \Omega_k(n) \), (respectively, \( \overline{\Omega}_k(n) \)) with additional conditions \( x_i^{t_i} \neq 1 \), \( i = 0, 1, \ldots, k \).

Also we denote by \( \mathcal{W}_k^* \) the class of all groups \( G \) such that for any infinite subsets \( X \) and \( Y \) there exist \( x \in X, y \in Y \) such that \([x_0, x_1, x_2^{t_2}, \ldots, x_k^{t_k}] = 1\), where \( t_i \) is an integer, \( x_i \in \{x, y\}, x_0 \neq x_1, i = 2, 3, \ldots, k \). If the integers \( t_2, t_3, \ldots, t_k \) are the same, we obtain the class \( \overline{\mathcal{W}}_k^* \). Note that

\[
(\mathcal{N}_k, n) \subseteq \mathcal{E}_k(n) \subseteq \mathcal{U}_k(n) \subseteq \Omega_k(n) \subseteq \Omega_k(n + 1),
\]

and

\[
\overline{\mathcal{U}}_k(n) \subseteq \overline{\Omega}_k(n) \subseteq \Omega_k(\infty) \quad \text{and} \quad \overline{\mathcal{W}}_k^* \subseteq \mathcal{W}_k^* \subseteq \Omega_k(\infty).
\]
Endimioni [9] proved that if $G \in (\mathcal{N}, n)$, $n \leq 3$, is a finite group, then $G$ is nilpotent. Abdollahi [2] considered $\mathcal{E}(2)$-groups and proved that if $G \in \mathcal{E}(2)$ is a finite group, then $G$ is nilpotent. We generalise this result.

**Theorem 1.** Let $G$ be a finite group with the condition $\mathcal{U}_k(2)$, then $G$ is nilpotent.

Note that $\mathcal{E} \subseteq \mathcal{U}(1) \subseteq \mathcal{U}(2)$. Thus Theorem 1 is a generalisation of a well known result due to Zorn (see for example [22, Theorem 12.3.4]) which states that a finite Engel group is nilpotent. Trabelsi [27] proved that a finitely generated soluble group $G$ is nilpotent-by-finite if and only if for every pair $X, Y$ of infinite subsets of $G$ there exist $x$ in $X$, $y$ in $Y$ and two positive integers $m = m(x, y)$, $n = n(x, y)$ satisfying $[x, n y^m] = 1$. We generalise this result and prove that

**Theorem 2.** Let $G \in \Omega_k(\infty)$ be a finitely generated soluble group. Then $G$ is nilpotent-by-finite.

Longobardi and Maj [16] (see also [8]) proved that a finitely generated soluble group $G \in \mathcal{E}(\infty)$ if and only if $G$ is finite-by-nilpotent. Theorem 2 and [2, Lemma 7] gives another proof for this result. Abdollahi [2] has proved that a finitely generated residually finite $\mathcal{E}_k(n)$-group, $n$ a positive integer, is finite-by-nilpotent. We consider the weaker condition $\Omega_k(n)$ and obtain the following result.

**Theorem 3.** Let $G \in \Omega_k(n)$ be a finitely generated residually finite group. Then $G$ in nilpotent-by-finite.

Recall that a group $G$ is said to be locally graded whenever every finitely generated subgroup has a nontrivial finite quotient. We say that a group $G$ is an $\mathcal{E}_k^*$-group provided that whenever $X$, $Y$ are infinite subsets of $G$, there exist $x$ in $X$ and $y$ in $Y$, such that $[x, ky] = 1$. Note that $\mathcal{E}_k^* \subseteq \mathcal{W}_k^*$. Puglisi and Spiezia [20] proved that every infinite locally finite or locally soluble $\mathcal{E}_k^*$-group is a $k$-Engel group. Abdollahi [1] improved this result for locally graded groups. We consider $\mathcal{W}_k^*$.

**Theorem 4.** Let $G \in \mathcal{W}_k^*$ be a locally graded group. Then $G$ is nilpotent-by-finite.

2. **Proofs**

We begin by an easy lemma without proof.

**Lemma 1.** Let $G$ be a group with $A$ as an Abelian normal subgroup, and let $g$ be any element of $G$, then for all distinct elements $a$ and $b$ of $A$ we have

$$[x_0^{a_0}, x_1^{a_1}, \ldots, x_k^{a_k}] = [g^{a_0}, ab^{-1}, g^{a_1}, g^{a_2}, \ldots, g^{a_k}]$$

where $x_i \in \{g^a, g^b\}$, $x_0 = g^a, x_1 = g^b$. 
**LEMMA 2.** Let $G$ be an infinite group in $\Omega_k(\infty)$, and $A$ be a normal Abelian subgroup of $G$. If there exists a torsion free element $g$ of $G$ such that the centraliser of $g^m$ in $A$, $C_A(g^m) = 1$, for all integer $m$, then $A$ is finite.

**PROOF:** Suppose that $A$ is infinite. Then the set $g^A = \{g^a \mid a \in A\}$ is infinite, as $C_A(g) = 1$. Now, since $G \in \Omega_k(\infty)$, there exist distinct elements $a, b \in A$ and integers $t_i$ such that $[x_0^a, x_1^b, \ldots, x_k^b] = 1$, where $x_0 = g^a, x_1 = g^b$, and $x_i \in \{g^a, g^b\}$, $i = 0, 1, 2, \ldots k$. Thus, by Lemma 1, $[g^a, ab^{-1}, g^{x_2}, \ldots, g^{x_k}] = 1$. Now, since $A$ is normal Abelian, $u = [g^a, ab^{-1}, g^{x_2}, \ldots, g^{x_k-1}] \in C_A(g^a) = 1$. So $u = 1$. Continuing in this way we find that $[g^a, ab^{-1}] \in C_A(g^a) = 1$. So $ab^{-1} \in C_A(g^a) = 1$, a contradiction. $\square$

The following lemma is proved similarly.

**LEMMA 3.** Let $G$ be a group in $U_k(n)$, $A$ be a normal Abelian subgroup of $G$. If there exists $g \in G$, such that $C_A(g^m) = 1$, for all integers $m$ with $g^m \neq 1$, then $|A| \leq n$.

Now we are ready to prove Theorem 1.

**PROOF OF THEOREM 1:** Suppose the assertion of the Theorem is false and choose a counter-example $G$ of smallest order. Now $G$ is a finite minimal nonnilpotent group. By a result of Schmidt (see [22, Theorem 9.1.9]) $G$ is a $\{p, q\}$-group, where $p, q$ are distinct primes and $G$ has a normal Sylow $p$-subgroup $P$ and a cyclic Sylow $q$-subgroup $Q$. Let $x \in Q$ be an element of order $q$. Then, since the centre of $G$ $Z(G) = 1$, we have $C_{Z(P)}(x) = 1$, and therefore, by Lemma 3, $|Z(P)| \leq 2$. Thus $Z(P) \leq Z(G) = 1$, a contradiction. Therefore $G$ is nilpotent. $\square$

Note that $S_3 \in U_2(3)$, so that the bound 2 in Theorem 1 cannot be improved.

**COROLLARY 1.** Let $G \in U_k(2)$ be a finitely generated soluble group. Then $G$ is nilpotent.

**PROOF:** If $G$ is not nilpotent, then by a result of Robinson and Wehrfritz (see [22, Theorem 15.5.3]) $G$ has a nontrivial finite nonnilpotent image. This contradicts Theorem 1.

A result of Kropholler states that if a finitely generated soluble group $G$ has no section isomorphic to the restricted wreath product of a cyclic group of prime order with the infinite cyclic group, then $G$ has finite rank. Therefore to prove Theorem 2, we first prove the following.

**LEMMA 4.** Let $G = A \text{ Wr } B$ be the restricted wreath product of a cyclic group $A$ of prime order $p$ with the infinite cyclic group $B = \langle b \rangle$. Then $G \not\in \Omega_k(\infty)$.

**PROOF:** Let $R = A^B$, be the base group of $G = A \text{ Wr } B$. We shall write $r \cdot b$ for the conjugate of $r$ under $b$ and $r \cdot (b^{\lambda} + b^{\mu})$ for $r \cdot b^{\lambda} + r \cdot b^{\mu}$, for all $r \in R$ and integers $\lambda, \mu$. Every element of $R$ can be expressed in the form

$$f(b) = \sum_{i=0}^{m-1} r_i \cdot b^i$$
with $r_i \in R_0$, where $R_0$ is the first isomorphic copy of $A$ in $R$. Note that $R$ is a free \( \mathbb{F}_p \langle b \rangle \)-module with basis \{r\}, where $R_0 = \langle r \rangle$. Thus the above $f(b)$ can be written as

\[
f(b) = r \cdot \sum_{i=0}^{m-1} s_i b^i\]

where $s_i \in \mathbb{F}_p$. Elements of $G$ will be expressed in the form \((f(b), b^\lambda)\) with $f(b) \in R$ and $\lambda \in \mathbb{Z}$, with multiplication given by \((f(b), b^\lambda)(g(b), b^\mu) = (f(b) + g(b) \cdot b^{-\lambda} \cdot b^{\lambda+\mu})\).

Note that the conjugate \((b^{f(b)})^\lambda\) of $b^\lambda$ under an element $f(b)$ of $R$ is expressed in the form \((-f(b) + f(b) \cdot b^{-\lambda}, b^\lambda)\). Also we have the commutator identity \([f(b), b^x] = f(b) \cdot (-1 + b^x)\).

Now suppose, for a contradiction, that $G \in \Omega_k(\infty)$, and consider the elements $f_i(b) = r \cdot b^i, i = 1, 2, 3, \ldots$. Since $G \in \Omega_k(\infty)$ there exists $i \neq j$ such that \([x_0, x_1, x_2, \ldots, x_k] = 1\), where $x_0 = b^{f_i(b)}$, $x_1 = b^{f_j(b)}$, $x_s \in \{b^{f_i(b)}, b^{f_j(b)}\}$. Since $R$ is a normal Abelian subgroup of $G$ we have, by Lemma 1, that

\[
[b^{t_0}, f_i f_j^{-1}, b^{t_1}, b^{t_2}, \ldots, b^{t_k}] = 1,
\]

or in additive notation,

\[
0 = (f_i - f_j) \cdot (1 - b^{t_0})(-1 + b^{t_1})(-1 + b^{t_2}) \cdots (-1 + b^{t_k})
\]

\[
= r \cdot (b^i - b^j)(1 - b^{t_0})(-1 + b^{t_1})(-1 + b^{t_2}) \cdots (-1 + b^{t_k}).
\]

Since $R$ is a free \( \mathbb{F}_p \langle b \rangle \)-module with basis \{r\}, in the group ring $\mathbb{F}_p \langle b \rangle$, we have

\[
(b^i - b^j)(1 - b^{t_0})(1 - b^{t_1})(1 - b^{t_2}) \cdots (1 - b^{t_k}) = 0,
\]

a contradiction, as the order of $b$ is infinite.

The proof of Theorem 2 is similar to that of [11, Theorem 2]. We include it for completeness.

**Proof of Theorem 2:** Suppose $G$ is not nilpotent-by-finite. Since $\Omega_k(\infty)$ is a quotient closed class of groups and since finitely generated nilpotent-by-finite groups are finitely presented, it follows, by [21, Lemma 6.17], that we may assume that every proper quotient of $G$ is nilpotent-by-finite. Let $A$ be a nontrivial normal Abelian subgroup of $G$. Then $G/A$ is nilpotent-by-finite. Since, by Lemma 4, $G$ has no section isomorphic to the wreath product of a cyclic group of order prime $p$ with the infinite cyclic group, a result of [14] shows that $G$ has finite rank. This means that, for some positive integer $t$, every proper subgroup of $G$ can be generated by at most $t$ elements. Let $T$ be the torsion subgroup of $A$. Then $T$ is finite and so $C = C_G(T)$, the centraliser of $T$ in $G$, is of finite index in $G$. If $T \neq 1$ then $G/T$ is nilpotent-by-finite and thus $C/T$ is nilpotent-by-finite. Since $T \leq Z(C)$, then $C_G(T)$ and hence $G$ would be nilpotent-by-finite. Thus $T = 1$, and $A$ is torsion free Abelian, and by passing to a suitable subgroup of finite index in $G$, if necessary, we may assume further that $G/A$ is a finitely generated torsion
free nilpotent group. Thus there exists a finite set \( T = \{ t_1, \ldots, t_r \} \) of elements of \( G \) such that \( G = \langle A, T \rangle \), and
\[
A = G_0 \leq \langle G_0, t_1 \rangle = G_1 \leq \cdots \leq \langle G_{r-1}, t_r \rangle = G_r = G
\]
is a central series from \( A \) to \( G \) with torsion free factors. Suppose \( r = 1 \) then \( G = \langle A, t_1 \rangle \).
If \( C_A(t_1^m) = 1 \), for all \( m \), then by Lemma 2, \( A \) is finite, a contradiction. Thus there exists a positive integer \( m_1 \), such that \( C_A(t_1^{m_1}) \neq 1 \). Therefore \( Z(\langle A, t_1^{m_1} \rangle) \neq 1 \) and hence \( D = A \cap Z(\langle A, t_1^{m_1} \rangle) \) is a nontrivial normal subgroup of \( G \). So \( G/D \) is nilpotent-by-finite and hence \( G \) is nilpotent-by-finite. Now assume that we have established the result when \( r < s \), and suppose \( r = s \). Then \( G_{s-1} \) is nilpotent-by-finite and \( G = \langle G_{s-1}, t \rangle \). Let \( H = \langle A, G_{s-1}^m \rangle \) for some suitable \( m > 0 \), so that \( H \) is nilpotent. Let \( Y = A \cap Z(H) \), then \( Y \) is normal in \( \langle H, t_s \rangle \) which is of finite index in \( G \). Moreover \( Z(\langle Y, t_s^m \rangle) \neq 1 \), for some \( m_s \), by Lemma 2. So \( D_1 = Y \cap Z(\langle Y, t_s^m \rangle) \) is a nontrivial subgroup of \( G \) contained in the centre of \( \langle H, t_s^m \rangle \) which is of finite index in \( G \). We may replace \( \langle H, t_s^m \rangle \) by its normal interior in \( G \), if necessary; \( H \) still contains \( A \) and hence \( D_1 \). Now \( \langle H, t_s^m \rangle / D_1 \) is nilpotent-by-finite, \( D_1 \leq Z(\langle H, t_s^m \rangle) \) and \( \langle H, t_s^m \rangle \) is of finite index in \( G \), thus \( G \) is nilpotent-by-finite, a contradiction. \( \square \)

**Corollary 2.** Let \( G \) be a finitely generated soluble group. Then \( G \in \Omega(\infty) \) if and only if \( G \) is nilpotent-by-finite.

Now, we want to consider a finitely generated residually finite group in \( \mathcal{O}_k(n) \), \( n \) a positive integer. We use a result of Wilson [28] which states that if \( G \) is a finitely generated residually finite group and \( N \) is a positive integer such that \( G \) has no section isomorphic to the twisted wreath product \( A \text{twr}_C B \), where \( B \) is finite and cyclic, \( A \) is an elementary Abelian group acted on faithfully and irreducibly by \( C \), and \( |B : C| > N \), then \( G \) is virtually a soluble minimax group. For the definition of the twisted wreath product we refer readers to Neumann [18].

Suppose that \( t_0, t_1, \ldots, t_k \) are fixed integers. Recall that a group \( G \) is in the class \( \mathcal{O}_k(n) \) if for every subset \( X \) of \( G \) of cardinality \( n+1 \), there exist distinct elements \( x, y \in X \), such that \( [x_0^t x_1^t \cdots x_k^t] = 1 \), where \( x_0, x_1, \ldots, x_k \in \{ x, y \} \) with \( x_0 \neq x_1 \). In the following lemma we may assume that \( t_0, t_1, \ldots, t_k \) are positive.

**Lemma 5.** Let \( A \) be a nontrivial Abelian group, \( B = \langle b \rangle \) a finite cyclic group \( C \), a subgroup of \( B \) of index \( m \), and suppose that \( C \) acts on \( A \). Let \( W = A \text{twr}_C B \) be the twisted wreath product of \( A \) by \( B \) with respect to the action of \( C \) on \( A \). If \( G \in \mathcal{O}_k(n) \), \( n \) a positive integer, then \( m \leq n + t_0 + t_1 + \cdots + t_k \).

**Proof:** Suppose that \( C = \langle b^m \rangle \) and let \( Y \) be a transversal to \( C \) in \( B \) so that \( Y = \{ b, b^2, \ldots, b^{m-1} \} \). Then \( W = A \text{twr}_C B = B \times A^Y \) is the splitting extension of \( A^Y \) by \( B \). The action of \( B \) on \( A^Y \) is given by \( f^B(y) = f(y')c^{-1} \), where \( y' \in Y \) and \( c \in C \) are unique elements such that \( yb = cy' \).
Now assume, for a contradiction, that \( m > n + t_0 + t_1 + \cdots + t_k \) and consider \( n + 1 \) elements \( f_0, f_1, \ldots, f_n \) so that

\[
   f_i(b^j) = \begin{cases} 
   1 & j \neq i, \quad j = 1, 2, \ldots, m - 1, \\
   a & j = i,
   \end{cases}
\]

where \( a \) is a fixed nontrivial element of \( A \). Since \( G \in \mathcal{O}_k(n) \) there exists \( i \neq j \) such that

\[
   [x_0^i, x_1^i, \ldots, x_k^i] = 1,
\]

where \( x_0 = b^{i_0}, x_1 = b^{i_1}, x_s \in \{b^{i_0}, b^{i_j}\} \). Then, as in Lemma 4, we have \((f_j f_j^{-1})_{(1-b^{i_0})(1-b^{i_1}) \cdots (1-b^{i_s})} = 1\). If \( i > j \), then for all \( s \in \{0, 1, \ldots, m\} \) we have \( i + s \leq n + t_0 + t_1 + \cdots + t_k \leq m - 1 \), and thus \( b^{i+s} \in Y \). Therefore \( f_j(b^{i+s}) = 1 \), as \( i + s > j \)

and

\[
   (f_j f_j^{-1})^{b^s}(b^j) = f_j^{b^s}(b^j) (f_j^{b^s}(b^j))^{-1} = f_i(b^{i+s}) (f_j(b^{i+s}))^{-1} = \begin{cases} 
   1 & s \neq 0 \\
   a & s = 0.
   \end{cases}
\]

Hence \( (f_j f_j^{-1})_{(1-b^{i_0})(1-b^{i_1}) \cdots (1-b^{i_s})}(b^j) = a \), a contradiction. In the same way, if \( i < j \) we get a contradiction.

Now we are in the position to prove Theorem 3.

PROOF OF THEOREM 3: By Lemma 5, \( G \) has no section isomorphic to \( W = A \wr_{\mathfrak{c}} B \), where \( A \) is elementary Abelian, \( B \) is finite cyclic, and \( C \) is a subgroup of \( B \) which acts faithfully irreducibly on \( A \), such that \( |B : C| > n + t_0 + t_1 + \cdots + t_k \). Thus by a result of Wilson [28], \( G \) is virtually a soluble minimax. Hence there exist a normal subgroup \( H \) of \( G \) with finite index, such that \( H \) in a soluble minimax group. By Theorem 2, \( H \) is nilpotent-by-finite. Hence there exists a normal nilpotent subgroup \( K \) of \( H \) such that \( H/K \) is finite. Therefore \( K \) is finitely generated, and \( G \) is nilpotent-by-finite.

Let \( t_0, t_1, \ldots, t_k \) be fixed integers. We denote by \( \mathcal{O}_k \) the class of all groups \( G \) such that for any infinite subsets \( X \) and \( Y \) there exist \( x \in X \), \( y \in Y \) such that

\[
   [x_0^i, x_1^i, x_2^i, \ldots, x_k^i] = 1,
\]

where \( x_i \in \{x, y\}, x_0 \neq x_1, i = 0, 1, \ldots, k \). Note that \( \mathcal{W}_k \subseteq \mathcal{C}_k \subseteq \mathcal{O}_k(G) \). The proof of the following lemma is easy and hence it is omitted.

LEMA 6. Let \( N \) be an infinite normal subgroup of a \( \mathcal{O}_k \)-group \( G \), then \( G/N \in \mathcal{O}_k(1) \). In particular if \( G \) is an infinite residually finite \( \mathcal{O}_k \)-group, then \( G \in \mathcal{O}_k(1) \).

A result of Wilson (see [28, Theorem 2]) states that a finitely generated residually finite \( k \)-Engel group is nilpotent. As a consequence of Theorems 1 and 3 we can generalise this fact.

COROLLARY 3. Let \( G \) be an infinite finitely generated residually finite \( \mathcal{U}_k(2) \)-group, and \( k \) be a positive integer. Then \( G \) is nilpotent.

PROOF: By Theorem 3, there exists a normal nilpotent subgroup \( H \) of finite index in \( G \). Now \( G/H \in \mathcal{U}_k(2) \). Thus, by Theorem 1, \( G/H \) is nilpotent, and \( G \) is soluble. Now \( G \) is a finitely generated soluble \( \mathcal{U}_k(2) \)-group, and by Theorem 1, is nilpotent.\[\square\]
Recall that a group $G \in \mathcal{W}_k^*$ if for any infinite subsets $X$ and $Y$ there exist $x \in X$, $y \in Y$ such that $[x_0, x_1, x_2^t, \ldots, x_k^t] = 1$, where $t_i$ is an integer, $x_i \in \{x, y\}, x_0 \neq x_1, i = 2, 3, \ldots, k$. If the integers $t_2, t_3, \ldots, t_k$ are the same, we say that $G \in \mathcal{W}_k^*$. Following [12] we say that a group $G$ is restrained if $(x)^{(y)} = (x^y | i \text{ an integer})$ is finitely generated for all $x, y$ in $G$. If there is a bound for the number of generators of $(x)^{(y)}$, then we call $G$ strongly restrained.

**Lemma 7.** Let $G$ be a group in $\mathcal{W}_k^*$. Then $G$ is restrained.

**Proof:** Let $x, y \in G$. We want to show that $(x)^{(y)}$ is finitely generated. The result is clear if the order of $y$ is finite. So assume that $y$ is of infinite order. Consider the sets $X = \{x^n | n \text{ an integer}\}$ and $Y = \{y^m | m \text{ a positive integer}\}$. If $X$ is finite then $(x)^{(y)} = (X)$ is finitely generated, as required. So we may assume that $X$ is infinite.

Since $G \in \mathcal{W}_k^*$, there exist integers $i, j$ such that $[x_0, x_1, x_2^i, \ldots, x_k^i] = 1$ where $x_0 \neq x_1, x = \{x^y, y^j\}$. Hence $[z_0, z_1, z_2^j, \ldots, z_k^j] = 1$, where $z_0 = x$ and $z_1 = y^j$ or $z_0 = y^j$ and $z_1 = x$, and $z_2, \ldots, z_k \in \{x, y^j\}$.

Suppose that $z_1 = z_2 = \cdots = z^t = y^j$, and $z_t = x$, for all $t \neq t_r$. Let $T = \{t_1, t_2, \ldots, t_k\}$, and denote by $T^{(r)}$ the set of all sums of $r$ distinct elements of $T$, and denote by $S^{(r)}$ the set of all sums of $r$ distinct elements of $T \cup \{1\}$. Then it is easy to see that

$$\left\langle x, [z_0, z_1], [z_0, z_1, z_2^j], \ldots, [z_0, z_1, z_2^j, \ldots, z_k^j] \right\rangle = \left\langle x, x^{y^j}, x^{y^{j^r}} | r \in \bigcup_{i=1}^{k} T^{(i)} \cup \bigcup_{i=1}^{k} S^{(i)} \right\rangle.$$

Therefore $(x)^{(y)} \leq (x^{y^{j^r}} | |r| \leq k(t_1 + \cdots + t_k + 1))$. This completes the proof. \(\square\)

**Corollary 4.** Let $G$ be an infinite finitely generated soluble $\mathcal{W}_k^*$-group. Then $G$ is polycyclic.

**Proof:** This follows immediately from Lemma 7 and [12, Corollary 4]. \(\square\)

**Proof of Theorem 4:** As in Lemma 7, $G$ must be strongly restrained. Thus by [12, Theorem A] $G$ is polycyclic-by-finite, and therefore residually finite. Now, by Lemma 6, $G \in \mathcal{W}_k^*(1)$, and thus, by Theorem 3, $G$ is nilpotent-by-finite. \(\square\)

**References**


Nilpotent-by-finite groups


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