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Countability constraints in order-theoretic approaches to computability

Pedro Hack[12] iD, Daniel A. Braun[1] and Sebastian Gottwald[1]

[1]Institute of Neural Information Processing, Ulm University, Ulm, Germany and [2]School of CIT, Department of Computer Science, Technical University of Munich, Garching, Germany
Corresponding author: Pedro Hack; Email: pedro.hack@tum.de

Abstract

Computability on uncountable sets has no standard formalization, unlike that on countable sets, which is given by Turing machines. Some of the approaches to define computability in these sets rely on order-theoretic structures to translate such notions from Turing machines to uncountable spaces. Since these machines are used as a baseline for computability in these approaches, countability restrictions on the ordered structures are fundamental. Here, we show several relations between the usual countability restrictions in order-theoretic theories of computability and some more common order-theoretic countability constraints, like order density properties and functional characterizations of the order structure in terms of multi-utilities. As a result, we show how computability can be introduced in some order structures via countability order density and multi-utility constraints.

1. Introduction

The formalization of computation on the natural numbers was initiated by Turing (Turing 1937, 1938) with the introduction of Turing machines (Rogers 1987). Such an approach is taken as canonical today since other attempts to formalize it have proven to be equivalent (Cutland 1980; Rogers 1987). Because of that, Turing machines are deployed as a baseline for computation from which it is transferred to other spaces of interest. The theory of numbering (Badaev and Goncharov 2000; Ershov 1999), for example, deals with computability on countable sets in general. The case of uncountable sets is more involved. In fact, despite several attempts (Abramsky and Jung 1994; Weihrauch 2012a, b), no canonical way of introducing computability on uncountable sets has been established. This results, for example, in the absence of a formal definition of algorithm on the real numbers. More specifically, the choice of some model or another may result in changes regarding computability of certain elementary operations, like multiplication by 3 (Di Gianantonio 1996).

Among the most extended approaches to computability on uncountable spaces (Kreitz and Weihrauch 1985; Weihrauch 2012a), some rely on order-theoretic structures (Abramsky and Jung 1994; Ershov 1972; Hack et al. 2022a; Keimel 2017; Scott 1970). Of particular importance are those dealing with computability on the real numbers (Di Gianantonio 1996; Edalat and Sünderhauf 1999). These approaches are based on two main features: the mathematical structure they require and the countability restriction they impose on such a structure in order to translate computability

from Turing machines to uncountable sets. We address the general mathematical structure in the accompanying paper (Hack et al. 2022a) and deal with the countability restrictions here. In particular, we are interested in the relationship between such restrictions and both order density and multi-utilities.

More specifically, we begin in Section 2 by recalling a general order-theoretical approach to computable elements on uncountable sets, which was recently introduced in by Hack et al. 2022a). Right after, in Section 3, we relate the countability restrictions in that approach to order density properties. We continue, in Section 4, recalling the more extended, although narrower, approach to computability in domain theory, which we refer to as *uniform computability*, and relating it to the approach in Hack et al. 2022a). In Section 5, we connect order density properties with the countability restrictions in uniform computability. We follow this, in Section 6, linking order density with order completeness and to a weak form of computability for functions, namely, Scott continuity. We finish, in Section 7, addressing the relation between countability restrictions and multi-utilities in both the uniform and non-uniform approaches.

2. Computability via Ordered Sets

In this section, we briefly recall the fundamental notions of an order-theoretic approach to computability on uncountable sets, which was recently introduced by Hack et al. 2022a). We will define a structure that carries computability from Turing machines, namely directed complete partial orders with an effective weak basis. We do not address how computability can be translated from representatives of this structure to other spaces of interest (see Hack et al. 2022a and the references therein).

Before introducing directed complete partial orders, we include some definitions of the formal approach to computability on $\mathbb{N}$ based on Turing machines.

Definition 1 (Computable functions and recursively enumerable sets, Rogers 1987; Turing 1937). *A function $f : \mathbb{N} \to \mathbb{N}$ is* computable *if there exists a Turing machine, which, for all $n \in \mathbb{N}$, halts on input n, that is, finishes after a finite amount of time, and returns $f(n)$. Note what we call a computable function is also referred to as a total recursive function to differentiate it from functions $g : \mathbb{N} \to \mathbb{N}$ where $\mathrm{dom}(g) \subset \mathbb{N}$ holds (Rogers 1987), which we call* partially computable. *A subset $A \subseteq \mathbb{N}$ is said to be* recursively enumerable *if either $A = \emptyset$ or there exists a computable function f such that $A = f(\mathbb{N})$.*

Recursively enumerable sets are, thus, the subsets of $\mathbb{N}$ whose elements can be produced in finite time, as we can introduce the natural numbers one by one in increasing order in a Turing machine and it will output one by one, each in finite time, all the elements in A (possibly with repetitions). Note that there exist subsets of $\mathbb{N}$ which are not recursively enumerable (Rogers 1987). As we are also interested in computability on the subsets of $\mathbb{N}^2$, we translate the notion of recursively enumerable sets from $\mathbb{N}$ to $\mathbb{N}^2$ using pairing functions. A *pairing function* $\langle \cdot, \cdot \rangle$ is a computable bijective function $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.[1] Since it is a common practice (Rogers 1987), we fix in the following $\langle n, m \rangle = \frac{1}{2}(n^2 + 2nm + m^2 + 3n + m)$, the *Cantor pairing function*.

Before continuing, we introduce an important concept for the following, finite maps.

Definition 2 (Finite map, Hack et al. 2022a). *We say a map $\alpha : \mathrm{dom}(\alpha) \to A$, where $\mathrm{dom}(\alpha) \subseteq \mathbb{N}$ is a* finite map for A *or simply a* finite map *if α is bijective and both α and α^{-1} are effectively calculable.*

Finite maps aim to translate computability from the natural numbers to another countable set A. Note the definition of finite maps relies on the informal notion of *effective calculability*. This

is the case since no general formal definition for *computable* maps $\alpha : \mathbb{N} \to A$ is known (Cutland 1980). In fact, the struggle between formal and informal notions of computability, best exemplified by Church's thesis (Rogers 1987), lies at the core of computability theory and is responsible for the introduction of different formal notions of computability (Cutland 1980, Chapter 3). Finite maps are also known as *effective denumerations* (Cutland 1980) or *effective enumerations* (Scott 1970).

We define now the order structure on which we rely to introduce computability in some (potentially uncountable) set P and connect it, right after, with Turing machines.

Definition 3 (Partial order, Bridges and Mehta 2013). *A partial order $\preceq$ on a set P is a reflexive ($x \preceq x$ for all $x \in P$), transitive ($x \preceq y$ and $y \preceq z$ imply $x \preceq z$ for all $x, y, z \in P$), and antisymmetric ($x \preceq y$ and $y \preceq x$ imply $x = y$ for all $x, y \in P$) binary relation. We will call a pair $(P, \preceq)$ a partial order and denote it simply by P.*

We may think of P as a set of data and of $\preceq$ as a representation of the precision (or *information*) relation between different elements in the set. Given $x, y \in P$, we may read $x \preceq y$ like *y is at least as informative as x* or like *y is at least as precise as x*.

We intend now to introduce the idea of some $x \in P$ being the limit of other elements in P, that is, the idea that one can generate some element $y \in P$ via a process that outputs other elements of P (which approximate y to arbitrary precision). This notion is formalized by the least upper bounds of directed sets.

Definition 4 (Direct set and least upper bound, Abramsky and Jung 1994). *$A \subseteq P$ is a directed set if, given $a, b \in A$, there exist some $c \in A$ such that $a \preceq c$ and $b \preceq c$. If $A \subseteq P$ is a directed set, then $b \in P$ is the least upper bound of A if $a \preceq b$ for all $a \in A$ and, given any $c \in P$ such that $a \preceq c$ for all $a \in A$, then $b \preceq c$ holds. We denote the least upper bound of A by $\sqcup A$ and also refer to it as the supremum of A.*

Hence, we can generate some $x \in P$ by generating a directed set A whose upper bound is x, $x = \sqcup A$. We have restricted ourselves to directed sets since we can think of them as the output of some computational process augmenting the precision or information given that, for any pair of outputs, there is a third that contains their information and, potentially, more. Directed sets are, thus, a formalization of a computational process having a direction, that is, processes gathering information in a consistent way. Of particular importance are *increasing sequences* or *increasing chains*, subsets $A \subseteq P$ where $A = (a_n)_{n \geq 0}$, and $a_n \preceq a_{n+1}$ for all $n \geq 0$. We can interpret increasing sequences as the output of some process where information increases every step.

Any process whose outputs increase information should tend toward some element in P, that is, any directed set $A \subseteq P$ should have a supremum $\sqcup A \in P$. A partial order with such a property is called *directed complete* or a *dcpo*. Note, for example, that the partial order $P_0 = \big((0, 1), \leq\big)$ is not directed complete.

Some subsets $B \subseteq P$ are able to generate all the elements in P via the supremum of directed sets contained in B. We refer to them as *weak bases*.

Definition 5 (Weak basis, Hack et al. 2022a). *A subset $B \subseteq P$ of a dcpo P is a weak basis if, for each $x \in P$, there exists a directed set $B_x \subseteq B$ such that $x = \sqcup B_x$.*

We are particularly interested in dcpos where countable weak bases exist, since we intend to inherit computability from Turing machines. In case we have some computational process whose outputs are in B and which is approaching some $x \in P \setminus B$, we would like to be able to provide, after a finite amount of time, the best approximation of x so far. In order to do so, we need to distinguish the outputs we already have in terms of precision. This is possible if the weak basis is *effective*.

Definition 6 (Effective weak basis, Hack et al. 2022a). *A countable weak basis $B \subseteq P$ of a dcpo P is effective if there exist both a finite map for $B = (b_n)_{n \geq 0}$ and a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $f(\mathbb{N}) \subseteq \{\langle n, m \rangle | b_n \preceq b_m\}$ and, for each $x \in P$, there is a directed set $B_x \subseteq B$ such that $\sqcup B_x = x$ and, if $b_n, b_m \in B_x \setminus \{x\}$, then there exists some $b_p \in B_x$ such that $b_n, b_m \prec b_p$ and $\langle n, p \rangle, \langle m, p \rangle \in f(\mathbb{N})$.*

The intuition behind the effectivity is that we can, by finite means, get progressively more *informative* elements from some directed set. Note we may show a countable weak basis $B = (b_n)_{n \geq 0}$ is effective by proving the stronger property that

$$\{\langle n, m \rangle | b_n \preceq b_m\}$$

is recursively enumerable. If this stronger condition is satisfied, then, for any finite subset $(b_n)_{n=1}^N \subseteq B$ where all elements are related, we can find some $n_0 \leq N$ such that $b_n \preceq b_{n_0}$ for all $n \leq N$ and, since $\alpha : \mathbb{N} \to B$ is a finite map, we can determine the best approximation so far, $\alpha(n_0)$. We define now computable elements for dcpos with an effective weak basis.

Definition 7 (Computable element, Hack et al. 2022a). *If P is a dcpo, $B \subseteq P$ is an effective weak basis and α is a finite map for B, then an element $x \in P$ is computable if there exists some $B_x \subseteq B$ such that the properties in the definition of effectivity are fulfilled and $\alpha^{-1}(B_x) \subseteq \mathbb{N}$ is recursively enumerable.*

We have achieved the goal of deriving computability (for potentially uncountable sets) from Turing machines via dcpos. Note that computable elements generalize the approach by Turing to computability on $P_{inf}(\mathbb{N})$, the family of infinite subsets of $\mathbb{N}$ (see Hack et al. 2022a). The dependence of computability on the order-theoretic model is also addressed by Hack et al. 2022a).

To recapitulate, the main features of our picture are (1) a map from the natural numbers to some countable set of *finite* labels $B = (b_n)_{n \geq 0}$ and (2) a partial order $\preceq$ which can be somewhat encoded via a Turing machine and which allows us to both associate to some infinite element of interest $x \in P$ a subset of our labels $B_x \subseteq B$ which converges to it and, in some sense, to provide approximations of x to arbitrary precision. As a result, the computability of x reduces to whether B_x can be finitely described or not.

Note that the structure of the partial order P is fundamental to address higher type computability. In case P is *trivial* (also known as *discrete*), that is, $x \preceq y$ if and only if $x = y$ for all $x, y \in P$ (Abramsky and Jung 1994), we cannot extend computability beyond countable sets and we end up considering countable sets with finite maps toward the natural numbers. This situation, thus, reduces our approach to the *theory of numberings* (Badaev and Goncharov, 2000, 2008; Ershov 1999).

While the set of computable elements in a dcpo is countable, since the set of recursively enumerable subsets of $\mathbb{N}$ is countable (Rogers 1987), the cardinality of a dcpo with an effective weak basis is bounded by the cardinality of the continuum $\mathfrak{c}$ (see Hack et al. 2022a). In fact, it is in the uncountable case where the order structure is of interest since the theory of numbering is insufficient.

2.1 Examples
To conclude this section, we list three examples of dcpos with effective weak bases, which will be relevant in the following.

2.1.1 The Cantor domain
If Σ is any finite set of symbols, an *alphabet*, we denote by Σ^* the set of finite strings of symbols in Σ and by Σ^ω the set of countably infinite sequences of symbols. The union of these last two sets is

called the *Cantor domain* or the *Cantor set model* (Blanck 2008; Martin and Mislove 2000) when we equip it with the prefix order. That is, the Cantor domain is the pair $(\Sigma^\infty, \preceq_C)$, where

$$\Sigma^\infty := \left\{ x \;\middle|\; x : \{1, \ldots, n\} \to \Sigma, \; 0 \leq n \leq \infty \right\},$$
$$x \preceq_C y \iff |x| \leq |y| \text{ and } x(i) = y(i) \text{ for all } i \leq |x|, \tag{1}$$

$|s|$ is the cardinality of the domain of $s \in \Sigma^\infty$, and $|\Sigma| < \infty$. One can see Σ^* is an effective weak basis for Σ^∞ (Hack et al. 2022a).

2.1.2 The interval domain

The *interval domain* (Di Gianantonio 1996; Edalat and Sünderhauf 1999; Scott 1970) consists of the pair $(\mathscr{I}, \sqsubseteq)$, where

$$\mathscr{I} := \left\{ [a, b] \subseteq \mathbb{R} \;\middle|\; a, b \in \mathbb{R}, \; a \leq b \right\} \cup \left\{ \perp \right\}, \text{ and}$$
$$x \sqsubseteq y \iff x = \perp \text{ or } x = [a, b], y = [c, d], a \leq c \text{ and } d \leq b. \tag{2}$$

Note that one can see $B_{\mathscr{I}} := \{[p, q] \subseteq \mathbb{R} | p \leq q, \; p, q \in \mathbb{Q}\} \cup \{\perp\}$ is an effective weak basis for $(\mathscr{I}, \sqsubseteq)$. If P is a partial order, we will denote by $\perp$ an element $x \in P$, if it exists, such that $x \preceq y$ for all $y \in P$, as we just did for the interval domain.

2.1.3 Majorization

For any $n \geq 2$, *majorization* (Gottwald and Braun 2019; Martin 2006; Marshall et al. 1979) consists of the pair $(\Lambda^n, \preceq_M)$, where

$$\Lambda^n := \left\{ x \in [0, 1]^n \;\middle|\; \sum_{i=1}^n x_i = 1 \text{ and (for all } i < n) \; x_i \geq x_{i+1} \right\},$$
$$x \preceq_M y \iff (\text{for all } i < n) \; s_i(x) \leq s_i(y), \tag{3}$$

and $s_i(x) := \sum_{j=1}^i x_j$ for all $i < n$. Note that majorization has an effective weak basis, as the following proposition (which we prove in the Appendix A.2.1) states.

Proposition 1. *If $n \geq 2$, then $\mathbb{Q}^n \cap \Lambda^n$ is an effective weak basis for majorization.*

Proposition 1 relies on a stronger property fulfilled by majorization, which we state in Lemma 1 and prove in the Appendix A.2.4.

Lemma 1. *If $x \in \Lambda^n \setminus \{\perp\}$, then for all $\varepsilon > 0$ there exists some $q \in \mathbb{Q}^n \cap \Lambda^n$ such that $s_k(x) - \varepsilon < s_k(q) < s_k(x)$ for all $k < n$.*

3. Order Density and Weak Bases

In the approach to computability from Section 2, dcpos with countable weak bases are the fundamental structure allowing to define computability. In this section, we relate order density properties to countable weak bases. In order to do so, we need some extra terminology. If P is a partial order and $x, y \in P$, we say y is *strictly preferred* to x or x is *strictly below y* for $x, y \in P$ and denote it by $x \prec y$ if $x \preceq y$ and $\neg(y \preceq x)$ hold. In case we have $\neg(x \preceq y)$ and $\neg(y \preceq x)$, we say x and y are *incomparable* and denote it by $x \bowtie y$. We introduce now two order density properties, which will play a major role in the following.

Definition 8 (Order density properties). *A subset $D \subseteq P$ of a partial order P is* Debreu dense *if, for any $x, y \in P$ such that $x \prec y$, there exists some $d \in D$ such that $x \preceq d \preceq y$. We say P is* Debreu separable *if there exists a countable Debreu dense subset $D \subseteq P$ (Bridges and Mehta 2013; Debreu 1954). Moreover, we say $D \subseteq P$ is* Debreu upper dense *if, for all $x, y \in P$ such that $x \bowtie y$, there exists some $d \in D$ such that $x \bowtie d \preceq y$ (Hack et al. 2022b). Lastly, we say P is* Debreu upper separable *if there exists a countable subset $D \subseteq P$, which is Debreu dense and Debreu upper dense (Hack et al. 2022b).*

To exemplify the previous properties, note that Σ^* is a countable Debreu dense subset of the Cantor domain Σ^∞ since, given $a, b \in \Sigma^\infty$ such that $a \prec_C b$, we have $a \in \Sigma^*$ and, taking $d = a$, we get $a \preceq_C d \preceq_C b$. Moreover, note that Σ^* is a countable Debreu upper dense subset of the Cantor domain since, given $a, b \in \Sigma^\infty$ such that $a \bowtie b$, we can either pick $d = b$ if $b \in \Sigma^*$ or some $d \in \Sigma^*$ such that $a \bowtie d \prec b$ if $b \in \Sigma^\omega$. Lastly, in particular, the Cantor domain is Debreu upper separable.

As a first relation between order density and weak bases, we show, in Proposition 2, that having a countable weak basis implies there is a countable Debreu upper dense subset and also a countable set that fulfills a weak form of Debreu density. In fact, although we are primarily interested in the countable case, we show a more general relation, where countability plays no role.

Proposition 2. *If P is a dcpo and $B \subseteq P$ is a weak basis, then B is a Debreu upper dense subset. Furthermore, if $x \prec y$, then there exists some $b \in B$ such that $b \preceq y$ and either $x \prec b$ or $x \bowtie b$ holds.*

Proof. For the first statement, consider $x, y \in P$ such that $x \bowtie y$. By definition of weak basis, there exists $B_y \subseteq B$ such that $\sqcup B_y = y$. If we have $b \preceq x$ for all $b \in B_y$, then, by definition of supremum, we would have $y \preceq x$, a contradiction. There exists, thus, $b_0 \in B_y$ such that $\neg(b_0 \preceq x)$. Moreover, $x \preceq b_0$ also leads to contradiction as we would have, by transitivity, $x \preceq y$. Thus, we have $x \bowtie b_0 \preceq y$ and B is a Debreu upper dense subset of P. For the second statement, consider $x, y \in P$ such that $x \prec y$ and $B_y \subseteq B$ such that $\sqcup B_y = y$. Notice $b \preceq y$ for all $b \in B_y$ by definition while $b \preceq x$ for all $b \in B_y$ implies $y \preceq x$, a contradiction. There exists, thus, $b_0 \in B_y$ such that $\neg(b_0 \preceq x)$. Then, $((x \prec b_0) \cup (x \bowtie b_0)) \cap (b_0 \preceq y)$ holds. $\square$

Because of the proof of Proposition 2, it seems having a countable weak basis is insufficient for a countable Debreu dense subset to exist. This is indeed the case, as we show in Proposition 3 via a counterexample.

Proposition 3. *There exist dcpos with countable weak bases and no countable Debreu dense subset.*

Proof. Consider $P := ([0, 1] \cup [2, 3], \preceq)$ where

$$x \preceq y \iff \begin{cases} x \leq y \text{ and } x, y \in [0, 1], \text{ or} \\ x \leq y \text{ and } x, y \in [2, 3], \text{ or} \\ x + 2 \leq y. \end{cases}$$

(See Fig. 1 for a representation of P.) Since P is clearly a partial order, we show now it is directed completely. Take $A \subseteq P$ directed. If we have either $A \subseteq [0, 1]$ or $A \subseteq [2, 3]$, then $\sqcup A$ exists and is the supremum of A in the usual $\mathbb{R}, \leq)$ sense. If A has elements in both, then it is easy to see that the supremum of $A \cap [2, 3]$ with respect to $\leq$ is the supremum of A with respect to $\preceq$. Analogously, $B := \mathbb{Q} \cap P$ is a countable weak basis. To conclude, assume $D \subseteq P$ is Debreu dense set. There exist, then, $d_x \in D$ such that $x \preceq d_x \preceq x + 2$ for all $x \in [0, 1]$. By definition, we have $d_x \in \{x, x + 2\}$. In particular, $d_x \neq d_y$ for all $x, y \in [0, 1]$ $x \neq y$ and D is uncountable. $\square$
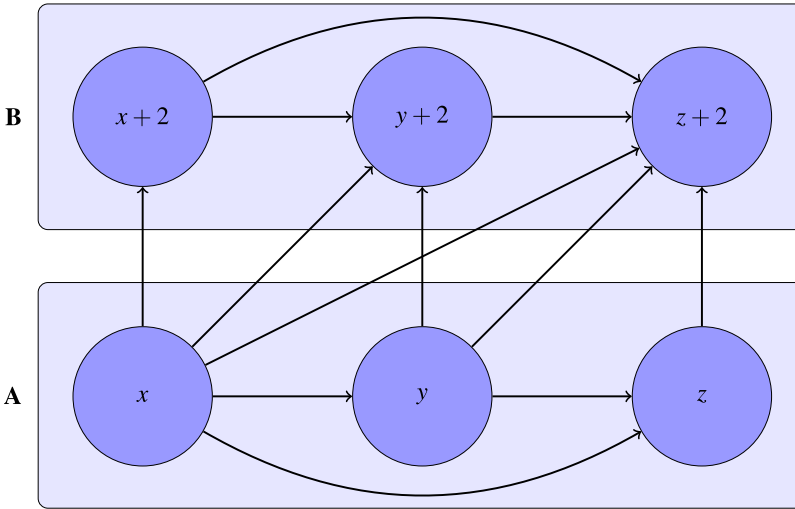
Figure 1. Representation of a dcpo, defined in Proposition 3, with a countable weak basis and no countable Debreu dense subset. In particular, we show $A := [0, 1]$, $B := [2, 3]$, and how $x, y, z \in A$, $x < y < z$, are related to $x + 2, y + 2, z + 2 \in B$. Notice an arrow from an element w to an element t represents $w \prec t$.

We consider now the converse of Proposition 2, that is, whether countable weak bases exist for any Debreu separable dcpo. Some extra terminology is needed for this purpose.

Definition 9 (Trivial directed sets and elements). *We say $x \in P$ has a* non-trivial *directed set if there exists a directed set $A \subseteq P$ such that $\sqcup A = x$ and $x \notin A$. Accordingly, given a weak basis $B \subseteq P$, we call the set of elements that have non-trivial directed sets $A \subseteq B$ the* non-trivial *elements of B and denote it by $\mathcal{N}_B$. We define the* trivial *elements of B equivalently and denote them by $\mathcal{T}_B$. In case we take $B = P$ as weak basis, we refer to the trivial (non-trivial) elements of B simply as trivial (non-trivial) elements.*

Intuitively, non-trivial (trivial) elements are those for which there is (there is not) a computational process that converges to them without ever actually outputting them. This is important since we may have some element that requires infinite precision, like say π in decimal representation, which may be achievable to arbitrary precision via some algorithm that only outputs elements with a finite representation.

We prove, in Theorem 1, that if P is a dcpo with a weak basis $B \subseteq P$ and a countable Debreu dense subset, then there exists a dcpo Q with a countable weak basis such that for the non-trivial set of B is included in Q, $\mathcal{N}_B \subseteq Q$. In order to do so, we first need two lemmas. We start, in Lemma 2, recalling the straightforward fact that, whenever the supremum of a directed set A is not contained in the set, we can find for each $a \in A$ some $b \in A$ that is strictly preferred to a, $a \prec b$ (see Hack et al. 2022a, Lemma 1). Right after, in Lemma 3, we show that, if $x \in P$ has a non-trivial directed set $A \subseteq P$ and there exists a countable Debreu dense subset $D \subseteq P$, then x has a non-trivial increasing sequence contained in D. In order to do so, we first consider the elements from D, which are *between* (in $\preceq$) those in A, and then profit from the countability of D to build an increasing sequence converging to the supremum of A.

Lemma 2 (Hack et al. 2022a). *If P is a dcpo and $A \subseteq P$ is a directed set such that $\sqcup A \notin A$, then, for all $a \in A$, there exists some $b \in A$ such that $a \prec b$.*

Lemma 3. *If P is a dcpo with a countable Debreu dense subset $D \subseteq P$, $x \in P$ is an element with a non-trivial directed set $A \subseteq P$, and*

$$D_A := \{d \in D \mid \exists a, b \in A \text{ s.t. } a \preceq d \preceq b\} \tag{4}$$

is a subset of D with a numeration $D_A = (d_n)_{n \geq 0}$, then $(d'_n)_{n \geq 0}$ is a non-trivial increasing sequence for x, where

$$\begin{aligned}
d'_0 &:= d_0, \\
d'_n &:= d_{m_n} \text{ for all } n \geq 1,
\end{aligned} \tag{5}$$

and $m_n \geq 0$ fulfills $d'_{n-1} \preceq d_{m_n}$ and $d_n \preceq d_{m_n}$ for all $n \geq 1$.

Proof. Take $x \in P$ and $A \subseteq P$ a non-trivial directed set for x. Notice, since $\sqcup A \notin A$, then for every $a \in A$ there is some $b \in A$ such that $a \prec b$ by Lemma 2. Consider, also, the set D_A defined in (4). Notice D_A is countable and $x \notin D_A$. We notice, first, D_A is directed. Given $d, d' \in D_A$, there exist $b, b' \in A$ such that $d \preceq b$ and $d' \preceq b'$ by definition. Since A is directed, there exists $c \in A$ such that $b \preceq c$ and $b' \preceq c$. Also, by Lemma 2, there exists some $e \in D$ such that $c \prec e$. Thus, there exists $d'' \in D_A$ such that $c \preceq d''$ by Debreu separability and, by transitivity, we have $d, d' \preceq d''$. We conclude D_A is directed. To finish, we will show that the increasing sequence $D'_A = (d'_n)_{n \geq 0} \subseteq D_A$ from (5) is well-defined and fulfills $\sqcup D'_A = x$. To show D'_A is well-defined it suffices to notice that we can consider a numeration $D_A = (d_n)_{n \geq 0}$ since $D_A \subseteq D$ and that m_n exists for all $n \geq 1$ since, as we showed, D_A is directed. Since D'_A is an increasing sequence by construction, we only need to show $\sqcup D'_A = \sqcup A$. Notice $d'_n \preceq \sqcup A$ by definition. Moreover, given any $a \in A$, there exists some $n \geq 0$ such that $a \preceq d'_n$ by Lemma 2, Debreu separability and definition of D'_A. In particular, if there is some $z \in P$ such that $d'_n \preceq z$ for all $n \geq 0$, then $a \preceq z$ for all $a \in A$ and, as a result, $\sqcup A \preceq z$. Thus, $\sqcup D'_A = \sqcup A$. $\qquad\square$

Note, as a consequence of Lemma 3, whenever a dcpo P is Debreu separable, any directed set $A \subseteq P$ contains an increasing sequence $(a_n)_{n \geq 0} \subseteq A$ such that $\sqcup (a_n)_{n \geq 0} = \sqcup A$ (see Proposition 24 in the Appendix 1).

Using Lemmas 2 and 3, we can now prove Theorem 1, our first construction of countable weak bases using order density properties. Intuitively, Theorem 1 considers Q, a subset of some dcpo P, and takes all the elements in some basis that can be achieved non-trivially and profits from the sequences in Lemma 3 to achieve them via a countable Debreu dense subset. Lastly, it adds elements to Q in order to assure it is a dcpo.

Theorem 1. *If P is a dcpo with a countable Debreu dense set $D \subseteq P$ and a weak basis $B \subseteq P$, then the dcpo $(Q, \preceq)$ fulfills $D \cup \mathcal{N}_B \subseteq Q$ and has D as countable weak basis, where*

$$Q := D \cup \{x \in P \setminus D \mid \exists \text{ a directed set } A \subseteq D \text{ s.t. } \sqcup A = x\}$$

and $\preceq$ is the restriction to Q of the partial order in P.

Proof. We will show $(Q, \preceq)$ is a dcpo with a countable weak basis such that $D \cup \mathcal{N}_B \subseteq Q$. Note $\mathcal{N}_B \subseteq Q \setminus D$ by Lemma 3. To conclude the proof, we only need to show Q is directed completely as, after this is established, it is clear by definition that $D \subseteq Q$ is a countable weak basis of Q. Take, thus, a directed set $A \subseteq Q$ with $\sqcup A \notin A$, as the opposite is straightforward. Note $\sqcup A \in P$, since P is a dcpo, and we intend to show $\sqcup A \in Q$. To begin with, consider

$$A' := A \cup \{d \in D \mid \exists a, b \in A \text{ s.t. } a \preceq d \preceq b\} \subseteq Q,$$

which is straightforwardly a directed set. Note $\sqcup A'$ exists in P and, actually, $\sqcup A' = \sqcup A$, as we have (1) for all $a' \in A'$ there exists some $a \in A$ such that $a' \preceq a$, which leads to $a' \preceq \sqcup A$ for all $a' \in A$, and (2) if $a' \preceq y$ for all $a' \in A'$ then $a \preceq y$ for all $a \in A$ and we get $\sqcup A \preceq y$. Consider now

$$A'' := \left(A' \cap D \right) \bigcup_{x \in A' \setminus D} B_x,$$

where, for all $x \in A' \setminus D$, $B_x \subseteq D$ is a directed set, which exists by definition of Q, such that $\sqcup B_x = x$. We conclude by showing that (1) A'' is directed and that (2) $\sqcup A'' = \sqcup A$, which imply, since $A'' \subseteq D$, that $\sqcup A = \sqcup A'' \in Q$. As a result, Q is directed complete, as we intended to prove. To show (1), we take $a, b \in A''$ and consider four cases: (a) $a, b \in B_x$ for some $x \in A' \setminus D$, (b) $a, b \in A' \cap D$, (c) $a \in B_x$, $b \in B_{x'}$ with $x, x' \in A' \setminus D$ $x \neq x'$ and (d) $a \in A' \cap D$ and $b \in B_x$ for some $x \in A' \setminus D$. In (a) there exists some $c \in B_x$ such that $a, b \preceq c$ since B_x is a directed set. (b) holds as there exists some $y \in A$ such that $a, b \preceq y$ and some $y' \in A$ such that $y \prec y'$ by Lemma 2. We obtain there exists some $c \in A' \cap D$ such that $a, b \preceq y \preceq c \preceq y'$. (c) holds as there exists some $y \in A$ such that $x, x' \preceq y$ and we can follow (b) to get some $c \in A' \cap D$ such that $a \preceq x \preceq c$ and $b \preceq x' \preceq c$. (d) holds similarly to (c). In order to finish, we only need to show (2) holds. Since for all $a'' \in A''$ there exists some $a' \in A'$ such that $a'' \preceq a'$, we have $a'' \preceq \sqcup A' = \sqcup A$ for all $a'' \in A''$. Moreover, if we have $a'' \preceq z$ for all $a'' \in A''$, then we have, by definition of A'', $a' \preceq z$ for all $a' \in A' \cap D$ and, by definition of $B_{a'}$, $a' \preceq z$ for all $a' \in A' \setminus D$. Thus, $\sqcup A = \sqcup A' \preceq z$ and, hence, $\sqcup A'' = \sqcup A' = \sqcup A$. □

Remark 1 (Implication for computability). By of Theorem 1, we can define computable elements (in the sense of Definition 7) on $D \cup \mathcal{N}_B \subseteq P$, where $D \subseteq P$ is a countable Debreu dense subset and $\mathcal{N}_B$ are the non-trivial elements of some weak basis $B \subseteq P$ (which may be uncountable), whenever D is effective.

Theorem 1 has the following straightforward strengthening.

Corollary 1. *If P is a dcpo with a countable Debreu dense subset $D \subseteq P$ and $B \subseteq P$ is a weak basis, then the set of trivial elements of B, $\mathscr{T}_B$, is countable if and only $D \cup \mathscr{T}_B$ is a countable weak basis for P.*

From the proof of Corollary 1, which relies on Lemma 3, Debreu separability alone seems to be insufficient to build a countable weak basis. This is precisely the case, as we show in Proposition 4 via a counterexample. In fact, our counterexample shows that even strengthening the hypothesis to Debreu upper separability is insufficient.

Proposition 4. *There exist Debreu upper separable dcpos without countable weak bases.*

Proof. Take $P := (\Sigma^* \cup \Sigma^\omega, \preceq)$, where Σ is finite and

$$x \preceq y \iff \begin{cases} x = y, \text{ or} \\ x \in \Sigma^*, \ y \in \Sigma^\omega \text{ and } x \preceq_C y, \end{cases}$$

with $\preceq_C$ the partial order from the Cantor domain. (See Fig. 2 for a representation with $\Sigma = \{0, 1\}$.) As we directly see P is a partial order, we begin by showing P is directed complete. Consider a directed set $A \subseteq P$ with $|A| \geq 2$. (If $A = \{a\}$ for some $a \in P$, then $\sqcup A = a$ and we have finished.) If $A \cap \Sigma^\omega = \emptyset$, notice A is not directed, since given $x, y \in A \cap \Sigma^*$ $x \neq y$, there is no $z \in \Sigma^*$ such that $x, y \preceq z$. Thus, there exists some $x_A \in A \cap \Sigma^\omega$. Notice we actually have $A \cap \Sigma^\omega = \{x_A\}$, as given $y \in \Sigma^\omega$ $y \neq x_A$ there is no $z \in P$ such that $x_A, y \preceq z$. Analogously, we obtain $y \preceq x_A$ for all $y \in A$ and, thus, $\sqcup A = x_A$. We conclude P is directed complete. We notice now Σ^* is a countable Debreu dense and Debreu upper dense subset of P. If $x \prec y$ for some $x, y \in P$, then $x \in \Sigma^*$ by definition and we conclude Σ^* is Debreu dense. If $x \bowtie y$ with $x, y \in \Sigma^\omega$, then there exists some $d \in \Sigma^*$ such that $x \bowtie_c d$ and $d \preceq_c y$. If $y \in \Sigma^\omega$ and $x \in \Sigma^*$, we consider some $d \in \Sigma^*$ such that $d \preceq_C y$ and get $x \bowtie d \preceq y$. If either $x \in \Sigma^\omega$ and $y \in \Sigma^*$ or $x, y \in \Sigma^*$, then we take $d = y$. We obtain P is Debreu
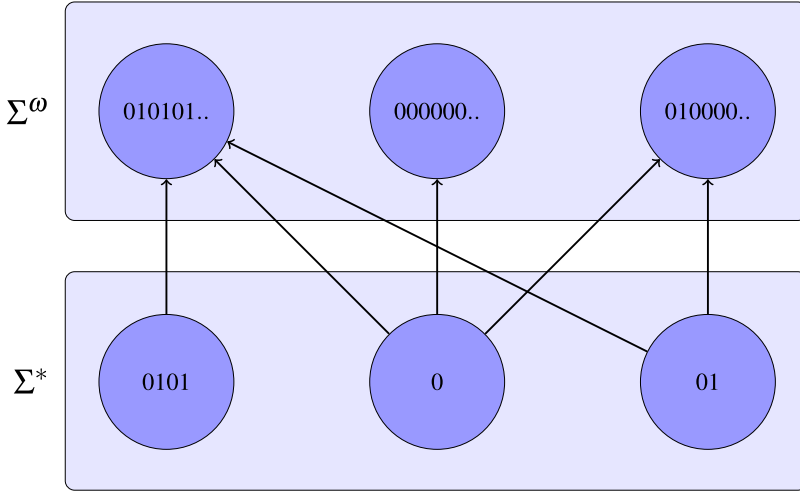
Figure 2. Representation of a dcpo, defined in Proposition 4, which is Debreu upper separable and has no countable weak basis. In particular, we show Σ^ω and Σ^* for $\Sigma = \{0, 1\}$ and how $010101\ldots, 000000\ldots, 010000\ldots \in \Sigma^\omega$ are related to $0101, 0, 01 \in \Sigma^*$. Notice an arrow from an element w to an element t represents $w \prec t$.

upper separable. To finish, we will show any weak basis $B \subseteq P$ is uncountable. By definition of weak basis, there exists some directed set $A_x \subseteq B$ such that $x = \sqcup A_x$ for all $x \in \Sigma^\omega$. As discussed above, this implies $x \in A_x$. Hence, $\Sigma^\omega \subseteq B$ and B is uncountable. □

Note that, in the proof of Hack et al. 2023, Proposition 4 (iii)), we introduced the counterexample we used in Proposition 4. Note, also, $\Sigma^\omega \subseteq \mathcal{T}_B$ for any weak basis B of the dcpo in the proof of Proposition 4. Before we continue relating weak bases to order density properties, we define the minimal elements of a partial order. If P is a partial order, then the *set of minimal elements of P*, denoted by $\min(P)$, is

$$\min(P) := \{x \in P \mid \text{there is no } y \in P \text{ such that } y \prec x\}.$$

Notice, for the Cantor domain, $\min(P) = \{\bot\}$. $\min(P)$ is related to order density properties since, as we note in Lemma 4, it is countable whenever countable Debreu upper dense subsets exist.

Lemma 4. *If P is a partial order with a countable Debreu upper dense subset $D \subseteq P$, then $\min(P)$ is countable.*

Proof. If $|\min(P)| \leq 1$, we are done. If there exist $x, y \in \min(P)$ $x \neq y$, then $x \bowtie y$, since $x \prec y$ ($y \prec x$) contradicts the fact $y \in \min(P)$ ($x \in \min(P)$). Thus, there exists some $d \in D$ such that $x \bowtie d \preceq y$. By definition of $\min(P)$, we have $d = y$ and, hence, $\min(P) \subseteq D$ is countable. □

Clearly, we cannot eliminate the hypothesis in Lemma 4 as, for example, $\min(P)$ is uncountable for any uncountable set with the trivial partial order.

Since, by Proposition 4, Debreu upper separability is not enough for countable weak bases to exist, we consider stronger order density properties.

Definition 10 (Order density properties II). *We say $D \subseteq P$ is* order dense *if, for any pair $x, y \in P$ such that $x \prec y$, there exists some $d \in D$ such that $x \prec d \prec y$ (Ok et al. 2002). We say P is* order separable *if it has a countable order dense subset (Mehta 1986).*

Notice that, although the Cantor domain Σ^∞ is Debreu separable, it is not order separable since, if $s \in \Sigma^*$ and $\beta \in \Sigma$, then we have $s \prec s\beta$ and, for all $t \in \Sigma^*$ such that $t \prec s\beta$, $t \preceq s$ also holds. As we show in Proposition 5, order separability is sufficient to build a countable weak basis for $P \setminus \min(P)$ the basic idea is similar to the one for Lemma 3 and, taking into account Lemma 4, we can extend the result to P by also assuming the existence of a countable Debreu upper dense subset.

Proposition 5. *If P is a dcpo, then countable order dense subsets $D \subseteq P$ are countable weak bases for $P \setminus \min(P)$. Furthermore, if P also has a countable Debreu upper dense subset, then $D \cup \min(P)$ is a countable weak basis for P.*

Proof. We begin showing the first statement. Notice, if P is a dcpo, then $P \setminus \min(P)$ is a dcpo as well. Take $D \subseteq P$ a countable order dense subset of P, which we can choose w.l.o.g. such that $D \cap \min(P) = \emptyset$. We will show D is a countable weak basis for $P \setminus \min(P)$. In particular, we will show the following lemma.

Lemma 5. *If P is a dcpo, $D \subseteq P$ is a countable order dense subset and $x, y \in P$ such that $x \prec y$, then $D' = (d'_n)_{n \geq 0} \subseteq D$ is an increasing chain such that $\sqcup D' = y$, where*

$$
\begin{aligned}
d'_0 &:= d_{m_0} \text{ and } m_0 := \min\{n \geq 0 \mid x \prec d_n \prec y\}, \\
d'_n &:= d_{m_n} \text{ and } m_n := \min\{n \geq m_{n-1} \mid d'_{n-1} \prec d_n \prec y\} \text{ for all } n \geq 1.
\end{aligned}
\tag{6}
$$

Proof. Take some $y \in P \setminus \min(P)$ and $D = (d_n)_{n \geq 0}$ a numeration of D. Since $y \notin \min(P)$, there exists some $x \in P$ such that $x \prec y$. Moreover, by definition of order separability, we have that m_n exists for all $n \geq 0$ and, hence, $D' = (d'_n)_{n \geq 0} \subseteq D$ in (6) is well-defined. Since $D' = (d'_n)_{n \geq 0}$ is an increasing sequence by construction, it has a supremum $\sqcup D'$. Given that y is an upper bound of $(d'_n)_{n \geq 0}$ by construction, we have $\sqcup D' \preceq y$. To finish the proof of the first statement, we assume $\sqcup D' \prec y$ and get a contradiction. If that was the case, there would be some $\overline{n} \geq 0$ such that $\sqcup D' \prec d_{\overline{n}} \prec y$ by order separability. Consider, thus, $\overline{m} := \max\{n < \overline{n} \mid d_n \in D'\}$. Since $d_{\overline{m}+1}, \ldots, d_{\overline{n}-1} \notin D'$ by definition of $\overline{m}$ and $d_{\overline{m}} \prec d_{\overline{n}} \prec y$ by transitivity, we would have, assuming $d'_n = d_{\overline{m}}$ for some $n \geq 0$ w.l.o.g., $d'_{n+1} = d_{\overline{n}}$, a contradiction. Thus, D' is an increasing chain such that $\sqcup D' = y$.   □

Hence, by Lemma 5, D is a countable weak basis for $P \setminus \min(P)$. Regarding the second statement, notice we can take $A_x = \{x\}$ as directed set with $\sqcup A_x = x$ for all $x \in \min(P)$, and, since $\min(P)$ is countable by Lemma 4, we have $D \cup \min(P)$ is a countable weak basis for P.   □

Remark 2 (Implication for computability). By Proposition 5, we can define computable elements (in the sense of Definition 7) on a dcpo P, even if it is uncountable, whenever we have a countable order dense subset D_1 and a countable Debreu upper dense subset D_2, provided $D_1 \cup D_2$ is effective.

Since order density of P is enough to introduce computability on $P \setminus \min(P)$ while, again by Proposition 4, Debreu separability is not, we take a stronger version of Debreu separability as hypothesis in Proposition 6. In particular, we show there are countable weak bases for $P \setminus \min(P)$ whenever countable Debreu dense subset satisfying a specific property exists. To express such a property, we recall some definitions in by Bridges and Mehta (2013). If $x, y \in P$, then we say y is an *immediate successor* of x and x is an *immediate predecessor* of y if both $x \prec y$ and $(x, y) := \{z \in P \mid x \prec z \prec y\} = \emptyset$ hold. In this scenario, (x, y) is called a *jump*.

Proposition 6. *If P is a dcpo and $D \subseteq P$ is a countable Debreu dense subset whose elements have a finite number of immediate successors, then $(Q_1 \cup D) \setminus \min(P)$ is a countable weak basis for $P \setminus \min(P)$, where*

$$Q_1 := \left\{ y \in P \mid \exists x \in P \text{ s.t. } (x, y) = \emptyset \right\}. \tag{7}$$

Furthermore, if P also has a countable Debreu upper dense subset, then $Q_1 \cup D \cup \min(P)$ is a countable weak basis for P.

Proof. We begin showing the first statement. Notice, if P is a dcpo, then $P \setminus \min(P)$ is also a dcpo. Take $D \subseteq P$ a countable Debreu dense subset of P whose elements have a finite number of immediate successors and $y \in P \setminus \min(P)$. Notice $y \in Q_0 \cup Q_1$, where

$$Q_0 := \{ y \in P \mid \text{for all } x \in P \text{ s.t. } x \prec y \, \exists z \in P \text{ s.t. } x \prec z \prec y \}$$

and Q_1 is defined as in (7). If $y \in Q_0$, we take some $x \prec y$ and we can emulate the construction of $(d'_n)_{n \geq 0}$ in Lemma 5, (6), to construct an increasing sequence contained in D whose supremum is y. We just have to notice, by Debreu separability and definition of Q_0, whenever $x \prec y$ for $x, y \in P$, there exist $z \in P$ and $m \geq 0$ such that $x \prec z \preceq d_m \prec y$. Conversely, if $y \in Q_1$, there exists some $x \in P$ such that $(x, y) = \emptyset$. If $y \notin D$, then $x \in D$ by Debreu separability. If we denote by $(s_n)_{n \geq 0}$ a numeration of the immediate successors of the elements in D, which can be constructed since each member of D has a finite number of immediate successors and D is countable by hypothesis, we have $Q_1 \subseteq D \cup (s_n)_{n \geq 0}$ and, thus, Q_1 is countable. We conclude $(Q_1 \cup D) \setminus \min(P)$ is a countable weak basis for $P \setminus \min(P)$. For the second statement, since $\min(P)$ is countable by Lemma 4, we conclude $Q_1 \cup D \cup \min(P)$ is a countable weak basis for P. $\square$

Remark 3 (Implication for computability). By Proposition 6, we can define computable elements (in the sense of Definition 7) on a dcpo P with a countable Debreu dense subset D_1 and a countable Debreu upper dense subset D_2 whenever the elements in D_1 have a finite number of immediate successors and $D_1 \cup D_2$ is effective.

Note the hypothesis in the first statement of Proposition 6 is weaker than order separability since, whenever a partial order is dense, immediate successors do not exist.[2] Notice, also, the Cantor domain satisfies the hypotheses in Proposition 6, since Σ^* is a countable Debreu dense subset of Σ^∞ and each $s \in \Sigma^*$ has exactly $|\Sigma|$ immediate successors. The converse of Proposition 6 is, again by Proposition 3, false.

To summarize, the main results in this section are Theorem 1 and Propositions 5 and 6. In the first one, we show how one can profit from Debreu density to introduce a dcpo that includes computability on the non-trivial elements of any weak basis B. In the second, we use the stronger property of order separability to extend computability to all non-minimal elements. In the last one, we show that we can achieve the same results as in Proposition 5 by asking for the existence of a countable Debreu separable subset whose elements only have a finite number of immediate successors. Lastly, we extend computability to the whole dcpo in Propositions 5 and 6 by also requiring the existence of a countable Debreu upper dense subset.

4. Uniform Computability via Ordered Sets

Following Hack et al. 2022a), in Section 2, we considered an element x in some dcpo P with a countable weak basis $B \subseteq P$ to be computable if there exists some directed set $B_x \subseteq B$ such that $\sqcup B_x = x$ and whose associated subset of the natural numbers $\alpha^{-1}(B_x)$ is recursively enumerable. We consider now a stronger approach, where we associate to each $x \in P$ a unique directed set $B_x \subseteq B$ such that $\alpha^{-1}(B_x)$ is recursively enumerable if and only if x is computable. In order to do so, B_x should be fundamentally related to x, that is, the information in every $b \in B_x$ should be gathered by any process which computes x. The differences between the approach in Section 2 and the one here are discussed in Hack et al. 2022a). Crucially, while the more general approach only enables to introduce computable elements, computable functions can also be defined in this

stronger framework, which was introduced by Scott in Scott (1970). In fact, the uniform approach to computability we discuss in this section is known as *domain theory* (Cartwright et al. 2016; Edalat and Sünderhauf 1999; Gierz et al. 2012; Mislove 1998; Scott 1982; Stoltenberg-Hansen 1994, 2001). Note that, as in Section 2, we only introduce the order structure used to define computability and do not address how to translate it to other spaces of interest (see Hack et al. 2022a and the references therein).

Before we continue, we introduce the Scott topology, which will play a major role in the following.

Definition 11 (Scott topology, Abramsky and Jung 1994; Scott 1970). *If P is a dcpo, we say a set $O \subseteq P$ is* open *in the* Scott topology *if it is* upper closed *(if $x \in O$ and there exists some $y \in P$ such that $x \preceq y$, then $y \in O$) and* inaccessible by directed suprema *(if $A \subseteq P$ is a directed set such that $\sqcup A \in O$, then $A \cap O \neq \emptyset$). We denote by $\sigma(P)$ the Scott topology on P.*

Note that the Scott topology characterizes the partial order in P, that is,

$$x \preceq y \iff x \in O \text{ implies } y \in O \text{ for all } O \in \sigma(P) \tag{8}$$

(Abramsky and Jung 1994, Proposition 2.3.2. Note that, by (8), the Scott topology satisfies the T_0 topological separation axiom. (We say a topological space (X, τ) is a T_0 space or a *Kolmogorov* space if, for every pair of distinct points $x, y \in X$, there exists an open set $O \in \tau$ such that either $x \in O$ and $y \notin O$ or $y \in O$ and $x \notin O$ hold Kelley 2017.)

We can think of the Scott topology as the family of properties on the data set P which allows us to distinguish the elements in P (Smyth 1983). In particular, by definition, if some computational processes have a limit, then any property of the limit is verified in finite time, and, since $\sigma(P)$ is T_0, these properties are enough to distinguish between the elements of P.

In order to attach to each element in a dcpo, a unique subset of $\mathbb{N}$, which is equivalent to it for all computability purposes, we recall the way-below relation.

Definition 12 (Way-below(-above) relation, Abramsky and Jung 1994; Scott 1972). *If $x, y \in P$, we say x is* way-below y *or y is* way-above x *and denote it by $x \ll y$ if, whenever $y \preceq \sqcup A$ for a directed set $A \subseteq P$, then there exists some $a \in A$ such that $x \preceq a$. The set of element way-below (way-above) some x is denoted by $\downarrow x$ ($\uparrow x$).*

We think of an element way-below another as containing *essential information* about the latter, since any computational process producing the latter cannot avoid gathering the information in the former. For example, if $A_1 \subseteq \downarrow x$ is a directed set such that $\sqcup A_1 = x$, then, if $A_2 \subseteq P$ is a directed set where $\sqcup A_2 = x$, there exists for all $a_1 \in A_1$ some $a_2 \in A_2$ such that $a_1 \preceq a_2$. We can regard A_1, thus, as a canonical computational process that yields x. (To be more precise, one can show that the computability, in the sense of Definition 15, of an element is equivalent to the computability of a specific subset of the basis. This is formalized in by Hack et al. 2022a, Proposition 2.) The significance of the introduction of $\ll$ for computability is discussed by Hack et al. 2022a). We simply note, for the moment, that $x \ll y$ implies $x \preceq y$ and that, if $x \preceq y$ and $y \ll z$, then $x \ll z$ for all $x, y, z \in P$.

We introduce now, in a canonical way, computable elements in a dcpo. To do so, we first define effective bases.

Definition 13 (Basis, Abramsky and Jung 1994). *A subset $B \subseteq P$ is called a* basis *if, for any $x \in P$, there exists a directed set $B_x \subseteq \downarrow x \cap B$ such that $\sqcup B_x = x$.*

Note that bases are exactly like weak bases except for the fact that they achieve any element $x \in P$ using elements of P, which contain essential information about x. Recall, if B is a basis, then

it is a weak basis, and $(\Uparrow b)_{b \in B}$ is a topological basis for $\sigma(P)$ (Abramsky and Jung 1994; Hack et al. 2022a). A dcpo is called *continuous* or a *domain* if bases exist. As in the case of weak bases, we are interested in dcpos with a countable basis or ω-*continuous* dcpos, since computability can be introduced via Turing machines there. Note that Σ^{∞} is ω-continuous, as $B = \Sigma^*$ is a countable basis.

Definition 14 (Effective basis, Edalat and Sünderhauf 1999; Stoltenberg-Hansen and Tucker 2008). *We say a basis $B \subseteq P$ is* effective *if there is a finite map, which enumerates B, $B = (b_n)_{n \geq 0}$, there is a bottom element $\perp \in B$ and*

$$\{\langle n, m \rangle \mid b_n \ll b_m\}$$

is recursively enumerable.

Effectivity for bases has the same purpose as for weak bases, although it poses stronger requirements. A discussion in this regard can be found in Hack et al. (2022a). We assume w.l.o.g. $b_0 = \perp$ in the following. Note that the relevance of the bottom element is discussed in Hack et al. 2022a). We can now define computable elements.

Definition 15 (Computable element, Edalat and Sünderhauf 1999). *If P is a dcpo with an effective basis $B = (b_n)_{n \geq 0}$, we say $x \in P$ is* computable *provided*

$$\{n \in \mathbb{N} \mid b_n \ll x\}$$

is recursively enumerable.

The dependence of the set of computable elements on the model in this approach is addressed in Hack et al. 2022a).

Before introducing computable functions, we need some extra terminology. We call a map $f : P \to Q$ between dcpos P, Q a *monotone* if $x \preceq y$ implies $f(x) \preceq f(y)$. Furthermore, we call f *continuous* if it is monotone and, for any directed set $A \subseteq P$, we have $f(\sqcup A) = \sqcup f(A)$ (Abramsky and Jung 1994; Scott 1970). Note that this definition is equivalent to the usual topological definition of continuity (see Kelley 2017) applied to the Scott topology. In fact, dcpos constitute the objects of the category DCPO, whose morphisms are the functions that are continuous in the Scott topology (Lawson 1998). Note, also, if $B \subseteq P$ is a weak basis and f is continuous, then $f(P)$ is determined by $f(B)$. This is the case since, if $x \in P$, then there exists some directed set $B_x \subseteq B$ such that $\sqcup B_x = x$. Thus, by continuity of f, $f(x) = f(\sqcup B_x) = \sqcup f(B_x)$. We can consider the monotonicity of f in the definition of continuity as a mere technical requirement to make sure $\sqcup f(A)$ exists for any directed set $A \subseteq P$.

We introduce now computable functions.

Definition 16 (Computable function, Edalat and Sünderhauf 1999). *We say a function $f : P \to Q$ between two dcpos P and Q with effective bases $B = (b_n)_{n \geq 0}$ and $B' = (b'_n)_{n \geq 0}$, respectively, is* computable *if f is continuous and the set*

$$\{\langle n, m \rangle \mid b'_n \ll f(b_m)\} \tag{9}$$

is recursively enumerable.

Intuitively, the idea that any formal definition of computable function is meant to capture is that of a map that sends computable inputs to computable outputs (Braverman 2005; Ko 2012). Indeed, this is the case for this definition (see Edalat and Sünderhauf 1999, Theorem 9) and Hack et al. 2022a). Moreover, we can think of continuity in $\sigma(P)$ as a weak form of computability. The

dependence of the set of computable functions on the model in this approach is addressed in Hack et al. 2022a).

To conclude, we can return to the examples from Section 2.1. In particular, one can see Σ^* is an effective basis for Σ^∞ (Hack et al. 2022a) and $B_\mathscr{I}$ is known to be an effective basis of $(\mathscr{I}, \sqsubseteq)$ (Edalat and Sünderhauf 1999). Moreover, $B_n := \Lambda^n \cap \mathbb{Q}^n$ is a countable basis for majorization for any $n \geq 2$ and the following lemma, which we prove in Appendix A.2.2 (see also Appendix A.2.3), holds.

Lemma 6. *The following statements hold:*

1. *Although $(\mathscr{I}, \sqsubseteq)$ is ω-continuous, any Debreu dense subset $Z \subseteq \mathscr{I}$ has the cardinality of the continuum $\mathfrak{c}$.*

2. *If $n = 2$, then $\mathbb{Q}^n \cap \Lambda^n$ is a countable basis for $(\Lambda^n, \preceq_M)$ and $(\Lambda^n, \preceq_M)$ is Debreu upper separable.*

3. *If $n \geq 3$, then $\mathbb{Q}^n \cap \Lambda^n$ is a countable basis for $(\Lambda^n, \preceq_M)$ and any Debreu dense subset $Z \subseteq \Lambda^n$ has the cardinality of the continuum $\mathfrak{c}$.*

Furthermore, one can slightly modify the proof of Proposition 1 in the Appendix A.2.1 to conclude B_n is an effective basis. The key property we use is Martin and Panangaden 2008, Lemma 5.1), where it was shown

$$x \ll_M y \iff x = \perp \text{ or } s_k(x) < s_k(y) \text{ for all } k < n \tag{10}$$

for all $x, y \in \Lambda^n$. Notice, Lemma 6 improves upon (Martin 2006, Theorem 1.3), establishing ω-continuity instead of just continuity for all $(\Lambda^n, \preceq_M)$ with $n \geq 2$.

5. Order Density and Bases

In this section, we continue relating order density properties to computability, specifically, to the uniform approach introduced in Section 4. We begin, in Section 5.1, mimicking the relations in Section 3, this time for bases. After establishing the discrepancy between bases and order density in general, we introduce, in Section 5.2, a property under which they coincide, namely, conditional connectedness. Importantly, the Cantor domain fulfills this property. Notice, for generality, we will assume bases do not necessarily have a bottom element in this section.

We can summarize the relation between Sections 3 and 5 as follows: First, we show, in Propositions 8 and 9, respectively, that the results regarding the existence of countable weak bases in Propositions 5 and 6 can be emulated for bases by restricting ourselves to *continuous* dcpos. Then, we show, in Theorem 2, that the positive and negative results regarding the relation between weak bases and order density properties (see Propositions 2, 3, and 4, and Theorem 1) can be improved upon to reach equivalence by additionally asking for conditional connectedness.

5.1 Continuous Debreu separable dcpos

We consider here continuous dcpos and extend the relationship between order density and computability from Section 3, focusing on connecting the former with countable bases. We begin introducing a definition and two lemmas that will be useful in the following. We say an element $x \in P$ is *compact* if $x \ll x$ holds and denote by $K(P)$ the set of compact elements of P (Abramsky and Jung 1994). Note that $K(P) \subseteq B$ for any basis $B \subseteq P$.

Lemma 7. *All compact elements in a dcpo P are trivial $K(P) \subseteq \mathscr{T}_P$. The converse, however, is not true.*

Proof. Take $x \in \mathrm{K}(P)$ and a directed set A such that $\sqcup A = x$. By definition of compact element, there exists some $a \in A$ such that $x \preceq a$. As $a \preceq \sqcup A$ by definition, then, by antisymmetry, we get $x = \sqcup A = a \in A$. To show the converse is false, take the dcpo $([0,1], \preceq)$, where $x \preceq y$ if $x \leq y$ for all $x, y \in (0, 1]$, $0 \preceq 0$ and $0 \preceq 1$. Note that $0 \bowtie y$ for all $y \in (0, 1)$. While the only directed set A with $\sqcup A = 0$ is $A = \{0\}$, we also have $A' = (0, 1)$ is directed with $\sqcup A' = 1$ and $0 \bowtie a$ for all $a \in A'$. Thus, $\neg(0 \ll 1)$ and $0 \notin \mathrm{K}(P)$ as, otherwise, we would have $0 \ll 0 \preceq 1$, hence $0 \ll 1$. $\qquad \square$

Lemma 8. *If P is a continuous dcpo, then $\sigma(P)$ is second countable if and only if P is ω-continuous. However, there exist dcpos without bases where $\sigma(P)$ is second countable.*

Proof. The first statement is already known, see Lawson 1998, Proposition 3.4). For the second statement, take $P := ([0, 1], \preceq)$, where, for all $x, y \in [0, 1]$,

$$x \preceq y \iff \begin{cases} x \leq y & \text{if } x, y \in \left[0, \frac{1}{2}\right], \\ y \leq x & \text{if } x, y \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

It is easy to see P has no basis (Hack et al. 2022a). To conclude, we note that $\{(p, q), [0, q), (p, 1] \,\big|\, 0 \leq p < \frac{1}{2} < q \leq 1, p, q \in \mathbb{Q}\}$ is a countable basis for $\sigma(P)$. Take, thus, $x \in O \in \sigma(P)$ and assume w.l.o.g. $x < \frac{1}{2}$. If $x > 0$, since $O \in \sigma(P)$ and $x \in O$, then there exists some $y < x$ such that $y \in O$, otherwise $D_x := \{y \in P | y < x\}$ fulfills $\sqcup D_x = x$ and $D_x \cap O = \emptyset$, a contradiction. Arguing analogously, we have there exists some $z > \frac{1}{2}$ such that $z \in O$. Since O is upper closed, $x \in (p_x, q_x) \subseteq O$, where $p_x, q_x \in \mathbb{Q}$, $y \leq p_x \leq x$ and $\frac{1}{2} \leq q_x \leq z$. If $x = 0$, we can follow argue analogously and conclude $x \in [0, q) \subseteq O$ for some $q \in \mathbb{Q}$, $\frac{1}{2} < q$. $\qquad \square$

Regarding the Cantor domain, Note that $(\mathord{\uparrow} x)_{x \in \Sigma^*}$ is a countable topological basis for its Scott topology. Lemma 8 establishes that, in order to relate order density with countable bases, we can relate the former to countability axioms on $\sigma(P)$. As a starting link, we relate, in Proposition 7, continuous Debreu separable dcpos with these axioms. In particular, we show Debreu separability of P implies $\sigma(P)$ is first countable for continuous dcpos. Furthermore, whenever the set of compact elements is countable, we show P is ω-continuous, that is, its Scott topology is second countable.

Proposition 7. *Take a dcpo P and a countable Debreu dense subset $D \subseteq P$. If P is continuous, then $\sigma(P)$ is first countable. Moreover, if $\mathrm{K}(P)$ is countable, then $D \cup \mathrm{K}(P)$ is a countable basis for P.*

Proof. We begin with the first statement. Take $x \in P$, a basis $B \subseteq P$ and a Debreu dense subset $D \subseteq P$. We will show there exists a countable family $(U_n^x)_{n \geq 0} \subseteq \sigma(P)$ such that, if $x \in O \in \sigma(P)$, then there exists some $n_0 \geq 0$ such that $x \in U_{n_0}^x \subseteq O$. If $x \in \mathrm{K}(P)$, we take $(U_n^x)_{n \geq 0} := \{\mathord{\uparrow} x\} \subseteq \sigma(P)$ and notice, if $O \in \sigma(P)$, then we have $\mathord{\uparrow} x \subseteq O$ whenever $x \in O$, as $\mathord{\uparrow} x = \mathord{\uparrow} x \subseteq O$ by definition. If $x \notin \mathrm{K}(P)$, then there exists some directed set $B_x \subseteq \mathord{\downarrow} x \cap B$ such that $\sqcup B_x = x \notin B_x$ by definition of basis. By Lemma 3, there exists an increasing sequence $(d_n')_{n \geq 0} \subseteq D$ (defined by (5) and (4)) such that $\sqcup(d_n')_{n \geq 0} = x \notin (d_n')_{n \geq 0}$. Since, by construction, there exists some $b_n \in B_x$ such that $d_n \preceq b_n \ll x$, we have $d_n' \ll x$ for all $n \geq 0$. Thus, if $x \notin \mathrm{K}(P)$, we take $U_n^x := \mathord{\uparrow} d_n'$ for all $n \geq 0$ since, given some $O \in \sigma(P)$ such that $x \in O$, then, by definition, there exists some $n_0 \geq 0$ such that $d_{n_0}' \in O$ because $\sqcup(d_n')_{n \geq 0} = x$. In particular, $x \in \mathord{\uparrow} d_{n_0}' \subseteq O$. Regarding the second statement, we notice, following the first part, for all $x \in P$, $O \in \sigma(P)$ such that $x \in O$ there exists some $n_0 \geq 0$ such that $x \in U_{n_0} \subseteq O$, where

$$(U_n)_{n \geq 0} := (\mathord{\uparrow} d)_{d \in D} \cup (\mathord{\uparrow} x)_{x \in \mathrm{K}(P)} \tag{11}$$

since $\mathrm{K}(P)$ is countable by assumption. We conclude $\sigma(P)$ is second countable and, by Lemma 8, P is ω-continuous. In particular, $D \cup \mathrm{K}(P)$ is a countable basis. $\qquad \square$

Remark 4 (Implication for computability). By Proposition 7, we can define computable elements (in the sense of Definition 15) on a continuous dcpo P with a countable Debreu dense subset D and countable compact elements $K(P)$ whenever $D \cup K(P)$ is effective. Moreover, if that is the case for two dcpos P_0, P_1, then we can define computable functions (in the sense of Definition 16) between them $f : P_0 \to P_1$.

Note that there exist Debreu separable dcpos where $\sigma(P)$ is not first countable (take, e.g., the dcpo in Hack et al. 2022a, Proposition 5) and continuous Debreu separable dcpos where $K(P)$ is uncountable (like $P := (A \cup B, \preceq)$ with $A := \mathbb{R}$, $B := (0, 1]$ and $x \preceq y$ if and only if $x \leq y$ with $x, y \in B$ or $x \in A$ and $y \in B$, where $\mathbb{Q} \cap B$ is a Debreu dense subset and $K(P) = A$ is uncountable). Strengthening the order density assumption in Proposition 7, we can improve upon Proposition 5 and construct countable *bases* instead of *weak bases*, as we show in Proposition 8.

Proposition 8. *If P is a continuous dcpo with a countable order dense subset $D \subseteq P$, then D is a countable basis for $P \setminus \min(P)$. Furthermore, if P also has a countable Debreu upper dense subset, then $D \cup \min(P)$ is a countable basis for P.*

Proof. We begin with the first statement. In order to do so, we start by showing the following lemma.

Lemma 9. *If P is a continuous dcpo with a countable order dense subset $D \subseteq P$, then*
$$K(P) = \min(P).$$

Proof. ($\subseteq$) If $y \in P \setminus \min(P)$, then we can construct an increasing chain $(d'_n)_{n \geq 0} \subseteq D$ (defined as in Lemma 5, (6)) with $D \subseteq P$ a countable order dense subset chosen w.l.o.g. with the property $D \cap \min(P) = \emptyset$, such that $\sqcup(d'_n)_{n \geq 0} = y$ and $y \notin (d'_n)_{n \geq 0}$. Thus, $y \notin K(P)$ by Lemma 7 and, hence, $K(P) \subseteq \min(P)$. ($\supseteq$) This is the case since P is continuous and, hence, we must have, for all $x \in \min(P)$, $B_x := \{x\} \subseteq \downarrow x \cap B$ for some basis $B \subseteq P$. That is, $x \in K(P)$. $\qquad \square$

We finish noticing $Q := P \setminus \min(P)$ is a dcpo where $K(Q) = \emptyset$. Thus, by (11), $D \subseteq Q$ is a countable basis for Q. For the second statement, we notice, if P has a countable Debreu upper dense subset, then $\min(P)$ is countable by Lemma 4 and $K(P)$ is also countable, as $K(P) = \min(P)$ by Lemma 9. Thus, by Proposition 7, P has a countable basis. In fact, we have that $D \cup \min(P)$ is a countable basis for P. $\qquad \square$

Remark 5 (Implication for computability). By Proposition 8, we can define computable elements and functions (in the sense of Definitions 15 and 16) on a continuous dcpo P with a countable order dense subset D and a countable Debreu upper dense subset whenever $D \cup \min(P)$ is effective.

As we did with Proposition 6 for Proposition 5, we complement Proposition 8 with Proposition 9, where we substitute order separability by a weaker property, namely, the existence of a countable Debreu dense subset where each element has a finite number of immediate successors. Notice, unlike Proposition 8, Proposition 9 holds for the Cantor domain.

Proposition 9. *If P is a continuous dcpo with a countable Debreu dense subset $D \subseteq P$ whose elements have a finite number of immediate successors, then $(D \cup K(P)) \setminus \min(P)$ is a countable basis for $P \setminus \min(P)$. Furthermore, if P also has a countable Debreu upper dense subset, then $D \cup K(P)$ is a countable basis for P.*

Proof. The first statement can be shown by emulating Lemma 5 (as in the proof of Proposition 6) to obtain, after applying Lemma 7 as in Lemma 9, that $K(P) \subseteq \min(P) \cup Q_1$, with Q_1 defined as in (7), that is,

$$Q_1 = \{y \in P \mid \exists x \in P \text{ s.t. } (x, y) = \emptyset\}.$$

Since Q_1 is countable, we follow the proof of Proposition 7 and conclude $P \setminus \min(P)$ has a countable basis. For the second statement, we rely again on Lemma 4 and obtain that $\min(P)$ is countable. We conclude that $K(P)$ is also countable and, thus, that $D \cup K(P)$ is a countable basis for P. □

Remark 6 (Implication for computability). By Proposition 9, we can define computable elements and functions (in the sense of Definitions 15 and 16) on a continuous dcpo P with a countable Debreu dense subset D whose elements have a finite number of immediate successors and a countable Debreu upper dense subset whenever $D \cup K(P)$ is effective.

As we show in Proposition 10, we cannot improve upon Proposition 9 eliminating the assumption about the finiteness of the immediate successors of D. In particular, there are continuous Debreu upper separable dcpos where $K(P)$, and, hence, any basis, is uncountable. Actually, the result remains false if we add the assumption that $x \preceq y$ implies $x \ll y$ for all $x, y \in P$. We will return to this assumption in Section 5.2.

Proposition 10. *There exist continuous Debreu upper separable dcpos P where $x \preceq y$ implies $x \ll y$ for all $x, y \in P$, but for which $K(P)$ is uncountable and, thus, no countable bases exist.*

Proof. Take the counterexample from the proof of Proposition 4, P. We will show every element in P is compact, $K(P) = P$. If $x \in \Sigma^\omega$ and $x \preceq \sqcup A$ for some directed set A, then $x = \sqcup A$, as there is no other element above x. Thus, $x \in A$ since, as we showed in Proposition 4, A would not be directed in the opposite case. Hence, $x \in K(P)$ and $\Sigma^\omega \subseteq K(P)$. If $x \in \Sigma^*$ and $x \preceq \sqcup A$ for some directed set A, then either $\sqcup A = x$ or $\sqcup A \in \Sigma^\omega$. If $\sqcup A = x$, then $A = \{x\}$ as there is no other element below x. If $\sqcup A \in \Sigma^\omega$, then $\sqcup A \in A$, as we argued above, and we have $x \in K(P)$. Thus, $\Sigma^* \subseteq K(P)$. We conclude $K(P) = P$. To finish, we only need some remarks. $K(P) = P$ implies P is continuous, as we can take the directed set $A = \{x\} \subseteq \downarrow x$ for all $x \in P$. Moreover, since $K(P) \subseteq B$ for any basis B and $K(P)$ is uncountable, P is not ω-continuous. Notice, also, if $x \preceq y$, then we have $x \in K(P)$, which means $x \ll x \preceq y$ and $x \ll y$ for all $x, y \in P$. □

Since we tried to construct countable bases in Propositions 8 and 9 using order density properties, we intend now to do the converse. Notice, by Proposition 2, ω-continuous dcpos have countable Debreu upper dense subsets. However, there are ω-continuous dcpos which are not Debreu separable. To see this, we can take the counterexample in Proposition 3 (as the countable weak basis defined there is actually a basis). Alternatively, we can use either majorization for any $n \geq 3$ or the interval domain since, as we show in Lemma 6 in the Appendix A.2.2, both are ω-continuous and any of their Debreu dense subsets has cardinality $\mathfrak{c}$. We summarize this paragraph in the following statement.

Proposition 11. *If P is a dcpo with a basis $B \subseteq P$, then B is a Debreu upper dense subset of P. However, there are ω-continuous dcpos which are not Debreu separable.*

5.2 Conditionally connected Debreu upper separable dcpos

Although Debreu upper separability and w-continuity are not equivalent for general continuous dcpos (cf. Propositions 10 and 11), they do coincide for the Cantor domain Σ^∞. In this section, we use one of its properties, *conditional connectedness*, to show, in Theorem 2, Debreu upper

separability and w-continuity are equivalent for any conditionally connected dcpo. We begin by defining conditional connectedness.

Definition 17 (Conditionally connected partial order). *A partial order P is conditionally connected if, for any pair $x, y \in P$ for which there exists some $z \in P$ such that $x \preceq z$ and $y \preceq z$, we have x and y are comparable, that is $\neg(x \bowtie y)$.*

Note that conditional connectedness amounts to comparable elements conforming to equivalent classes. We could have named partial orders fulfilling Definition 17 *upward* conditionally connected to distinguish them from *downward* conditionally connected, which has been used in order-theoretical approaches to thermodynamics (Giles 2016; Landsberg 1970; Roberts and Luce 1968) and is referred to as *conditionally connected*. Note that it is straightforward to characterize conditionally connected partial orders as those where every directed set is a chain (see Lemma 12 in the Appendix A.1).

An example of a conditionally connected partial order is the Cantor domain since, if $x, y, z \in \Sigma^\infty$ are three strings such that x and y are prefixes of z, then it is clear that either x is a prefix of y or vice versa. A negative example comes from majorization for all $n \geq 3$, as one can pick, for example, the pair $x, y \in \Lambda^n$ where $x = (0.6, 0.2, 0.2, 0, \ldots, 0)$ and $y = (0.5, 0.4, 0.1, 0, \ldots, 0)$, and, although $x, y \preceq (1, 0, \ldots, 0) \in \Lambda^n$ holds, we have $x \bowtie y$. Note that conditional connectedness weakens a common property in order theory and its applications, namely, totality (Bridges and Mehta 2013; Debreu 1954; Lieb and Yngvason 1999).[3]

Even though conditionally connected dcpos may be more suitable to be interpreted in terms of information, many relevant order-theoretical models of uniform computability are not conditionally connected, like majorization, and have no conditionally connected straightforward variation. Take for example the interval domain (2). It is not conditionally connected since, for any $x \in \mathbb{R}$ and $0 < \varepsilon_i, \varepsilon_i'\ i = 1, 2$ with $\varepsilon_1 > \varepsilon_1'$ and $\varepsilon_2 < \varepsilon_2'$, we have $a := [x - \varepsilon_1, x + \varepsilon_2] \sqsubset x$ and $b := [x - \varepsilon_1', x + \varepsilon_2'] \sqsubset x$ although $a \bowtie b$. However, in case $\varepsilon_1 + \varepsilon_2 < \varepsilon_1' + \varepsilon_2'$, one would say a contains more information than b about *any* $y \in a \cap b$, which is not reflected by $\sqsubseteq$ although it is meant to represent information. The natural modification of the interval domain, which takes care of this, is, however, not a dcpo. We would like to define the binary relation $(\mathscr{I}, \sqsubseteq_2)$, where $[a, b] \sqsubseteq_2 [c, d] \iff [a, b] \cap [c, d] \neq \emptyset$ and $\ell([a, b]) \geq \ell([c, d])$ with $\ell([a, b]) := b - a$ being the length of the interval. Notice in order to turn $\sqsubseteq_2$ into a partial order, we need to identify all intervals which share some intersection and have the same length. One can easily see, however, that transitivity does not hold, hence, $\sqsubseteq_2$ is not even a partial order.

In the remainder of this section, we study the properties of conditionally connected dcpos to end up showing, in Theorem 2, the equivalence of Debreu upper separability and w-continuity for them. From Proposition 10, we know Debreu upper separability is insufficient to assure K(P) is countable in a continuous dcpo, which is necessary in order to hope for a countable basis $B \subseteq P$, since K$(P) \subseteq B$. Nevertheless, as we will see in Proposition 14, K(P) is countable for conditionally connected Debreu separable dcpos. In order to arrive at that result, we first characterize both the way-below relation (Proposition 12) and K(P) (Proposition 13) for conditionally connected dcpos.

Proposition 12 (Conditionally connected characterization of $\ll$). *If P is a conditionally connected dcpo, then we have, for all $x, y \in P$,*

$$x \ll y \iff \begin{cases} x \prec y, \text{ or} \\ x = y \text{ and there is no directed set } A \subseteq P \setminus \{x\} \\ \text{such that } \sqcup A = x. \end{cases}$$

Proof. Take $x, y \in P$. If $x \ll y$, then by definition we have $x \preceq y$ and either $x \neq y$, implying $x \prec y$, or $x \in \mathrm{K}(P)$ and there is no directed set $A \subseteq P \setminus \{x\}$ such that $\sqcup A = x$ by Lemma 7. For the converse, we first show, if P is a conditionally connected dcpo, $x \preceq y$ and A is a directed set such that $y \preceq \sqcup A$, then either there exists $a \in A$ such that $x \preceq a$ or $x = y = \sqcup A$. Consider, thus, a directed set A such that $y \preceq \sqcup A$ and $x \preceq y$. We have $x \preceq \sqcup A$ by transitivity and $\neg(x \bowtie a)$ for all $a \in A$ by conditional connectedness. If there exists some $a \in A$ such that $x \preceq a$, then we have finished. Otherwise, we have $a \prec x$ for all $a \in A$, thus, $\sqcup A \preceq x$ by definition of supremum and $x = y = \sqcup A$ by antisymmetry. We can now use this property to finish the proof. If $x \prec y$, then, given any directed set A such that $y \preceq \sqcup A$, we get there exists some $a \in A$ such that $x \preceq a$ by the property we proved above, since we have $x \neq y$. As a consequence, $x \ll y$ by definition of $\ll$. Consider now some $x \in P$ such that there is no directed set $A \subseteq P \setminus \{x\}$ fulfilling $\sqcup A = x$. Given some directed set A such that $x \preceq \sqcup A$, we get $\neg(x \bowtie a)$ for all $a \in A$ by conditional connectedness. If there exists $a \in A$ such that $x \preceq a$, then $x \ll x$. If the contrary holds, then $a \prec x$ for all $a \in A$ and $\sqcup A \preceq x$ by definition of supremum. Thus, $x = \sqcup A$ by antisymmetry and $x \in A$ by assumption, contradicting the fact $a \prec x$ for all $a \in A$. $\square$

Note that the characterization of compact elements in Proposition 12 also holds under the hypothesis that P is continuous, which is, as we show in Proposition 15, weaker than conditional connectedness. However, $x \prec y$ does not necessarily imply $x \ll y$ in case P is just continuous. (To convince ourselves about this, we can take, for example, majorization and compare (3),

$$x \preceq_M y \iff (\text{for all } i < n) \; s_i(x) \leq s_i(y),$$

with (10),

$$x \ll_M y \iff x = \bot \text{ or } s_k(x) < s_k(y) \text{ for all } k < n.)$$

Notice, also, there is no non-essential information in conditionally connected dcpos, that is, whenever a process converges to some $x \in P$, it must eventually gather the information of any element which contains information about x, since $y \prec x$ implies $y \ll x$ for all $x, y \in P$. Using Proposition 12, we can characterize the compact elements $\mathrm{K}(P)$ for conditionally connected dcpos. We do so in Proposition 13, for which we need another definition. We say $x \in P$ is *isolated* if there exists some $v_x \in P$ such that $v_x \prec x$ and, for all $y \in P$, we have $y \preceq v_x$ provided $y \prec x$. We denote the set of isolated elements by $\mathrm{I}(P)$ and note $I(\Sigma^\infty) = \Sigma^* \setminus \{\bot\}$.

Proposition 13 (Conditionally connected characterization of $\mathrm{K}(P)$). *If P is a conditionally connected dcpo, then*

$$\mathrm{K}(P) = \mathrm{I}(P) \cup \min(P).$$

Proof. $(\supseteq)$ $\min(P) \subseteq \mathrm{K}(P)$ holds by the characterization of $\mathrm{K}(P)$ in Proposition 12 since, if $x \in \min(P)$, the only directed set A such that $\sqcup A = x$ is $A = \{x\}$. Take $x \in \mathrm{I}(P)$ and assume $x \notin \mathrm{K}(P)$. Then, there exists a directed set $A \subseteq P \setminus \{x\}$ such that $\sqcup A = x$ by Proposition 12. Since $a \preceq v_x$ for all $a \in A$ by definition of an isolated element, we would have $x = \sqcup A \preceq v_x$ by definition of supremum, a contradiction. Thus, $\mathrm{I}(P) \subseteq \mathrm{K}(P)$. $(\subseteq)$ Take $x \notin \mathrm{I}(P) \cup \min(P)$ and define $A_x := \{z \in P | z \prec x\}$. Since $x \notin \min(P)$, we have $A_x \neq \emptyset$ and, by conditional connectedness, A_x is a chain. In particular, $\sqcup A_x$ exists and, since $a \prec x$ for all $a \in A_x$, $\sqcup A_x \preceq x$ by definition of supremum. If $\sqcup A_x = x$, then $x \notin \mathrm{K}(P)$ by Proposition 12, since $x \notin A_x$. Conversely, $\sqcup A_x \prec x$ would contradict the fact $x \notin \mathrm{I}(P)$ as, by definition of $\sqcup A_x$, any $y \in P$ such that $y \prec x$ would fulfill $y \preceq \sqcup A_x$. Thus, we could take $v_x := \sqcup A_x$ and conclude $x \in \mathrm{I}(P)$, a contradiction. In summary, $x \in \mathrm{K}(P)$ implies $x \in \mathrm{I}(P) \cup \min(P)$. $\square$

Notice, although $\mathrm{I}(P) \cup \min(P) \subseteq \mathrm{K}(P)$, $\mathrm{K}(P) \nsubseteq \mathrm{I}(P) \cup \min(P)$ if we weaken the hypothesis from *conditionally connected* to *continuous* in Proposition 13 (take, e.g., the dcpo in Proposition 10). As we already know from Lemma 4, $\min(P)$ is countable whenever countable

Debreu upper dense subsets exist. Moreover, for conditionally connected dcpos, the set of isolated elements $I(P)$ is countable as well if Debreu upper separability holds. Because of the characterization of the compact elements in Proposition 13, these two facts result in Proposition 14, where we show conditionally connected dcpos which are Debreu upper separable have a countable number of compact elements.

Proposition 14. *If P is a conditionally connected and Debreu upper separable dcpo, then $K(P)$ is countable.*

Proof. Since $K(P) = \min(P) \cup I(P)$ by Proposition 13 and $\min(P)$ is countable whenever there exists some countable Debreu upper dense set by Lemma 4, we only need to show $I(P)$ is countable to get the result. Take $D \subseteq P$ a countable Debreu dense subset and $D' := I(P) \setminus D$, where $I(P)$ the set of isolated points. If $x \in D'$, then there exists some $d \in D$ such that $v_x \preceq d \preceq x$, where $v_x \in P$ has the property that for all $y \in P$ such that $y \prec x$ we have $y \preceq v_x$. By definition of v_x and since $x \notin D$, $v_x = d$ holds. We define now a map

$$f : D' \to D$$
$$x \mapsto v_x.$$

If we show f is injective, then we get D' is countable and, thus, $I(P)$ also, since we have $I(P) \subseteq D' \cup D$. Take, thus, $x, y \in D'$ such that $f(x) = f(y)$ and assume $x \neq y$. Notice $x \prec y$ contradicts the definition of v_y, since it would mean $v_y = v_x \prec x \prec y$. Equivalently, we can discard $y \prec x$ by definition of v_x. If $x \bowtie y$, then, by Debreu upper density, there exists some $d \in D$ such that $x \bowtie d \preceq y$. Since $y \notin D$, we have $x \bowtie d \prec y$. Thus, $d \preceq v_y = v_x \prec x$ which contradicts, by transitivity, the fact that $x \bowtie d$. Since $x \neq y$ leads to contradiction, we conclude $f(x) = f(y)$ implies $x = y$ for all $x, y \in D'$ and, hence, f is injective. $\square$

To conclude $I(P)$ is countable in Proposition 14, it is sufficient for P to be a partial order instead of a dcpo. Notice, also, that we do not need the elements of D in its proof to have a finite number of immediate successors as in Proposition 9. In fact, there exist conditionally connected dcpos where any such a D has elements with an infinite number of immediate successors (take, e.g., P the dcpo defined analogously to (1) but using the natural numbers as alphabet $\Sigma := \mathbb{N}$ and note that any such $D \subseteq P$ will have elements with a countably infinite number of immediate successors). Actually, Proposition 14 shows any conditionally connected and Debreu upper separable partial order has a countable number of jumps, since the set of jumps is equinumerous to the set of isolated elements for conditionally connected dcpos (one can see that the map

$$\phi : J(P) \to I(P)$$
$$(x, y) \mapsto y$$

is bijective, where $J(P)$ denotes the set of jumps of P). We improve, thus, on the classical result that the number of jumps is countable for any Debreu separable *total* order (Debreu 1954) (see also Bridges and Mehta 2013, Proposition 1.4.4), as total partial orders are conditionally connected and Debreu separability coincides with Debreu upper separability for total orders. Notice, if we eliminate conditional connectedness, the number of jumps could be uncountable even if Debreu upper separability holds (take, for example, the dcpo in the proof of Proposition 4). In fact, there are continuous Debreu upper separable dcpos where $K(P)$ is uncountable (see Proposition 10) and conditionally connected dcpos where $K(P)$ is uncountable (for example, any uncountable set with the trivial partial order).

Before proving Theorem 2, we show, in Proposition 15, that conditionally connected dcpos are continuous. This fact plus the countability of $K(P)$ from Proposition 14 and some results from Sections 3 and 5.1 will allow us to derive Theorem 2.

Proposition 15. *If P is a conditionally connected dcpo, then P is a basis for P.*

Proof. We will show $B := P$ is a basis whenever P is conditionally connected. We need to show there exists a directed set $B_x \subseteq \downarrow x \cap P$ such that $x = \sqcup B_x$ for all $x \in P$. If $x \in K(P)$, we can take $B_x := \{x\}$. If $x \notin K(P)$, then, by Proposition 12, there exists a directed set $B_x \subseteq P \setminus \{x\}$ such that $\sqcup B_x = x$. Thus, $b \prec x$ and, again by Proposition 12, we have $b \ll x$ for all $b \in B_x$.   □

Notice, if P is conditionally connected, then, by the characterization of $K(P)$ in Proposition 13 plus the fact it has a basis $B \subseteq P$ (Proposition 15), any $x \in P \setminus (I(P) \cup \min(P))$ is a non-trivial element of B. If we also assume P has a countable Debreu dense subset $D \subseteq P$, we can then find an increasing sequence $(x_n)_{n \geq 0} \subseteq D$ such that $x = \sqcup(x_n)_{n \geq 0}$ and $x \notin (x_n)_{n \geq 0}$ (see Lemma 3) and we can easily see $x_n \ll x$ for all $n \geq 0$. Thus, if we assume P is a conditionally connected Debreu upper separable dcpo, we can construct a countable basis using $K(P)$ (countable by Proposition 14) and a countable Debreu dense subset. We show this and its converse in Theorem 2.

Theorem 2. *If P is a conditionally connected dcpo, then the following statements are equivalent:*

1. *P is Debreu upper separable.*
2. *P is ω-continuous.*
3. *P is Debreu separable and $K(P)$ is countable.*

Proof. We begin showing (1) implies (2). Given that $K(P)$ is countable by Proposition 14, we can apply Proposition 7, since conditionally connected dcpos are continuous by Proposition 15, and get the result. We proceed now to show (2) implies (3). If $B \subseteq P$ is a countable basis, then $K(P) \subseteq B$ is also countable. Take $x, y \in P$ such that $x \prec y$. We can follow Proposition 2 to get some $b_0 \in B$ such that $b_0 \preceq y$ and $\neg(b_0 \preceq x)$. Since we have both $b_0 \preceq y$ and $x \prec y$, then $\neg(x \bowtie b_0)$ holds by conditional connectedness. As a result, we have $x \prec b_0 \preceq y$. Thus, B is a countable Debreu dense subset of P. We finish showing (3) implies (1). We will show $D' := D \cup K(P)$ is a countable Debreu upper dense subset, where $D \subseteq P$ is a countable Debreu dense subset. Given any pair $x, y \in P$ such that $x \bowtie y$, we need to show there exists some $d \in D'$ such that $x \bowtie d \preceq y$. If $y \in K(P)$, then we take $d := y$ and have finished. If $y \notin K(P)$, then there exists a directed set $A \subseteq P$ such that $\sqcup A = y$ and $y \notin A$ by Proposition 12. By Lemma 3, thus, there is a directed set $D_y \subseteq D$ such that $\sqcup D_y = y$ and $y \notin D_y$. Analogously to the proof of Proposition 2, we can conclude there exists some $d \in D_y$ such that $d \prec y$ and $\neg(d \preceq x)$. Since $x \prec d$ would imply $x \prec y$ by transitivity, which is a contradiction, we obtain $x \bowtie d \prec y$.   □

Remark 7 (Implication for computability). The proof of Theorem 2 shows that we can introduce uniform computability (for both elements and functions) in conditionally connected dcpos which have a countable subset D, which is both Debreu dense and Debreu upper dense provided the countable basis $D \cup I(P) \cup \min(P)$ is effective.

We conclude from Theorem 2 that the coincidence between Debreu upper separability and *w*-continuity is not a specific fact of the Cantor domain but it holds for any conditionally connected dcpo. Note that we cannot substitute *conditional connectedness* by the weaker pair of hypotheses, which includes both continuity of P and the fact $x \preceq y$ implies $x \ll y$ for all $x, y \in P$. In that scenario, (1) does not imply (2) in Theorem 2 (see Proposition 10). This is relevant, since the second hypothesis in the pair is a rather strong property of conditionally connected dcpos.

In Corollary 2, we refine Theorem 2, characterizing a stronger order density property for conditionally connected dcpos. We say a partial order is *upper separable* if there exists a countable subset $D \subseteq P$, which is order dense and such that, for any pair $x, y \in P$ where $x \bowtie y$, there exists

some $d \in D$ such that $x \bowtie d \prec y$ (Ok et al. 2002). Note that the Cantor domain is not upper separable, as it is not order separable. Moreover, if $s \in \Sigma^*$ and $\beta, \gamma \in \Sigma$ $\beta \neq \gamma$, then we have $s\beta \bowtie s\gamma$, although, for all $t \in \Sigma^*$ such that $t \prec s\beta$, we have $t \prec s\gamma$.

Corollary 2. *If P is a conditionally connected dcpo, then the following statements are equivalent:*

1. *P is upper separable*
2. *P is ω-continuous and either $K(P) = \{\bot\}$ or $K(P) = \emptyset$ holds.*

Proof. We begin by showing (1) implies (2). Take $D \subseteq P$ a countable upper dense subset. By Theorem 2, we have P is ω-continuous and, by Lemma 9, we obtain that $K(P) = \min(P)$. We finish by noticing, if we have $x, y \in \min(P)$ $x \neq y$, then $x \bowtie y$ by definition and there exists $d \in D$ such that $x \bowtie d \prec y$ by upper separability, contradicting the fact $y \in \min(P)$. Thus, we have $|\min(P)| \leq 1$. Assume there exists some $x \in \min(P)$ and take $y \in P \setminus \min(P)$. Notice $y \prec x$ contradicts the fact $x \in \min(P)$ and so does, by upper separability, $y \bowtie x$. Thus, we have $x \prec y$ and $x = \bot$. Hence, we either have $K(P) = \{\bot\}$, or $K(P) = \emptyset$. We finish showing (2) implies (1). Take $B \subseteq P$ a countable basis and $x, y \in P$ such that $x \prec y$. We can follow the argument in Theorem 2 ((2) implies (3)) to get some $b_0 \in B$ such that $x \prec b_0 \ll y$. Since $y \neq \bot$, we have $y \notin K(P)$ and, thus, $x \prec b_0 \prec y$. Take now $x, y \in P$ such that $x \bowtie y$. Since we have a countable basis, we fulfill the hypotheses in Theorem 2 (3) and we can follow the argument there (when proving (3) implies (1)) to conclude there exists some $b_0 \in B$ such that $x \bowtie b_0 \prec y$. $\square$

Notice, a dcpo to which Corollary 2 applies can be found in the proof of Proposition 16. To finish this section, we notice the Cantor domain is not only ω-continuous but ω-algebraic. A dcpo P is *algebraic* if it has a basis $B \subseteq P$ consisting of compact elements $B = K(P)$ and *ω-algebraic* if such a basis is countable (Abramsky and Jung 1994). The Cantor domain is actually ω-algebraic, as $B := \{\bot\} \cup \Sigma^*$ is a countable basis and $B = \min(P) \cup I(P) = K(P)$. Nevertheless, we cannot substitute *ω-continuous* by *ω-algebraic* in Theorem 2, as we show in Proposition 16.

Proposition 16. *There exist conditionally connected Debreu upper separable dcpos which are not ω-algebraic.*

Proof. We can consider P the interval $[0, 1]$ with the usual order $\leq$. Clearly, it is conditionally connected (it is total) and Debreu upper separable (because of $\mathbb{Q} \cap [0, 1]$). However, while $B := \mathbb{Q} \cap [0, 1]$ is a countable basis, we have $K(P) = \{0\}$. Thus, P is not ω-algebraic. $\square$

To summarize, the main results in this section are Propositions 7, 8, and 9, and Theorem 2. In the first one, we show how one can profit from Debreu density and the countability of $K(P)$ to introduce computability on a continuous dcpo. In the second, we avoid the requirement on $K(P)$ and use the stronger property of order separability to introduce computability on all non-minimal elements. In the third one, we show that we can achieve the same results as in Proposition 8 by asking for the existence of a countable Debreu separable subset whose elements only have a finite number of immediate successors. Moreover, we extend computability to the whole dcpo in Propositions 8 and 9 by also requiring the existence of a countable Debreu upper dense subset. Lastly, in Theorem 2, we use conditional connectedness to show the equivalence between having a countable basis and being Debreu upper separable.

6. Relating Order Density with Completeness and Continuity

We have focused, in Sections 3 and 5, on the relationship between order density properties and both countable weak bases and bases. However, they are related to other properties of these

approaches to computability. In this section, we relate them to both order completeness and to a weak form of computable functions, namely, continuity in the Scott topology.

6.1 Completeness

Regarding completeness, we show, in Proposition 17, Debreu separable partial orders where increasing sequences have a supremum and are directed complete. Note that this implication was obtained in Martin and Mislove 2000, Proposition 2.5.1) under a different hypothesis, namely, second countability of the Scott topology. Since second countability of $\sigma(P)$ is equivalent to ω-continuity for continuous dcpos P by Lemma 8, there are Debreu separable partial orders whose Scott topology is not second countable (see Proposition 10) and vice versa (see Proposition 11).

Proposition 17. *If P is a Debreu separable partial order, then the following conditions are equivalent:*

1. *Every directed set has a supremum.*
2. *Every increasing sequence has a supremum.*

Proof. Since (1) implies (2) by definition, we only show the converse is true. Take $A \subseteq P$ a directed set and $D \subseteq P$ a countable Debreu dense subset. If there exists $a' \in A$ such that $a \preceq a'$ for all $a \in A$, then $\sqcup A = a'$ and we have finished. In the opposite case, we can argue as in Lemma 2 and obtain that, for every $a \in A$, there exists some $b \in A$ such that $a \prec b$. We can, thus, construct an increasing sequence $(d'_n)_{n \geq 0} \subseteq D$ as in Lemma 3 for which $\sqcup (d'_n)_{n \geq 0}$ exists by hypothesis. Since, for all $a \in A$, there exists some $n \geq 0$ such that $a \preceq d'_n$ by construction of $(d'_n)_{n \geq 0}$, we have, by transitivity, that $a \preceq \sqcup (d'_n)_{n \geq 0}$ for all $a \in A$. If we assume there exists some $x \in P$ such that for all $a \in A$ we have $a \preceq x$, then, as for all $n \geq 0$ there exists some $a_n \in A$ such that $d'_n \preceq a_n$ by construction of $(d'_n)_{n \geq 0}$, we have $d'_n \preceq x$ for all $n \geq 0$ by transitivity. Thus, by definition of supremum, $\sqcup (d'_n)_{n \geq 0} \preceq x$. We conclude that $\sqcup A = \sqcup (d'_n)_{n \geq 0}$. In particular, $\sqcup A$ exists. $\square$

Notice, whenever a partial order is Debreu separable, we can associate to each directed set an increasing sequence (defined by (4) and (5)), which is intimately related to it in the sense that, by Proposition 17, the directed set has some element as supremum if and only if the increasing sequence has the same supremum.

6.2 Continuity

Debreu separability is also closely related to the Scott topology. In particular, it is related to computable maps between dcpos as well. More precisely, to the wider set of continuous functions in the Scott topology. As we show in Theorem 3, continuous functions between dcpos $f : P \to Q$ coincide with sequentially continuous functions whenever P is Debreu separable. In fact, under the same hypothesis, they also coincide with monotones which preserve suprema of increasing sequences. Note that we say a monotone function $f : P \to Q$ *preserves suprema of increasing sequences* if, given any increasing sequence $(x_n)_{n \geq 0}$, we have $\sqcup (f(x_n))_{n \geq 0} = f(\sqcup (x_n)_{n \geq 0})$.

Theorem 3. *If P is a Debreu separable dcpo, Q is a dcpo, and $f : P \to Q$ is a map, then the following are equivalent:*

1. *f is sequentially continuous.*
2. *f is monotone and preserves suprema of increasing sequences.*
3. *f is continuous.*

Proof. We begin showing (1) implies (2). We first prove f is monotone. We will do so by contrapositive, that is, we will show, if f is not monotone, then it is not sequentially continuous. If f is not monotone, there exist $x, y \in P$ such that $x \preceq y$ and $\neg(f(x) \preceq f(y))$. Consider, then, the sequence $(x_n)_{n \geq 0}$, where $x_n := y$ for all $n \geq 0$. Note that $(x_n)_{n \geq 0}$ converges to x since, given some $O \in \sigma(P)$ such that $x \in O$, we have $x_n = y \in O$ for all $n \geq 0$ since $x \preceq y = x_n$. However, since we have $\neg(f(x) \preceq f(y))$, then, as we know from (8),

$$x \preceq y \iff x \in O \text{ implies } y \in O \text{ for all } O \in \sigma(P)$$

and there exists some $O \in \sigma(Q)$ such that $f(x) \in O$ and $f(y) \notin O$. Hence, $f(x_n) = f(y) \notin O$ and $(f(x_n))_{n \geq 0}$ does not converge to $f(x)$. Thus, f is not sequentially continuous, and, by contrapositive, any sequentially continuous f is monotone. We show now f preserves suprema of increasing sequences. Take an increasing sequence $(x_n)_{n \geq 0}$ and notice it converges to $\sqcup(x_n)_{n \geq 0}$ in $\sigma(P)$. This is the case since, given $O \in \sigma(P)$ such that $\sqcup(x_n)_{n \geq 0} \in O$, then, by definition of $\sigma(P)$, there exists some $n_0 \geq 0$ such that $x_{n_0} \in O$ and, since $(x_n)_{n \geq 0}$ is increasing and O is upper closed, $x_n \in O$ for all $n \geq n_0$. Since f is monotone, we have $f(x_n) \preceq f(\sqcup(x_n)_{n \geq 0})$ for all $n \geq 0$ and $(f(x_n))_{n \geq 0}$ is an increasing sequence, which implies $\sqcup(f(x_n))_{n \geq 0}$ exists. Define, for simplicity, $x := \sqcup(f(x_n))_{n \geq 0}$ and $y := f(\sqcup(x_n)_{n \geq 0})$. By definition of supremum, we have $x \preceq y$. Consider now $O \in \sigma(P)$ such that $y \in O$. Since $(x_n)_{n \geq 0}$ converges to $\sqcup(x_n)_{n \geq 0}$ in $\sigma(P)$, we have $(f(x_n))_{n \geq 0}$ converges to y in $\sigma(Q)$ by sequential continuity. Thus, there exists some $n_0 \geq 0$ such that $f(x_n) \in O$ for all $n \geq n_0$. In particular, since $f(x_n) \preceq x$ for all $n \geq 0$ and O is upper closed, we have $x \in O$. Hence, as we have shown, $y \in O$ implies $x \in O$ for all $O \in \sigma(P)$ and, by (8), $y \preceq x$. We obtain, by antisymmetry,

$$\sqcup(f(x_n))_{n \geq 0} = x = y = f(\sqcup(x_n)_{n \geq 0}).$$

In summary, f preserves suprema of increasing sequences.

We show now (2) implies (3). To do so, it is sufficient to show, under the hypotheses in (2), that $f(\sqcup A) = \sqcup f(A)$ for any directed set $A \subseteq P$. Assume first $\sqcup A \in A$. Since f is monotone, we have $f(a) \preceq f(\sqcup A)$ for all $a \in A$. If there exists some $z \in Q$ such that $f(a) \preceq z$ for all $a \in A$, then $f(\sqcup A) \preceq z$. Thus, $\sqcup f(A) = f(\sqcup A)$. Assume now $\sqcup A \notin A$ and take, as in Lemma 3, an increasing sequence $(d_n')_{n \geq 0}$, such that $\sqcup(d_n')_{n \geq 0} = \sqcup A$, where $\sqcup A \notin (d_n')_{n \geq 0}$. Notice, by monotonicity of f, $(f(d_n'))_{n \geq 0}$ is an increasing sequence, which means $\sqcup(f(d_n'))_{n \geq 0}$ exists and, since f preserves suprema of increasing sequences by hypothesis, we have

$$f(\sqcup A) = f(\sqcup(d_n')_{n \geq 0}) = \sqcup(f(d_n'))_{n \geq 0}.$$

Thus, if we show $\sqcup(f(d_n'))_{n \geq 0} = \sqcup f(A)$, then the proof is finished. Define, for simplicity, $z_0 := \sqcup(f(d_n'))_{n \geq 0}$ and $z_1 := \sqcup f(A)$. Since for all $a \in A$ there exists some $n \geq 0$ such that $a \preceq d_{n_0}'$ and f is monotone, we have $f(a) \preceq f(d_n') \preceq z_0$. Thus, by transitivity, $f(a) \preceq z_0$ for all $a \in A$ and, by definition of supremum, $z_1 \preceq z_0$. Conversely, since for all $n \geq 0$ there exists some $a \in A$ such that $d_n' \preceq a$ and f is monotone, we have $f(d_n') \preceq f(a) \preceq z_1$ for all $n \geq 0$. Thus, by transitivity, $f(d_n') \preceq z_1$ for all $n \geq 0$ and, by definition of supremum, $z_0 \preceq z_1$. By antisymmetry, we conclude $z_0 = z_1$. Hence,

$$f(\sqcup A) = f(\sqcup d_n') = \sqcup(f(d_n'))_{n \geq 0} = \sqcup f(A),$$

that is, f preserves suprema of directed sets.

Lastly, the fact that (3) implies (1) is a well-known topological fact (see Kelley 2017) for which Debreu separability of P is not needed. □

Remark 8 (Computability interpretation). Theorem 3 points toward the fact that, provided P is Debreu separable, a function between dcpos $f : P \to Q$ is computable (in the sense that of Definition 16) if and only if it sends computable elements (in particular, sequences of elements converging toward another one) to computable elements. (Note that this is not exactly so, since computable functions also require an effectivity property by (9).)

Note that Theorem 3 applies to the Cantor domain Σ^∞. More importantly, it is interesting to consider the relation between Theorem 3 and the topological countability axioms, like first and second countability. In particular, the equivalence in Theorem 3 between (1) and (3) also holds under the stronger hypotheses in the first statement of Proposition 7, since any function $f : X \to Y$ between topological spaces X and Y is sequentially continuous if and only if it is continuous whenever X is a first countable topological spaces (Munkres 2019). However, there exist second countable spaces (thus first countable) which are not Debreu separable (see Proposition 10). Moreover, there are Debreu separable dcpos which are not first countable (see Hack et al. 2022a, Proposition 5) and also some which are not topologically separable (like $P := (\mathbb{R}, =)$, where, if $B \subseteq P$ is a topologically dense subset of P, then $B = \mathbb{R}$, since $\{x\} \in \sigma(P)$ for all $x \in P$). Lastly, Note that the equivalence in Theorem 3 holds for any computable function between dcpos since, whenever they are defined, countable bases exist and, by Lemma 8, $\sigma(P)$ is second countable.

To clarify the relation between Debreu separable dcpos and the topological countability axioms, we need a couple more definitions. In order to achieve them, we use the following notation: Given a sequence $(x_n)_{n \geq 0} \subseteq X$ and a topological space (X, τ), we denote by $x_n \to x$ the fact that $(x_n)_{n \geq 0}$ converges to $x \in X$ according to τ.

Definition 18 (Sequentially open sets and sequential spaces). *If (X, τ) is a topological space, $O \subseteq X$ is a* sequentially open set *if, for every sequence $(x_n)_{n \geq 0} \subseteq X$ such that $x_n \to x \in O$, there exists some n_0 such that $x_n \in O$ for all $n \geq n_0$. Moreover, we say (X, τ) is a* sequential space *if every sequentially open set $O \subseteq X$ is an open set $O \in \tau$.*

Intuitively, sequential spaces are those for which convergence is completely determined by sequences (as opposed to the more general scenario, where they are characterized by nets (Kelley 2017). The equivalence between (1) and (3) in Theorem 3 points toward the fact that the Scott topology of Debreu separable dcpos is completely determined by sequences since continuity is. This is indeed the case, as we show directly, that is, without using (2), in Proposition 18.

Proposition 18. *If P is a Debreu separable dcpo, then P is a sequential space with respect to its Scott topology $\sigma(P)$.*

Proof. To prove the result, we only need to show that, if a set $O \subseteq P$ is sequentially open, then it is upper closed and, for all directed sets $A \subseteq P$ such that $\sqcup A \in O$, we have $A \cap O \neq \emptyset$. Take, thus, some $x \in O$ and some $y \in P$ such that $x \preceq y$. We note that $(x_n)_{n \geq 0}$, where $x_n := y$ for all $n \geq 0$, converges to x in $\sigma(P)$. Thus, by hypothesis, there exists some n_0 such that $x_n = y \in O$ for all $n \geq n_0$. Hence, O is upper closed. Take now a directed set $A \subseteq P$ such that $\sqcup A \in O$. If $\sqcup A \in A$, then $A \cap O \neq \emptyset$ and we have finished. If $\sqcup A \notin A$ and $D \subseteq P$ is a countable Debreu dense subset of P, then, by Lemma 3, we can construct a sequence $(d'_n)_{n \geq 0} \subseteq D$, defined by (4) and (5), such that $\sqcup(d'_n)_{n \geq 0} = \sqcup A$ and $\sqcup A \notin (d'_n)_{n \geq 0}$. Since $(d'_n)_{n \geq 0}$ fulfills $d'_n \to \sqcup d'_n = \sqcup A \in O$ by construction, then, by hypothesis, there exists some $n_0 \geq 0$ such that $d'_n \in O$ for all $n \geq n_0$. Hence, by definition of $(d'_n)_{n \geq 0}$, there exists some $a \in A$ such that $d'_{n_0} \preceq a$ and, since O is upper closed as we showed before, we conclude $a \in A \cap O \neq \emptyset$. We conclude $O \in \sigma(P)$. $\square$

To summarize, this section shows that Debreu separability allows us to characterize both the completeness (Proposition 17) and the Scott topology (Theorem 2 and Proposition 18) of a partial order using only sequences.

7. Functional Countability Restrictions

We conclude, in this section, relating the usual countability restriction in the order-theoretical approaches to computability with some usual functional restrictions in order theory, namely, countable multi-utilities.

A family V of real-valued functions $v: X \to \mathbb{R}$ is called a *multi-utility (representation) of* $\preceq$ (Evren and Ok 2011) if

$$x \preceq y \iff v(x) \leq v(y) \text{ for all } v \in V.$$

Whenever a multi-utility consists of strict monotones, it is called a *strict monotone* (or *Richter-Peleg* Alcantud et al. 2016) *multi-utility (representation) of* $\preceq$. In this section, we relate multi-utilities and strict monotone multi-utilities to both bases and weak bases.

We begin, in Proposition 19, addressing the existence of multi-utilities for dcpos and their relation to both bases and weak bases. Note that we say a real-valued function $f : X \to \mathbb{R}$ is lower semicontinuous if $f^{-1}((a, \infty))$ is an open set of τ_X, the topology of X.

Proposition 19. *If P is a dcpo and, for all $x \in P$,*

$$d(x) := \{y \in P \mid y \preceq x\}, \tag{12}$$

then $(u_x)_{x \in P}$ is a lower semicontinuous multi-utility, where

$$u_x(y) = \begin{cases} 1 & \text{if } y \in d(x)^c, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x, y \in P$. Moreover, if P has a (basis) weak basis $B \subseteq P$, then $(v_b)_{b \in B}$ is a (lower semicontinuous) multi-utility, where

$$v_b(x) = \begin{cases} 1 & \text{if } b \preceq x \ (b \ll x), \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in P$, $b \in B$. However, the converse of the second statement is not true.

Proof. For the first statement, it is easy to see that $(u_x)_{x \in P}$ is a multi-utility for $\sigma(P)$. To conclude, we show $d(x)$ is closed in $\sigma(P)$ for all $x \in P$, which implies u_x is lower semicontinuous for all $x \in P$. Since $d(x)$ is a lower set for all $x \in P$, we only need to show, given any directed set $A \subseteq d(x)$, we have $\sqcup A \in d(x)$. Take $A \subseteq d(x)$ a directed set. Note that $\sqcup A$ exists and $a \preceq x$ for all $a \in A$, which means, by definition of least upper bound, $\sqcup A \preceq x$ and $\sqcup A \in d(x)$. Thus, $d(x)$ is closed in $\sigma(P)$ for all $x \in P$.

For the second statement, note that $\{v_b\}_{b \in B}$ is a multi-utility by Proposition 2, where $B \subseteq P$ is a weak basis. If B is a basis, then $\{v_b\}_{b \in B}$ is a multi-utility again by Proposition 2 (in this case, we take $v_b(x) = 1$ if $b \ll x$ for all $x \in P$, $b \in B$ and $v_b(x) = 0$ otherwise), and v_b is lower semicontinuous for all $b \in B$ since B is a basis, which implies $\mathord{\uparrow} b \in \sigma(P)$ (Abramsky and Jung 1994).

For the third statement, take P the dcpo from the proof of Proposition 4 as a counterexample. As shown there, P has no countable weak basis. However, $(w_x)_{x \in \Sigma^*}$ is a countable multi-utility, where $w_x(y) := 1$ if $x \preceq y$ and $w_x(y) := 0$ otherwise. Note that w_x is lower semicontinuous for all $x \in \Sigma^*$ since $\Sigma^* \subseteq \mathrm{K}(P)$ as we showed in the proof of Proposition 10. $\qquad \square$

Note that the second statement in Proposition 19 also holds whenever $B \subseteq P$ is a Debreu dense subset (Hack et al. 2022b). However, as we showed in Propositions 3 and 4, there exist dcpos where either of these hypotheses holds for some countable B and the other does not.

As shown in Proposition 19, there exist dcpos where lower semicontinuous multi-utilities exist and the Scott topology is not second countable (note the counterexample in Proposition 19 is a continuous dcpo and, hence, the existence of a countable basis and second countability of the Scott topology are equivalent Abramsky and Jung 1994). As we show in the following proposition, the equivalence holds when considering a coarser topology, namely, the lower topology. (Given a partial order $P = (X, \preceq)$, the closed sets of the *lower topology* $\tau^l_{\preceq}$ are the intersections of finite unions of elements in the family $(d(x))_{x \in X}$, which we defined in (12).

Before proving that proposition, we recall some topological concepts that we will use for both its proof and that of the following result, Theorem 4.

Definition 19 (Subbasis, net, and net convergence, Munkres 2019). *If (X, τ) is a topological space, then a* subbasis *is a family of subsets $(O_i)_{i \in I} \subseteq X$ whose union is X and such that any element in the topology τ can be generated as the union of finite intersections of elements in the subbasis. Moreover, if I is a directed set (in the sense of some preorder $\preceq_I$), a* net $\{x_\alpha\}_{\alpha \in I} \subseteq X$ *is a function $g : I \to X$, $\alpha \mapsto x_\alpha$. Lastly, we say a net $\{x_\alpha\}_{\alpha \in I} \subseteq X$* converges *to $x \in X$ provided there exists, for each $O \in \tau$, some $\alpha_O \in I$ such that $x_\alpha \in O$ whenever $\alpha_0 \preceq_I \alpha$.*

Note that subbases contrast with the more common topological notion of bases, where the elements in τ are generated using only *unions* of elements in a basis. Regarding net convergence, it is clear that, if we take a dcpo P, we fix $I = D$, we take the identity as g, and we use $(P, \sigma(P))$ as our topological space, then any directed set $D \subseteq P$ is a net which converges to $\sqcup D$.

Proposition 20. *If $(P, \tau_\preceq^l)$ is the topological space consisting of a dcpo P equipped with the lower topology $\tau_\preceq^l$, then there exists a lower semicontinuous countable multi-utility if and only if $\tau_\preceq^l$ is second countable.*

Proof. Assume first there exists a countable lower semicontinuous multi-utility $(u_n)_{n \geq 0}$. Note, by definition of the lower topology, $(A_x)_{x \in P}$ is a subbasis for $\tau_\preceq^l$, where

$$A_x := d(x)^c = \{y \in P \mid \neg(y \preceq x)\}.$$

Take $y \in A_x$ for some $x \in P$ and note there exist a pair $n \geq 0$ and $q \in \mathbb{Q}$ such that $y \in O_{n,q} \subseteq A_x$, where $O_{n,q} := u_n^{-1}((q, \infty))$ for all $n \geq 0$ and $q \in \mathbb{Q}$. This is the case since we have, by definition, $\neg(y \preceq x)$. Hence, there exists some $n \geq 0$ such that $u_n(x) < u_n(y)$ and, as a result, some $q \in \mathbb{Q}$ such that $y \in O_{n,q} \subseteq A_x$. Note $(O_{n,q})_{n \geq 0, q \in \mathbb{Q}} \subseteq \tau_\preceq^l$ given that u_n is lower semicontinuous for all $n \geq 0$. As a result, $(O_{n,q})_{n \geq 0, q \in \mathbb{Q}}$ is a countable subbasis of $\tau_\preceq^l$ and, hence, $\tau_\preceq^l$ is second countable. For the converse, take $(B_n)_{n \geq 0}$ a countable basis for $\tau_\preceq^l$ and note $(u_n)_{n \geq 0}$ is a lower semicontinuous countable multi-utility, where for all $n \geq 0$ we have $u_n(x) := 1$ if $x \in B_n$ and $u_n(x) := 0$ otherwise. To see this holds, take $x, y \in P$. If $x \preceq y$ and $y \notin B_n$ for some $n \geq 0$, then $y \in O$, where $O \subseteq B_n^C$ is some finite union of intersections of sets in the family $(d(x))_{x \in P}$. Note such an O exists by definition of $\tau_\preceq^l$. By transitivity of $\preceq$, $x \in O$. Thus, $x \notin B_n$ and u_n is monotone for all $n \geq 0$. If $\neg(x \preceq y)$, then $x \in A_y \in \tau_\preceq^l$ and there exists some $n \geq 0$ such that $x \in B_n \subseteq A_y$. Hence, we have $u_n(x) > u_n(y)$ and $(u_n)_{n \geq 0}$ is a multi-utility. Note u_n is lower semicontinuous since B_n is open for all $n \geq 0$. □

In particular, note, if $\tau_\preceq^l$ is second countable, then there exists a lower semicontinuous countable multi-utility for $\sigma(P)$.

In the following theorem, we show the main results of this section. In particular, we note we can partially reproduce the implications in Theorem 2 by requiring the existence of a finite strict monotone multi-utility instead of conditional connectedness.

Theorem 4. *If P is a dcpo with a finite lower semicontinuous strict monotone multi-utility, then the following hold:*

1. *P is ω-continuous if and only if $\mathrm{K}(P)$ is countable.*

2. *If P has a (countable) basis $B \subseteq P$, then B is a (countable) Debreu dense and Debreu upper dense subset.*

Proof. (1) If P is ω-continuous, then, clearly, it is continuous and $\mathrm{K}(P)$ is countable. (The latter follows since $\mathrm{K}(P) \subseteq B$ for any basis $B \subseteq P$ Abramsky and Jung 1994.) To show the converse holds, we begin proving any dcpo P is continuous whenever a finite lower semicontinuous strict monotone multi-utility exists. In order to do so, we establish first that, under the same hypothesis, $x \prec y$ implies $x \ll y$ for all $x, y \in P$.

Lemma 10. *If P is a dcpo with a finite lower semicontinuous strict monotone multi-utility, then $x \prec y$ implies $x \ll y$ for all $x, y \in P$.*

Proof. Fix $(v_n)_{n \leq N}$ a finite lower semicontinuous strict monotone multi-utility and take $x, y \in P$ such that $x \prec y$ and $D \subseteq P$ a directed set such that $y \preceq \sqcup D$. We intend to show there exists some $d \in D$ such that $x \preceq d$. Note $x \prec \sqcup D$ by transitivity and $v_n(x) < v_n(\sqcup D)$ for all $n \leq N$ by definition of $(v_n)_{n=1}^N$. If $\sqcup D \in D$, then we can take $d = \sqcup D$ and we have finished. If $\sqcup D \notin D$, then there exists some $d_1 \in D$ such that $v_1(x) < v_1(d_1) < v_1(\sqcup D)$, since D converges as a net to $\sqcup D$ (see the comment after Definition 19) and v_1 is lower semicontinuous. For the same reason, there exists a set $\{d_1, \ldots, d_N\} \subseteq D$ such that $v_n(x) < v_n(d_n) < v_n(\sqcup D)$ for all $n \leq N$. Given the fact D is directed, there exists some $c_1 \in D$ such that $d_1 \preceq c_1$ and $d_2 \preceq c_1$. We define c_n recursively in the same way using d_n and c_{n-1} for all n such that $1 < n \leq N$. Note that $d := c_{N-1}$ has all the desired properties. In particular, by definition, $d \in D$ and $v_n(x) < v_n(d)$ for all $n \leq N$. By definition of $(v_n)_{n=1}^N$, we have $x \prec d \in D$. Thus, $x \ll y$.

Alternatively, we can also show that $x \prec y$ implies $x \ll y$ for all $x, y \in P$ as follows: Given that $x \prec y$, we have

$$y \in U := \bigcap_{n \leq N} v_n^{-1}\big((\alpha_n, \infty)\big) \in \sigma(P),$$

where U is open since it is a finite intersection of open sets (which are open by lower semicontinuity of $(v_n)_{n=1}^N$) and $(\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N$ such that $v_n(x) < \alpha_n < v_n(y)$ for $n = 1, \ldots, N$. Hence, given a directed set D such that $y \preceq \sqcup D$, we have $\sqcup D \in U \in \sigma(P)$ and, thus, there exists some $d \in D$ such that $d \in U$. Since $v_n(x) < v_n(d)$ for all $n \leq N$, we have $x \prec d$ and we have finished the alternative argument supporting that $x \ll y$ for all $x, y \in P$ such that $x \prec y$. $\qquad\square$

We proceed now to show P is continuous whenever a finite lower semicontinuous strict monotone multi-utility exists.

Lemma 11. *If P is a dcpo with a finite lower semicontinuous strict monotone multi-utility, then P is continuous.*

Proof. Take some $x \in P$. We ought to show there exists some directed set $D_x \subseteq P \cap \downarrow x$ such that $\sqcup D_x = x$. If there exists some directed set $D \subseteq P \setminus \{x\}$ such that $\sqcup D = x$, then we have finished, since we have $d \ll x$ for all $d \in D$ given that $d \prec x$ for all $d \in D$. Assume now there is no directed set $D \subseteq P \setminus \{x\}$ such that $\sqcup D = x$. We will show, in this case, $x \in \mathrm{K}(P)$, which concludes the proof that P is continuous by taking $D_x := \{x\}$. Consider, hence, a directed set $D \subseteq P$ such that $x \preceq \sqcup D$. If $\sqcup D = x$, then $x \in D$ and we have finished. If $x \prec \sqcup D$, then $x \ll \sqcup D$ by Lemma 10 and there exists some $d \in D$ such that $x \preceq d$. Thus, $x \in \mathrm{K}(P)$ in this scenario and P is continuous. $\qquad\square$

To conclude, we ought to show P is ω-continuous whenever there exists a finite lower semicontinuous strict monotone multi-utility and $\mathrm{K}(P)$ is countable. To establish this, we show that the family consisting of (a) the union of the sets of elements way above each compact element and (b) the sets of elements strictly above each set of rational values a finite lower semicontinuous strict monotone multi-utility may take,

$$B := \left(\uparrow x \right)_{x \in K(P)} \bigcup \left(\bigcap_{n \leq N} v_n^{-1} \big((q_n, \infty) \big) \right)_{(q_1, \ldots, q_N) \in \mathbb{Q}^N}, \tag{13}$$

is a countable basis for $\sigma(P)$ whenever $K(P)$ is countable. In order to show that (13) is indeed a basis, take $x \in O \in \sigma(P)$ and notice, since P is continuous as we showed, there exists some $y \in O$ such that $y \ll x$. If $y = x$, then $x \in K(P)$. Thus, $x \in \uparrow x \subseteq O$. If $y \neq x$, then $y \prec x$. As a result, there exists some $(q_1, \ldots, q_N) \in \mathbb{Q}^N$ such that $v_n(y) < q_n < v_n(x)$ for all $n \leq N$ and $x \in U$, where

$$U := \bigcap_{n \leq N} v_n^{-1} \big((q_n, \infty) \big).$$

Note $U \subseteq O$ since any $z \in U$ fulfills $v_n(y) < v_n(z)$ for all $n \leq N$ and, by definition of $(v_n)_{n=1}^N$, we get $y \prec z$, thus $z \in O$ as O is upper closed by definition.

(2) We show a (countable) basis $B \subseteq P$ is a Debreu dense and Debreu upper dense subset of P. Note B is a Debreu upper dense subset of P by Proposition 2. To conclude, we show B is also Debreu dense. Take, thus, $x, y \in P$ such that $x \prec y$. As we showed in (1), we have $x \ll y$ since there exists a finite lower semicontinuous strict monotone multi-utility. Hence, by the interpolation property (Abramsky and Jung 1994, Lemma 2.2.15), there exists some $b \in B$ such that $x \ll b \ll y$. In particular, we have $x \preceq b \preceq y$. Hence, B is a (countable) Debreu dense subset of P. $\square$

In order to interpret Theorem 4 (1) in terms of computability, we provide an explicit basis (in the dcpo sense) in the following proposition. We do so since the proof of Theorem 4 (1) only shows that $\sigma(P)$ is second countable whenever a dcpo has a finite lower semicontinuous strict monotone multi-utility and $K(P)$ is countable.

Proposition 21. *Take P a dcpo with a finite lower semicontinuous strict monotone multi-utility $(v_i)_{i=1}^N$, $T := \{(q, r) \in \mathbb{Q}^N \times \mathbb{Q}^N | q_i < r_i$ for $i = 1, \ldots, N\}$, a numeration of T, $\gamma : \mathbb{N} \to T$, whose first (last) N components we denote by γ_1 (γ_2),*

$$m_0 := \min\{n \geq 0 \mid \exists x \in P \text{ s.t. } v_i(\alpha_1(n)) < v_i(x) < v_i(\alpha_2(n))\} \text{ and}$$
$$m_n := \min\{n \geq m_{n-1} + 1 \mid \exists x \in P \text{ s.t. } v_i(\alpha_1(n)) < v_i(x) < v_i(\alpha_2(n))\} \text{ for all } n \geq 1.$$

If $K(P)$ is countable, then $K(P) \cup (t_n)_{n \geq 0}$ is a countable basis, where, for all $n \geq 0$, we take as t_n some $x \in P$ such that $v_i(\gamma_1(m_n)) < v_i(x) < v_i(\gamma_2(m_n))$.

Proof. We ought to show that, for each $x \in P$ and $B := K(P) \cup (t_n)_{n \geq 0}$, there exists a directed set $B_x \subseteq \downarrow x \cap B$ such that $\sqcup B_x = x$. If $x \in K(P)$, then we take $B_x = \{x\}$ and we have finished. If $x \notin K(P)$, then consider some $(q_1^0, \ldots, q_N^0) \in \mathbb{Q}^N$ such that $q_i^0 < v_i(x) < q_i^0 + 1$ for $i = 1, \ldots, N$. Since $(v_i)_{i=1}^N$ is lower semicontinuous and P is continuous by Lemma 11, there exists some $y_0 \in \downarrow x \cap P$ such that $q_i^0 < v_i(y_0) < v_i(x)$ for $i = 1, \ldots, N$ (the latter inequalities follow since $x \notin K(P)$ and, hence, $y_0 \prec x$). We can then consider some $(r_1^0, \ldots, r_N^0) \in \mathbb{Q}^N$ such that $v_i(y_0) < r_i^0 < v_i(x)$ for $i = 1, \ldots, N$. By construction, there exists some $x_0 \in (t_n)_{n \geq 0}$ such that $q_i^0 < v_i(x_0) < r_i^0$ for $i = 1, \ldots, N$. We can then take, for all $m > 0$, $q_i^m = r_i^{m-1}$ for $i = 1, \ldots, N$ (we do this provided $r_i^{m-1} < v_i(x) < r_i^{m-1} + 2^{-m}$, otherwise we take some $(q_1^m, \ldots, q_N^m) \in \mathbb{Q}^N$ fulfilling these inequalities), find some $y_m \in \downarrow x \cap P$ such that $q_i^m < v_i(y_n) < v_i(x)$ for $i = 1, \ldots, N$ and some $(r_1^m, \ldots, r_N^m) \in \mathbb{Q}^N$ such that $v_i(y_m) < r_i^m < v_i(x)$ for $i = 1, \ldots, N$. Finally, by construction, there exists, for all $m > 0$, some $x_m \in (t_n)_{n \geq 0}$ such that $q_i^m < v_i(x_m) < r_i^m$ for $i = 1, \ldots, N$. To conclude, we show that $\sqcup (x_m)_{m \geq 0} = x$. By construction, for all $m \geq 0$, $v_i(x_m) < v_i(x_{m+1})$ for $i = 1, \ldots, m$. Hence, $(x_m)_{m \geq 0}$ is an increasing sequence and $\sqcup (x_m)_{m \geq 0}$ exists. Moreover, by construction, $\sqcup (x_m)_{m \geq 0} \preceq x$. Now, if $\sqcup (x_m)_{m \geq 0} \prec x$, then $v_i(\sqcup (x_m)_{m \geq 0}) < v_i(x)$ for $i = 1, \ldots, N$, which contradicts the definition of $(x_m)_{m \geq 0})$. Hence, since $x \prec y$ implies $x \ll y$ by Lemma 10, $B = K(P) \cup (t_n)_{n \geq 0}$ is a countable basis. $\square$

We can now interpret Theorem 4 (1) in terms of computability.

Remark 9 (Implication for computability). By Theorem 4, we can define computable elements and functions (in the sense of Definitions 15 and 16) on a dcpo P with a finite lower semicontinuous strict monotone multi-utility whenever $K(P)$ is countable and $K(P) \cup (t_n)_{n \geq 0}$ is effective. (Note that $(t_n)_{n \geq 0}$ was defined in Proposition 21.)

Note we have shown in the proof of Theorem 4 (1) that the equivalence in Proposition 12 also holds when substituting conditional connectedness by the existence of a finite lower semicontinuous strict monotone multi-utility. Note, as we stated in Proposition 11, there exist ω-continuous dcpos which are not Debreu separable. The inclusion of ω-continuity in both clauses of Theorem 4 is necessary in order for $K(P)$ to be countable and for P to be Debreu upper separable, as we show in Proposition 22 (1). Moreover, in Proposition 22 (2), we show the converse of Theorem 4 (2) is false. That is, although the equivalence between the clauses (2) and (3) in Theorem 2 and the fact they imply Theorem 2 (1) are achieved requiring the existence of a finite lower semicontinuous strict monotone multi-utility instead of conditional connectedness, Theorem 2 (1) does not imply either Theorem 2 (2) or Theorem 2 (3). Lastly, note that there are dcpos where $K(P)$ is countable and P is not even continuous, like the one in Lemma 7, where $K(P) = \emptyset$ and, as argued there, there is no $x \in P$ such that $x \ll 0 \in P$.

Proposition 22. *There exist dcpos P with finite lower semicontinuous strict monotone multi-utilities and either of the following properties:*

1. $K(P)$ *is uncountable and P is not Debreu upper separable.*
2. P *is Debreu upper separable and $K(P)$ is uncountable.*

Proof. (1) Take the dcpo P which consists of the set $[0, 1]$ endowed with the trivial ordering and note $V := \{i_d, -i_d\}$ is a strict monotone multi-utility, where i_d is the identity function. Both functions in V are lower semicontinuous in $\sigma(P)$ since $\preceq$ is the trivial ordering and, whenever $\sqcup D = x$ for some directed set, we have $D = \{x\}$. Because of that, $K(P) = [0, 1]$ and P is not Debreu upper separable.

(2) Take the dcpo $P := ([0, 1], \preceq)$, where, for all $x, y \in P$, $x \preceq y$ if and only if $x = y$ or $x \in \mathbb{Q}$, $y \notin \mathbb{Q}$ and $x < y$. Note P is, essentially, the counterexample in Proposition 4. As we showed there, $K(P) = P$ is uncountable, and P is Debreu upper separable. To conclude, one can see $V := \{v_1, v_2\}$ is a strict monotone multi-utility, where v_1 is the identity function and $v_2(x) := -x$ if $x \notin \mathbb{Q}$ and $v_2(x) := -x - 1$ if $x \in \mathbb{Q}$. Note the functions in V are lower semicontinuous for the same reason the functions in (1) are. □

Note that Theorem 4 (2) can be used to conclude certain dcpos have no finite strict monotone multi-utility that is lower semicontinuous in their Scott topology. We can apply this, in particular, to the examples in Section 2.1 that are not Debreu upper separable (see Lemma 6).

Corollary 3. *Both $(\Lambda^n, \preceq_M)$ for $n \geq 3$ and $(I, \sqsubseteq)$ have no finite lower semicontinuous strict monotone multi-utility representation.*

The examples in Corollary 3 are also useful to show we cannot improve on Theorem 4 (2) by weakening the hypothesis from finite strict monotone multi-utilities to multi-utilities, as we show in the following proposition.

Proposition 23. *There exist ω-continuous dcpos which, despite having finite lower semicontinuous multi-utilities, are not Debreu upper separable.*

Proof. We can take majorization $(\Lambda^n, \preceq_M)$ with any $n \geq 3$ as a counterexample. As we show in Lemma 6, $(\Lambda^n, \preceq_M)$ is ω-continuous and not Debreu separable for all $n \geq 3$. To conclude, we show $(\Lambda^n, \preceq_M)$ has a finite lower semicontinuous multi-utility for all $n \geq 2$. In particular, we show $(s_i)_{i=1}^{n-1}$ is a finite lower semicontinuous multi-utility of majorization $(\Lambda^n, \preceq_M)$ for all $n \geq 2$, where $s_i(x) := \sum_{j=1}^{i} x_j$.

For simplicity of notation, we define $P := (\Lambda^n, \preceq_M)$. We show s_k is lower semicontinuous for all $k < n$, that is, that $s_k^{-1}(r, \infty) \in \sigma(P)$ for all $r \in \mathbb{R}$, $k < n$. Take $k < n$ and some $r \in \mathbb{R}$ such that $k/n \leq r < 1$ and note the other cases are straightforward. (If $r < k/n$, then $s_k^{-1}(r, \infty) = \Lambda^n$ since $s_k(\perp) = k/n$ for $k = 1, \ldots, n-1$, and $\Lambda^n \in \sigma(P)$ by definition of topology. Moreover, if $1 \leq r$, then $s_k^{-1}(r, \infty) = \emptyset$ since $s_k((1, 0, \ldots, 0)) = 1$ for $k = 1, \ldots, n-1$, and $\emptyset \in \sigma(P)$ by definition of topology.) Notice, given $p \in s_k^{-1}(r, \infty)$, there exists some $q \in s_k^{-1}(r, \infty)$ such that $p \in \, \Uparrow q \subseteq s_k^{-1}(r, \infty)$. To see this, we can take some $\varepsilon < s_k(p) - r$ and apply Lemma 1, obtaining some $q \in \mathbb{Q}^n \cap \Lambda^n$ such that $s_k(p) - \varepsilon < s_k(q) < s_k(p)$ for all $k < n$. We have, in particular, $q \ll p$ by (10). This concludes the proof, since we have $s_k^{-1}(r, \infty) \in \sigma(P)$ given that $\Uparrow q \in \sigma(P)$ for all $q \in P$ (Abramsky and Jung 1994, Corollary 2.2.16). $\qquad \square$

Note Proposition 23 also shows we cannot improve on the result by asking for the existence of countably infinite lower semicontinuous strict monotone multi-utilities, since they also exist for the counterexample in Proposition 23. In fact, they exist whenever lower semicontinuous countable multi-utilities do (Alcantud et al. 2016).

To summarize, the main results in this section are Proposition 19 and Theorem 4. In the first one, we show that any dcpo has a lower semicontinuous multi-utility and, moreover, that we can pick one with the cardinality of any basis the dcpo may have. In the latter, we show that, whenever finite strict monotone multi-utilities exist, the existence of countable bases is equivalent to the countability of K(P) and, moreover, that any basis is both Debreu dense and Debreu upper dense.

8. Conclusion

In this paper, we have illustrated the role of countability restrictions in the attempt to translate computability from Turing machines to uncountable spaces using ordered structures. We have connected the countability restrictions in a general order-theoretic approach to computability that was recently introduced (Hack et al. 2022a) and the ones in domain theory (Abramsky and Jung 1994; Scott 1970) to the usual ones in order theory, namely, order density properties and multi-utilities. In particular, we have established several connections between order density properties, such as Debreu separability, order density, or Debreu upper separability, and the existence of countable weak bases in the more general approach. We have also explored the influence of order density properties in domain theory, establishing their equivalence with countable bases for the class of dcpos that are conditionally connected, which includes the prominent example of the Cantor domain. After connecting order density with both order completeness and continuity in the Scott topology, we finished relating bases, weak bases, and order density to multi-utilities. Regarding computability, we obtained several results that show, for a given dcpo with either some functional (multi-utility) or density countability restriction, how computability can be defined starting from these constraints. Several questions remain open. For example, it would be relevant to further clarify the role of multi-utilities in computability, since they play a leading role in the study of partial orders.

Competing interests. The authors declare none.

Notes

1 Notice, just like we defined computable functions $f : \mathbb{N} \to \mathbb{N}$, we can define computable functions $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ via Turing machines.

2 We say a partial order is *dense* if, for all $x, y \in X$ such that $x \prec y$, there exists some $z \in X$ such that $x \prec z \prec y$ (Bridges and Mehta 2013). That is, a partial order is *dense* if it has an order-dense subset.

3 A partial order P is said to be *total* if any pair $x, y \in P$ is comparable (Bridges and Mehta 2013).

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Appendix A
A.1 Proofs

Proposition 24. *If P is a Debreu separable dcpo and $A \subseteq P$ is a directed set, then there exists an increasing chain $(a_n)_{n \geq 0} \subseteq A$ with the same supremum as A, $\sqcup(a_n)_{n \geq 0} = \sqcup A$.*

Proof. Take $D \subseteq P$ as a Debreu dense subset of P. If $\sqcup A \in A$, then we take $(a_n)_{n \geq 0}$ with $a_n := \sqcup A$ for all $n \geq 0$ and we have finished. Otherwise, consider the increasing sequence $(d'_n)_{n \geq 0} \subseteq D$ such that $\sqcup(d'_n)_{n \geq 0} = \sqcup A$ from the proof of Lemma 3. By construction, there exists some $b_n \in A$ such that $d'_n \preceq b_n$ for all $n \geq 0$. Notice $(b_n)_{n \geq 0}$ is a directed set, since given $n, m \geq 0$ there exist some $c \in A$ such that $b_n, b_m \preceq c$ and, by construction, some $p \geq 0$ such that $b_n, b_m \preceq c \preceq d'_p \preceq b_p$. Thus, we construct an increasing chain $(a_n)_{n \geq 0} \subseteq A$ from $(b_n)_{n \geq 0}$, like we constructed D'_A from D_A in the proof of Lemma 3. Notice $\sqcup(a_n)_{n \geq 0}$ exists, since P is directed complete. We only need to show $\sqcup(a_n)_{n \geq 0} = \sqcup A$. By definition, $a_n \preceq \sqcup A$ for all $n \geq 0$. Assume there exists some $z \in P$ such that $a_n \preceq z$ for all n. Then, $d'_n \preceq z$ for all $n \geq 0$ and, thus, $\sqcup A = \sqcup(d'_n)_{n \geq 0} \preceq z$. Thus, $\sqcup(b'_n) = \sqcup A$. $\square$

Lemma 12. *If P is a partial order, then P is conditionally connected if and only if any directed set is a chain.*

Proof. Consider a directed set $A \subseteq P$ and $x, y \in A$. Since A is directed, there exists $z \in A$ such that $x \preceq z$ and $y \preceq z$. We have, by conditional connectedness, $\neg(x \bowtie y)$ and, thus, A is a chain. Conversely, take $x, y \in P$ such that there exists $z \in P$ where $x \preceq z$ and $y \preceq z$ hold. Take $A := \{x, y, z\}$. By construction, A is directed and, by hypothesis, a chain. In particular, $\neg(x \bowtie y)$ and P is conditionally connected. $\square$

A.2 Results for majorization and the interval domain

A.2.1 Proof of Proposition 1

By Lemma 6, $\mathbb{Q}^n \cap \Lambda^n$ is a countable weak basis for any $n \geq 2$. Hence, it is sufficient to show there exists a finite map $\alpha : \mathbb{N} \to \mathbb{Q}^n \cap \Lambda^n$ such that $\{\langle n, m \rangle | \alpha(n) \preceq \alpha(m)\}$ is recursively enumerable. We begin with a finite map $\alpha_0 : \mathbb{N} \to \mathbb{Q} \cap [0, 1]$ which, aside from 0 and 1, orders the rationals in $[0, 1]$ lexicographically, considering first the denominators and then the numerators (see α in Hack et al. 2022a, Proposition 1). Using α_0, we construct now α for the case $n = 2$ by just selecting some pairs in $\alpha_0(\mathbb{N}) \times \alpha_0(\mathbb{N})$. If $m = 0$, then $\alpha(m) = (\alpha_0(1), \alpha_0(0))$ and we define $p_0 = 1$ and $q_0 = 0$. Notice $\alpha_0(1) + \alpha_0(0) = 1$. If $m \geq 1$, we begin with $p_m = p_{m-1} - 1$ and $q_m = q_{m-1} + 1$, if $p_{m-1} > 0$, and with $p_m = q_{m-1} + 1$ and $q_m = 0$, if $p_{m-1} = 0$. If $\alpha_0(p_m) + \alpha_0(q_m) = 1$, then $\alpha(m) = (\alpha_0(p_m), \alpha_0(q_m))$ ordering them decreasingly, if necessary. Otherwise, we decrease p_m one unit and increase q_m one unit and continue doing so until we get either two rational numbers whose sum is one or $p_m = 0$. In the former case, if we achieved our goal after k decreases, then we fix $p_m = p_{m-1} - k$ and $q_m = k$ and take $\alpha(m) = (\alpha_0(p_m), \alpha_0(q_m))$, ordered if necessary. In the latter case, we consider $p_m = p_{m-1} + 1$ and $q_m = 0$ and repeat the one-unit decrease of p_m and one-unit increase of q_m process until we find a pair of rationals whose sum is one. Once ordered, we take this pair as $\alpha(m)$ and we fix p_m and q_m accordingly. Notice that we can follow an analogous procedure to construct a finite map for any $n > 2$. From now on, we consider an arbitrary $n \geq 2$. We now only need to show $\{\langle n, m \rangle | \alpha(n) \preceq \alpha(m)\}$ is recursively enumerable, that is, we need to construct some computable function $f : \mathbb{N} \to \{\langle n, m \rangle | \alpha(n) \preceq \alpha(m)\}$. Given $m \in \mathbb{N}$, we get $p.q \in \mathbb{N}$ such that $m = \langle p, q \rangle$ and calculate $\alpha(p), \alpha(q)$. If $s_k(\alpha(p)) \leq s_k(\alpha(q))$ for all $k \leq n$, then $f(m) = m$. Otherwise, $f(m) = 0$, since $0 = \langle 0, 0 \rangle$ and $\alpha(0) \preceq \alpha(0)$.

A.2.2 Proof of Lemma 6

(1) As we know from Edalat and Sünderhauf (1999), $(\mathscr{I}, \sqsubseteq)$ is ω-continuous. Take $Z \subseteq \mathscr{I}$ a Debreu dense subset of $\mathscr{I}$. Take for each any $x \in \mathbb{R}$ some $y_x < x$ and notice $[y_x, x] \sqsubset [x, x]$. By Debreu separability, there exists some $z \in Z$ such that $[y_x, x] \sqsubseteq z \sqsubseteq [x, x]$. Thus, defining $z := [z_1, z_2]$, we have $z_2 = x$, since, by definition, $x \leq z_2 \leq x$. If we fix for each x such a z and denote it by z_x, we have x determines z_x uniquely. Hence, the map $f : \mathbb{R} \to Z$, $x \mapsto z_x$ is injective and, by injectivity of f, $\mathfrak{c} \leq |Z|$. Thus, Z has the cardinality of the continuum.

(2), (3) By Hack et al. 2022b, Lemma 5 (i) and (ii), we know both that $(\Lambda^n, \preceq_M)$ is order separable, thus Debreu upper separable, if $n = 2$ and that any Debreu dense subset has the cardinality of the continuum if $n \geq 3$. To conclude, we show $(\Lambda^n, \preceq_M)$ is ω-continuous for all $n \geq 2$. In particular, we show for each $x \in \Lambda^n$ there exists some $B_x \subseteq \mathbb{Q}^n \cap \Lambda^n$ such that $x = \sqcup B_x$ and $b \ll x$ for all $b \in B_x$ and obtain, as a result, $\mathbb{Q}^n \cap \Lambda^n$ is a countable basis. If $x = \bot$, then $B_\bot := \{\bot\} \subseteq \mathbb{Q}^n \cap \Lambda^n$ does the job, since $\bot \in K(\Lambda^n)$. If $x \neq \bot$, then take

$$B_x := \{q \in \mathbb{Q}^n \cap \Lambda^n \mid s_k(q) < s_k(x) \text{ for all } k < n\}.$$

Notice that we have $q \ll x$ by (10), thus, $q \preceq x$ for all $q \in B_x$. To finish, we need to show $\sqcup B_x = x$. Assume there exits some $y \in \Lambda^n$ such that $q \preceq y$ for all $q \in B_x$. By Lemma 1, that would mean for any $\varepsilon > 0$, we have $s_k(x) - \varepsilon < s_k(y)$ for all $k < n$. Thus, $s_k(x) \leq s_k(y)$ for all $k \leq n$ and $x \preceq y$. As a

result, $\sqcup B_x = x$. An alternative proof that $(\Lambda^n, \preceq_M)$ is ω-continuous for all $n \geq 2$, where Lemma 1 is not used, can be found in Appendix A.2.3.

A.2.3 Second proof that majorization has a countable basis

We prove here majorization is ω-continuous for all $n \geq 2$ without using Lemma 1. By Martin 2006, Theorem 1.3), we know $(\Lambda^n, \preceq_M)$ is a continuous dcpo for all $n \geq 2$. We will show $B := \mathbb{Q}^n \cap \Lambda^n$ is a countable basis. In order to do so, it is sufficient to show that, for all $x, y \in \Lambda^n$ such that $x \ll y$, there exists some $b \in B$ such that $x \ll b \ll y$ (Mao and Xu 2018, Proposition 2.4). By (10), if $x \ll y$, then either $x = \perp$ or $\sum_{i=1}^k x_i < \sum_{i=1}^k y_i$ for all $k < n$. Since $\perp \in B$ and $\perp \ll \perp$, we can take $b = \perp$ in the first case. Assume the second case holds. Note $0 < x_i < 1$ for all $i \leq n$, since the opposite contradicts the fact $\sum_{i=1}^k x_i < \sum_{i=1}^k y_i$ for all $k < n$. Take $(\varepsilon_i)_{i=1}^{k-1}$, where

$$
\begin{cases}
0 < \varepsilon_i < \min\left\{y_i - x_i, \ s_{i+1}(y) - s_{i+1}(x)\right\} & \text{if } i = 1, \\[2mm]
0 < \varepsilon_i < \min\left\{s_i(y) - s_i(x) - s_{i-1}(\varepsilon), \right. \\[1mm]
\left. s_{i+1}(y) - s_{i+1}(x) - s_{i-1}(\varepsilon)\right\} & \text{if } 1 < i \leq n-2 \\[2mm]
0 < \varepsilon_i < s_i(y) - s_i(x) - s_{i-1}(\varepsilon) & \text{if } i = n-1
\end{cases}
$$

Note the first upper bound in the definition of ε_i is the property we will use here for all $i \leq n-2$, while the second upper bound is included to make sure ε_{i+1} is well-defined. Take for all $i < n$ some $b_i \in (x_i, x_i + \varepsilon_i) \cap \mathbb{Q}$ such that $b_i \geq b_{i+1}$ for all $i < n-1$ and define $b := (b_1, \ldots, b_n)$, where $b_n := 1 - \sum_{j=1}^{n-1} b_j$, implying $b \in \mathbb{Q}^n$. Note $b_n = 1 - \sum_{j=1}^{n-1} b_j < 1 - \sum_{j=1}^{n-1} x_j = x_n \leq x_{n-1} < b_{n-1}$, which implies $b \in \Lambda^n$. By definition of $(\varepsilon_i)_{i=1}^{k-1}$, we have

$$
\sum_{j=1}^k x_j < \sum_{j=1}^k b_j < \sum_{j=1}^k (x_j + \varepsilon_j) < \sum_{j=1}^k x_j + \sum_{j=1}^k (y_j - x_j) = \sum_{j=1}^k y_j
$$

for all $k < n$. Thus, $x \ll b \ll y$ and we have finished.

A.2.4 Proof of Lemma 1

Consider $x \in \Lambda^n \setminus \{\perp\}$. Assume first $x = (\frac{1}{m}, \ldots, \frac{1}{m}, 0, \ldots, 0)$ for some $m < n$. Fix w.l.o.g. $0 < \varepsilon < \frac{1}{m}$, consider some $\varepsilon' \in \mathbb{Q}$ such that

$$
0 < \varepsilon' < \min\left\{\frac{\varepsilon}{m}, \frac{\varepsilon}{n}(n - m)\right\}
$$

and define $\beta = \frac{m}{n-m}\varepsilon'$. We define then the components of q, $q_i = \frac{1}{m} - \varepsilon'$ for $1 \leq i \leq m$ and $q_i = \beta$ for $m < i \leq n$. To assure $q \in \Lambda^n$, we need to show $s_n(q) = 1$ and $q_i \leq q_{i-1}$ for $2 \leq i < n$. Notice $s_n(q) = s_n(x) - m\varepsilon' + (n-m)\beta = 1$ by definition of β while for the second part it suffices to show $\frac{1}{m} - \varepsilon' > \beta$, which holds as we have

$$
\frac{1}{m} - \varepsilon' > \beta \iff \frac{1}{m} - \varepsilon' > \frac{m}{n-m}\varepsilon' \iff \frac{1}{m} > \frac{n}{n-m}\varepsilon'
$$

and, by definition of ε',

$$
\varepsilon' < \varepsilon \frac{n-m}{n} < \frac{n-m}{nm}.
$$

We show now $s_k(q) > s_k(x) - \varepsilon$ for all $k < n$ holds. If $i \leq m$ we have $s_i(q) = s_i(x) - i\varepsilon' > s_i(x) - \varepsilon$, since $\varepsilon' < \frac{\varepsilon}{m}$ holds by definition. If $m < i < n$, then we get $s_i(q) = 1 - m\varepsilon' + (i-m)\beta > 1 - \varepsilon$, as the following holds

$$\varepsilon' < \frac{\varepsilon}{m} \implies \varepsilon' < \frac{\varepsilon(n-m)}{m(n-i)} \iff m\varepsilon' + (m-i)\beta < \varepsilon,$$

where the first inequality is true by definition of ε' and the first implication holds since $m < i < n$. Note $s_k(q) = \frac{1}{m} - k\varepsilon' < s_k(x)$ for $1 \leq k \leq m$ and $s_k(q) = 1 - (n-k)\beta < 1 = s_k(x)$ for $m < k < n$.

We assume now $x \neq (\frac{1}{m}, \ldots, \frac{1}{m}, 0, \ldots, 0)$ and define $k = \min\{i < n | x_j = x_i \text{ or } x_j = 0 \text{ for all } j \geq i\}$, if it exists, and, otherwise, $k = n$. We also define $h = \min\{k \leq i \leq n | x_{i+1} = 0\}$, if it exists, and, otherwise, $h = n$. Lastly, consider $\alpha = h - (k-1)$. We fix now $\varepsilon > 0$, assume $h = n$ and consider some ε' such that

$$0 < \varepsilon' < \min\left\{\frac{\varepsilon}{k}, \left(1 + \frac{k-1}{\alpha}\right)^{-1}(x_{k-1} - x_k)\right\}.$$

Notice ε' is well-defined as we have $x_{k-1} > x_k$. We take now $q_i \in (x_i - \varepsilon', x_i) \cap \mathbb{Q}$ for all $i < k$ such that $q_i \leq q_{i-1}$ for $2 \leq i < k$ and $q_i = \tau$ for $k \leq i \leq h$ where $\tau = \frac{1}{\alpha}(1 - \sum_{i=1}^{k-1} q_i)$. Notice as $h = n$ we have $s_n(q) = 1$ and we also have $\tau < q_{k-1}$ as

$$\tau = \frac{1}{\alpha}\left(1 - \sum_{i=1}^{k-1} q_i\right) < \frac{1}{\alpha}\left(1 - \sum_{i=1}^{k-1} x_i + (k-1)\varepsilon'\right) = x_k + \frac{k-1}{\alpha}\varepsilon'$$

$$\overset{(i)}{<} x_{k-1} - \varepsilon' < q_{k-1}$$

where we have used the definition of ε' in (i). We show now we have $s_i(q) > s_i(x) - \varepsilon$ for all $i < h$ which is sufficient as we are assuming $h = n$. If $i \leq k-1$ we have

$$s_i(q) > s_i(x) - i\varepsilon' > s_i(x) - \frac{i}{k}\varepsilon > s_i(x) - \varepsilon.$$

while if $k \leq i < h$ we have

$$s_i(q) = \sum_{j=1}^{k-1} q_j + \sum_{j=k}^{i} q_j \overset{(i)}{>} s_k(x) - \varepsilon + \sum_{j=k}^{i} x_j = s_i(x) - \varepsilon.$$

where in (i) we have used the fact $q_i > x_i$ for all $i \geq k$ as we have

$$x_i = x_k = \frac{1}{\alpha}\left(1 - s_{k-1}(x)\right) < \frac{1}{\alpha}\left(1 - s_{k-1}(q)\right) = \tau = q_i.$$

Thus, if $h = n$, we have finished. Note that we can see $s_i(q) < s_i(x)$ for $1 \leq i < n$ similarly to the previous case.

Assume now $h < n$. Notice defining some q' where $q'_i = q_i$ if $i \leq h$ and $q'_i = 0$ if $h < i \leq n$ does not work, since we would have $s_h(q) = s_h(x) = 1$. Thus, we need $q'_i > 0$ if $h < i \leq n$. Consider some $\beta \in \mathbb{Q}$ such that

$$0 < \beta < \min\left\{\frac{\varepsilon}{n-h}, \left(1 + \frac{n-h}{h - (k-1)}\right)^{-1} q_k, \frac{1}{n-h}\left(s_k(q) - \left(s_k(x) - \varepsilon\right)\right)\right\},$$

where q is defined as in the case $h = n$, and define $\beta' = \frac{n-h}{h-(k-1)}\beta$. We consider now q' where $q'_i = q_i$ if $i < k$, $q'_i = q_i - \beta'$ if $k \leq i \leq h$ and $q'_i = \beta$ if $h < i$. Notice we have $q'_i \leq q'_{i+1}$ for $2 \leq i < n$, since we have it already from q in case $i \leq h$, and for $h < i$ it suffices to show $q'_k > \beta$, which is true as

$$q'_k = q_k - \beta' = q_k - \frac{n-h}{h - (k-1)}\beta > \beta,$$

where the inequality holds by definition of β. Notice, also, we have $s_n(q') = s_h(q) - \beta'(h - (k-1)) + \beta(n-h) = s_h(q) = 1$. To conclude, we only need to show $s_i(q') > s_i(x) - \varepsilon$ for all $i < n$. If

$i < k$, then we already have it as $s_i(q') = s_i(q)$ and we know it holds for q. If $k \leq i \leq h$, then it also holds as

$$s_i(q') = s_i(q) - (i - (k-1))\beta' \geq s_i(q) - (h - (k-1))\beta' \overset{(i)}{>} s_i(x) - \varepsilon,$$

where, in (i), we apply the fact that for $k \leq i \leq h$, we have

$$\beta' < \frac{1}{h - (k-1)}\left(s_k(q) - \left(s_k(x) - \varepsilon\right)\right) \overset{(ii)}{\leq} \frac{1}{h - (k-1)}\left(s_i(q) - \left(s_i(x) - \varepsilon\right)\right),$$

where the first inequality holds by definition of β and β' and (ii) also does, as we have for $k \leq i \leq h$

$$\begin{aligned}
s_i(q) - (s_i(x) - \varepsilon) &= s_k(q) + (i - k)q_k - (s_k(x) - \varepsilon) - (i - k)x_k \\
&= s_k(q) - (s_k(x) - \varepsilon) + (i - k)(q_k - x_k) \\
&\leq s_k(q) - (s_k(x) - \varepsilon),
\end{aligned}$$

where we used the fact $q_k > x_k$ in the inequality. If $h < i < n$, then

$$s_i(q') = 1 - (n - i)\beta > 1 - \varepsilon = s_i(x) - \varepsilon,$$

where the inequality holds since we have $\beta < \frac{\varepsilon}{n-h} < \frac{\varepsilon}{n-i}$. Note that we can see $s_i(q) < s_i(x)$ for $1 \leq i < n$, similarly to both previous cases.