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Corrigendum

A curve selection lemma in spaces of arcs and the image of the Nash map

(Compositio Math. 142 (2006), 119–130)

Ana J. Reguera

The purpose of this note is to correct a mistake in the article 'A curve selection lemma in spaces of arcs and the image of the Nash map' (Compositio Math. 142 (2006), 119–130, cited here as [Reg06]). This is due to an overlooked hypothesis in the definition of *generically stable* subset of the space of arcs X_∞ of a variety X defined over a perfect field k. By a variety X over k we mean an integral separated k-scheme of finite type. Although not explicitly stated in [Reg06], we consider varieties defined over a perfect field k (see Remark C.10 for explanation). We retain the notation of the original article.

In Definition 3.1 in the original article the following extra property is needed: N *is not contained in* $(Sing\ X)_\infty$. This property is applied in the proof of Theorem 4.1 and its corollaries (Corollaries 4.4, 4.6 and 4.8). In addition, in Theorem 4.1 the *reduced structure in the space of arcs* has to be considered. All the statements in [Reg06] remain true while adding these conditions. We state below the corrected version of Definition 3.1 that has to be applied, and in C.3 the list of statements in which some of the two additional conditions are used.

Theorem 4.1 has been applied mainly through its Corollary 4.8 (curve selection lemma) to the sets N_E (Definition 2.1 in [Reg06]) where, assuming the existence of a resolution of singularities $p : Y \to X$ of X, E is an irreducible component of the exceptional locus of p. One occurrence is Theorem 5.1 in [Reg06] in order to study the image of the Nash map. Theorem 5.1 is a consequence of Theorem 4.1 applied to the Nash sets $N_\alpha := N_{E_\alpha}$ where E_α is an essential component on Y (Proposition 2.2 in [Reg06]). These sets N_E, and hence also the N_αs, satisfy Definition 3.1 in its corrected version (see C.5).

The original definition was as follows.

Definition 3.1. An irreducible subset N of X_∞ will be called *generically stable* if there exists an open affine subscheme W_0 of X_∞, such that $N \cap W_0$ is a nonempty closed subset of W_0 whose defining ideal is the radical of a finitely generated ideal.

To correct the mistake in [Reg06], we add the property 'N *not contained in* $(Sing\ X)_\infty$' in Definition 3.1. Thus the definition now reads as follows.

Definition 3.1 (corrected version). Let X be an algebraic variety over a perfect field k. An irreducible closed subset N of X_∞ not contained in $(Sing\ X)_\infty$ will be called *generically stable* if

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there exists an open affine subscheme W_0 of X_∞, such that $N \cap W_0$ is a nonempty closed subset of W_0 whose defining ideal is the radical of a finitely generated ideal.

The point is that there exist irreducible subsets of X_∞ satisfying Definition 3.1 in the original version which are contained in $(Sing\ X)_\infty$.

Example C.1. Let k be a perfect field of characteristic $p > 0$ and let us consider the variety

$$X := \{x_3^p + x_1 x_2^p = 0\} \subset \mathbb{A}_k^3 = Spec\ k[x_1, x_2, x_3].$$

Let $N = (Sing\ X)_\infty$, which is a subset of X_∞. Then N satisfies Definition 3.1 in the original version and it is of course contained in $(Sing\ X)_\infty$. In fact, set $f := x_3^p + x_1 x_2^p$; then, with notation as in 3.5 in [Reg06], we have $\mathbf{F}_0 = \mathbf{X}_{3,0}^p + \mathbf{X}_{1,0}\mathbf{X}_{2,0}^p$, $\mathbf{F}_1 = \mathbf{X}_{1,1}\mathbf{X}_{2,0}^p$, $\mathbf{F}_p = \mathbf{X}_{3,1}^p + \mathbf{X}_{1,0}\mathbf{X}_{2,1}^p + \mathbf{X}_{1,p}\mathbf{X}_{2,0}^p$, $\mathbf{F}_{p+1} = \mathbf{X}_{1,1}\mathbf{X}_{2,1}^p + \mathbf{X}_{1,p+1}\mathbf{X}_{2,0}^p, \ldots$. Hence

$$\left(\mathcal{O}_{X_\infty}\right)_{X_{1,1}} \cong \left(k[\underline{\mathbf{X}}_0, \ldots, \underline{\mathbf{X}}_n, \ldots]\right)_{\mathbf{X}_{1,1}} / \left(\left\{\mathbf{X}_{2,n}^p, \mathbf{X}_{3,n}^p\right\}_{n \geq 0}\right).$$

But $Sing\ X = \{x_2 = x_3 = 0\}$, hence $(Sing\ X)_\infty$ is the closed irreducible subset of X_∞ given by $X_{2,n} = X_{3,n} = 0$, $n \geq 0$. Therefore

$$\overline{X_\infty \setminus (Sing\ X)_\infty} \subseteq \{X_{1,1} = 0\}.$$

Equivalently, for the open subset of X_∞ defined by $W_0 := \{X_{1,1} \neq 0\}$, we have $N \cap W_0 = W_0$, i.e. $\mathcal{P}(\mathcal{O}_{X_\infty})_{X_{1,1}} = \sqrt{(0)}(\mathcal{O}_{X_\infty})_{X_{1,1}}$ where $\mathcal{P}$ is the generic point of $N = (Sing\ X)_\infty$. Thus N satisfies Definition 3.1 in the original version. However, N does not satisfy Theorem 4.1. In fact,

$$\mathcal{P}\left(\mathcal{O}_{X_\infty}\right)_{X_{1,1}} = \left(\{X_{2,n}, X_{3,n}\}_{n \geq 0}\right)\left(\mathcal{O}_{X_\infty}\right)_{X_{1,1}}$$

is not finitely generated.

Now, in the main result in [Reg06] we have to replace X_∞ by $(X_\infty)_{\mathrm{red}}$.

Theorem 4.1 [Reg06]. *Let X be a variety over k and let N be an irreducible generically stable subset of X_∞. There exists an open affine subscheme W of $(X_\infty)_{\mathrm{red}}$ such that $N \cap W$ is a nonempty closed subset of W whose ideal is finitely generated.*

Equivalently, for affine varieties the theorem reads as follows.

C.2. [Reg06]. *Let X be an affine variety over a perfect field k. Let $\mathcal{P}$ be a point in X_∞ (i.e. a prime ideal of $\mathcal{O}_{X_\infty}$) and let $Z(\mathcal{P})$ be the set of zeros of $\mathcal{P}$ in X_∞. Suppose that $Z(\mathcal{P})$ is a generically stable subset of X_∞. Then there exists $G \in \mathcal{O}_{X_\infty} \setminus \mathcal{P}$ such that the ideal $\mathcal{P}(\mathcal{O}_{(X_\infty)_{\mathrm{red}}})_G$ is a finitely generated ideal of $(\mathcal{O}_{(X_\infty)_{\mathrm{red}}})_G$.*

Proof of Theorem 4.1 (with the reduced structure $(X_\infty)_{\mathrm{red}}$ and the corrected version of Definition 3.1). Following the proof in [Reg06], one reduces to X affine; let $d := \dim X$. With notation as at the beginning of § 4 of [Reg06], let $\mathcal{P}$ be the prime ideal of $\mathcal{O}_{X_\infty}$ defining N and, for $n \geq 0$, let $\mathcal{P}_n$ be the prime ideal of $\mathcal{O}_n := \mathcal{O}_{\overline{j_n(X_\infty)}}$ defining $\overline{j_n(N)}$. Here $\overline{j_n(X_\infty)}$ (respectively, $\overline{j_n(N)}$) is the closure in the space X_n of arcs of order n of the image of X_∞ (respectively, N) by $j_n : X_\infty \to X_n$, both $\overline{j_n(X_\infty)}$ and $\overline{j_n(N)}$ considered with their reduced structure. We have $\mathcal{O}_n \subseteq \mathcal{O}_{n+1}$ and $\mathcal{O}_{(X_\infty)_{\mathrm{red}}} = \bigcup_n \mathcal{O}_n$ (*reduced structure* $(X_\infty)_{\mathrm{red}}$ has to be considered in line 6 of the first paragraph in § 4). It suffices to prove that there exist n_1, $G \in \mathcal{O}_{n_1}$, $G \in \mathcal{O}_{X_\infty} \setminus \mathcal{P}$ and, for all $n \geq n_1$, that there exist $H_{1,n+1}, \ldots, H_{m-d,n+1}$ in $\mathcal{O}_{n+1}$ satisfying

$$\mathcal{P}_{n+1}(\mathcal{O}_{n+1})_G = \big(\mathcal{P}_n + (H_{1,n+1}, \ldots, H_{m-d,n+1})\big)(\mathcal{O}_{n+1})_G \tag{1}$$

and the $H_{r,n+1}$ belong to a finitely generated ideal $\mathcal{J}'_N$ of $\mathcal{O}_{X_\infty}$ contained in $\mathcal{P}$ (Lemma 4.2 in [Reg06]). Then we will have that $\mathcal{P}(\mathcal{O}_{(X_\infty)_{\rm red}})_G$ (*reduced structure* $(X_\infty)_{\rm red}$ has to be considered in line 1 after Lemma 4.2) is finitely generated by the generators of $\mathcal{P}_{n_1}$ and the (finite number of) generators of $\mathcal{J}'_N$.

Let $J \subset \mathcal{O}_X$ be the Jacobian ideal of X and let $e = ord_N J$ (line 9 of the proof of Lemma 4.2 in [Reg06]). Then e *is finite* since $N \not\subseteq (Sing\ X)_\infty$ by the corrected version of Definition 3.1; it is at this point that the corrected version of Definition 3.1 is needed. Arguing as in Lemma 4.1 in [DL99] and the second paragraph of the proof of Lemma 4.2 in [Reg06], we reduce to the case where X is a complete intersection (not necessarily irreducible): there exists $l \in \mathcal{O}_X$ such that $Sing\ X \subset (l = 0)$, $ord_N l = e$ and there exists a complete intersection subscheme X' of $\mathbb{A}^m_k$ containing X and of dimension d such that $(\mathcal{O}_X)_l \cong (\mathcal{O}_{X'})_l$. We have $(\mathcal{O}_{(X_\infty)_{\rm red}})_{L_e} \cong (\mathcal{O}_{(X'_\infty)_{\rm red}})_{L_e}$ (with *reduced structure* in line 10 of the proof of Lemma 4.2). Then it suffices to prove the result for X'.

The proof ends as on page 125 of [Reg06]: set: $X' = Spec\ k[x_1, \ldots, x_m]/(f_1, \ldots, f_{m-d})$, J' its Jacobian ideal, $e' := ord_N J'$ and $\mathcal{J}'_N$ the ideal of $\mathcal{O}_{X_\infty}$ (finitely) generated by $\bigcup_{q'} \{Q'_0, \ldots, Q'_{e'-1}\}$, where q' runs over all the $(m-d) \times (m-d)$ minors of the Jacobian matrix, hence $\mathcal{J}'_N$ is contained in $\mathcal{P}$. We may suppose that $ord_N q = e'$ where q is the determinant of the matrix $(\frac{\partial f_i}{\partial x_j})_{1 \leq i,\ j \leq m-d}$, then consider its adjoint matrix M. Let W_0 be an open subset of X_∞ as in the corrected version of Definition 3.1. Then we consider the set $W_0 \cap D(Q_{e'}) \cap V(\mathcal{J}'_N)$ whose intersection with N is a nonempty open subset of N; we may suppose that it is $D(G) \cap N$. We multiply M by the column vector $(f_i)_{i=1}^{m-d}$ and apply $\mathbf{j}^\sharp_\infty$, i.e. Taylor's development (line 9 on page 125 in [Reg06]). Then for $n \geq e'$ we obtain elements $H_{1,n+1}, \ldots, H_{m-d,n+1} \in \mathcal{O}_{n+1}$ ((8) on page 125), $H_{r,n+1} \in \mathcal{J}'_N$, hence $(\mathcal{P}_n + (H_{1,n+1}, \ldots, H_{m-d,n+1}))\mathcal{O}_{n+1} \subseteq \mathcal{P}_{n+1}$, such that for $n \gg 0$, the schemes

$$Spec(\mathcal{O}_{n+1})_G/\mathcal{P}_{n+1} \subseteq Spec\big((\mathcal{O}_n)_G/\mathcal{P}_n\big)[\underline{\mathbf{X}}_{n+1}]/\big(\{H_{r,n+1}\}_{r=1}^{m-d}\big) \tag{2}$$

have the same $\bar{k}$-rational points. In addition, we can eliminate $X_{r,n+1}$ in $H_{r,n+1}$ because G and hence $Q_{e'}$ is a unit, and we obtain that the right-hand side of (2) is isomorphic to $Spec((\mathcal{O}_n)_G/\mathcal{P}_n)[\mathbf{X}_{m-d+1,n+1}, \ldots, \mathbf{X}_{m,n+1}]$. Therefore both schemes in (2) are reduced, and we conclude that they are isomorphic. This proves (1) and concludes the proof. □

C.3. We next list all the statements in [Reg06], apart from Theorem 4.1, for which either the corrected version of Definition 3.1 has to be applied or the reduced structure $(X_\infty)_{\rm red}$ in the space of arcs has to be considered.

- Lemma 3.2: N is an irreducible generically stable subset of X_∞ as in the corrected version of Definition 3.1. This lemma connects generically stable subsets of X_∞ with the stability property in [DL99, Lemma 4.1] (see C.6). This is explained in detail at the end of this note (see Proposition C.8).
- Definition 3.3: N is an irreducible generically stable subset of X_∞ as in the corrected version of Definition 3.1. This definition uses Lemma 3.2.

- Lemma 3.6: N is an irreducible generically stable subset of X_∞ as in the corrected version of Definition 3.1. The statement and proof of Lemma 3.6 remain unchanged: the condition $N \not\subset (\mathit{Sing}\ X)_\infty$ is implicit in the hypothesis of Lemma 3.6.
- *Remark* 3.7: The subset $\{\mathbf{X}_1 = 0\}$ of $\mathbb{A}^1_\infty$ is a generically stable subset as in the corrected version of Definition 3.1.
- Proposition 3.8: The set N_α is an irreducible generically stable subset of X_∞ as in the corrected version of Definition 3.1 (see C.5 to recall the definition of N_α and to complete the proof of Proposition 3.8). Read 'there exists an open affine subscheme W_0 of $(X_\infty)_{\mathrm{red}}$' in the second line of Proposition 3.8.
- Corollary 4.4: N is an irreducible generically stable subset of X_∞ as in the corrected version of Definition 3.1. This corollary is a consequence of Lemma 4.2.
- Corollary 4.6: N is an irreducible generically stable subset of X_∞ as in the corrected version of Definition 3.1. Besides, read 'the ring $\widehat{\mathcal{O}_{(X_\infty)_{\mathrm{red}},z}}$ is Noetherian' in (i) of Corollary 4.6. Read 'the maximal ideal $\mathcal{P}\mathcal{O}_{(X_\infty)_{\mathrm{red}},z}$ of $\mathcal{O}_{(X_\infty)_{\mathrm{red}},z}$ is finitely generated' in the first line of the proof. This result is a consequence of Theorem 4.1.
- Corollary 4.8 (curve selection lemma): N is generically stable as in the corrected version of Definition 3.1. The statement and proof of Corollary 4.8 remain unchanged.
- *Question* after Corollary 4.8: N is an irreducible generically stable subset of X_∞ as in the corrected version of Definition 3.1.
- *Proof* of Proposition 4.9, first line: N is an irreducible generically stable subset of X_∞ as in the corrected version of Definition 3.1. The proof remains unchanged.
- End of page 127: in the example given here the subset N of X_∞ is not a generically stable subset of X_∞ either with the original or with the corrected version of Definition 3.1. Due to its relevance in the article, we next recall Corollary 4.8 in [Reg06] and discuss the main idea in this example.

Corollary 4.8 (Curve selection lemma) [Reg06]. *Let N and N' be two irreducible subsets of X_∞ such that $N \subset N'$, $N \neq N'$. Suppose that N is generically stable, and let z be the generic point of N and k_z its residue field. Then there exists a morphism*

$$\phi : \mathit{Spec}\ K[[\xi]] \to N'$$

where K is a finite algebraic extension of k_z, such that the image of the closed point of $\mathit{Spec}\ K[[\xi]]$ is z, and the image of the generic point belongs to $N' \setminus N$.

Remark C.4. Recall that, in Corollary 4.8 above, the irreducible subset N of X_∞ is generically stable as in the corrected version of Definition 3.1. Recall also that in [Reg06], after Corollary 4.8, we proposed as a *question* a stronger version of this curve selection lemma: to know whether, under the same hypothesis, there exists Φ as in Corollary 4.8 and satisfying also that the image of the generic point of $\mathit{Spec}\ K[[\xi]]$ is the *generic point of* N'. This question is still open.

The idea in the following example was communicated to me by M. Lejeune-Jalabert. Let X be the Whitney umbrella $x_3^2 = x_1 x_2^2$ in $\mathbb{A}^3_{\mathbb{C}}$, and let z be the point in X_∞ determined by any arc $x_1(t),\ x_2(t),\ x_3(t)$ such that

$$ord_t x_1(t) = 1 \quad \text{and} \quad x_2(t) = x_3(t) = 0. \tag{3}$$

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Let N be the closure of z and let N' be the closure of $X_\infty^{Sing} \setminus (Sing\ X)_\infty$. Here X_∞^{Sing} denotes the set of arcs centered in some point of $Sing\ X$. Then $N \subset N'$ ([IK03], Lemma 2.12) but there does not exist $\phi : Spec\ K[[\xi]] \to N'$ which maps the closed point of $Spec\ K[[\xi]]$ to z and the generic point to the generic point of N'. In fact, if $x_1(\xi,t),\ x_2(\xi,t),\ x_3(\xi,t) \in K[[\xi,t]]$ satisfy $x_3^2 = x_1 x_2^2$ and $ord_t x_1(0,t) = 1$, $x_2(0,t) = x_3(0,t) = 0$, then $ord_{(\xi,t)} x_1(\xi,t) = 1$, thus $x_2(\xi,t) = x_3(\xi,t) = 0$. In [Reg06], at the end of page 127, we considered the point z of X_∞ defined by $x_1(t) = t$, $x_2(t) = x_3(t) = 0$. Its closure is not a generically stable subset of X_∞ either with the original or with the corrected version of Definition 3.1.

Let us now consider the generic point $\mathcal{P}$ of $(Sing\ X)_\infty \cap (X_{1,0} = 0)$, which also satisfies (3). We have

$$\left(\mathcal{O}_{X_\infty}\right)_{X_{1,1}} / \sqrt{(X_{1,0})} \cong \left(k[\underline{\mathbf{X}}_0, \ldots, \underline{\mathbf{X}}_n, \ldots]\right)_{\mathbf{X}_{1,1}} / \left(\{\mathbf{X}_{1,0}\} \cup \{\mathbf{X}_{2,n}, \mathbf{X}_{3,n}\}_{n \geq 0}\right).$$

This implies that, for the open subset of X_∞ defined by $W_0 := \{X_{1,1} \neq 0\}$, we have $N \cap W_0 = (X_{1,0} = 0) \cap W_0$, i.e. $\mathcal{P}(\mathcal{O}_{X_\infty})_{X_{1,1}} = \sqrt{(X_{1,0})}(\mathcal{O}_{X_\infty})_{X_{1,1}}$. Thus N satisfies Definition 3.1 in the original version, but $N \subset (Sing\ X)_\infty$, hence N is not a generically stable subset of X_∞ with the corrected version of Definition 3.1.

C.5. Section 5 in [Reg06] is devoted to characterizing the image of the Nash map in terms of wedges. We assume the existence of a resolution of singularities $p : Y \to X$ of X. Then the main result in § 5, Theorem 5.1, is consequence of Corollary 4.8 (curve selection lemma) applied to the sets N_E, where E is an irreducible component of the exceptional locus of p. Recall that N_E is the closure in X_∞ of $N_E(Y) := \{z \in X_\infty \setminus (Sing\ X)_\infty / \widetilde{h}_z(0) \in E\}$ (Definition 2.1 in [Reg06]). Here, given $z \in X_\infty \setminus (Sing\ X)_\infty$, $\widetilde{h}_z : Spec\ k_z[[t]] \to Y$ is a lifting of h_z to Y, which exists and is unique by the properness of p. Equivalently, N_E is the closure in X_∞ of the image by $p_\infty : Y_\infty \to X_\infty$ of the set Y_∞^E of arcs in Y centered in some point of E.

The set N_E is irreducible and only depends on the divisorial valuation ν_E (Proposition 2.2 in [Reg06]). We denote $N_\alpha := N_{E_\alpha}$ when $E = E_\alpha$ is an essential divisor, i.e. the center of the divisorial valuation $\nu_\alpha := \nu_{E_\alpha}$ on any resolution of singularities is an irreducible component of its exceptional locus.

Note that Y_∞^E is not contained in the space of arcs of any strict subvariety of Y, hence N_E is not contained in the space of arcs of any strict subvariety of X, equivalently, the image of the arc defined by the generic point $\mathcal{P}_E$ of N_E is dense in X. In particular, $N_E \not\subset (Sing\ X)_\infty$, i.e. N_E has finite order of contact with the Jacobian ideal J of X, which is precisely $\nu_E(J)$. This completes the proof of Proposition 3.8. in [Reg06]. We conclude then that N_E (hence also N_α for ν_α essential) is an irreducible generically stable subset of X_∞ as in the corrected version of Definition 3.1.

We will next connect generically stable subsets with *stable* subsets in X_∞ (see (2.7) in [DL99]). The term stable here is motivated by the foundational Denef–Loeser stability property in [DL99].

C.6. [DL99, Lemma 4.1] Let X be an algebraic variety over a perfect field k, let $d = \dim X$. There exists $c \in \mathbb{N} \setminus \{0\}$ such that, for all $n, e \in \mathbb{N}$ with $n \geq ce$, the map

$$j_{n+1}(X_\infty) \longrightarrow j_n(X_\infty)$$

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is a piecewise trivial fibration over $j_n(X_\infty \setminus j_e^{-1}((\mathit{Sing}\ X)_e))$ with fiber $\mathbb{A}_k^d$. That is, there exists a finite partition of $j_n(X_\infty \setminus j_e^{-1}((\mathit{Sing}\ X)_e))$ into subsets S which are locally closed in X_n such that the morphism restricted to the inverse image of S is a trivial fibration with fiber $\mathbb{A}_k^d$. Here $(\mathit{Sing}\ X)_e$ denotes the space of arcs of order e on $\mathit{Sing}\ X$.

Definition C.7 (3.1 in [Reg09]). Let X be an affine variety over a perfect field k and let $\mathcal{P}$ be a point of X_∞ (i.e. a prime ideal of $\mathcal{O}_{X_\infty}$). We say that $\mathcal{P}$ is a stable point of X_∞ if there exist $n_1 \in \mathbb{N}$, and $G \in \mathcal{O}_{X_\infty} \setminus \mathcal{P}$, $G \in \mathcal{O}_{X_{n_1}}$ such that, for $n \geq n_1$, the map $X_{n+1} \longrightarrow X_n$ induces a trivial fibration

$$\overline{j_{n+1}(Z(\mathcal{P}))} \cap (X_{n+1})_G \longrightarrow \overline{j_n(Z(\mathcal{P}))} \cap (X_n)_G \tag{4}$$

with fiber $\mathbb{A}_k^d$, where $d = \dim X$, $(X_n)_G$ is the open subset $X_n \setminus Z(G)$ of X_n and $\overline{j_n(Z(\mathcal{P}))}$ is the closure of $j_n(Z(\mathcal{P}))$ in X_n with the reduced structure.

This definition extends naturally to any variety.

Proposition C.8. *Let X be a variety over a perfect field and let $\mathcal{P}$ be a point in X_∞. The following assertions are equivalent:*

(i) *$Z(\mathcal{P})$ is a generically stable subset of X_∞;*
(ii) *$\mathcal{P}$ is a stable point of X_∞.*

Moreover, if the previous conditions hold, then the image of the arc $h_\mathcal{P} : \mathit{Spec}\ \kappa(\mathcal{P})[[t]] \to X$ defined by $\mathcal{P}$ is dense in X; here $\kappa(\mathcal{P})$ is the residue field of $\mathcal{P}$.

Proof. We may assume without loss of generality that X is affine. Let $d = \dim X$ and, for $n \geq 0$, let $\mathcal{P}_n$ be the prime ideal of $\mathcal{O}_{\overline{j_n(X_\infty)}}$ defining $\overline{j_n(Z(\mathcal{P}))}$.

The assertion (i) $\Rightarrow$ (ii) is Lemma 3.2 in [Reg06]. We have already proved it in this note, at end of the proof of Theorem 4.1. In fact, if $Z(\mathcal{P})$ is a generically stable subset of X_∞ then we have proved that

$$\mathit{Spec}(\mathcal{O}_{n+1})_G/\mathcal{P}_{n+1} \cong \mathit{Spec}\big((\mathcal{O}_n)_G/\mathcal{P}_n\big)[\mathbf{X}_{m-d+1,n+1}, \ldots, \mathbf{X}_{m,n+1}] \tag{5}$$

for $n \gg 0$ ((2) and the last five lines in the proof of Theorem 4.1). Thus (ii) holds.

Now let $\mathcal{P}$ be a stable point of X_∞. Then there exist $n_1 \in \mathbb{N}$ and $G \in \mathcal{O}_{X_\infty} \setminus \mathcal{P}$, $G \in \mathcal{O}_{X_{n_1}}$ such that, for $n \geq n_1$, the map (4) is a trivial fibration with fiber $\mathbb{A}_k^d$. Let Y be the closure of the image of the arc $h_\mathcal{P}$. If $d' := \dim Y < d$ then (4) would contradict the stability property C.6 for the variety Y, for which the piecewise trivial fibration has fiber $\mathbb{A}_k^{d'}$. Therefore $Y = X$, i.e. the image of the arc $h_\mathcal{P}$ is dense in X. In particular, $Z(\mathcal{P}) \not\subset (\mathit{Sing}\ X)_\infty$ and hence there exists $e \in \mathbb{N}$ such that $Z(\mathcal{P}) \subset X_\infty \setminus j_e^{-1}((\mathit{Sing}\ X)_e)$. By Lemma 4.1 in [DL99] (see C.6), there exist a constant c and a locally closed subset S of X_{n_0}, where $n_0 = \sup\{n_1, ce\}$, such that the generic point of $j_{n_0}(Z(\mathcal{P}))$ belongs to S and, for $n \geq n_0$,

$$j_{n+1}(X_\infty) \longrightarrow j_{n_0}(X_\infty)$$

induces a trivial fibration over S with fiber $\mathbb{A}_k^{d(n+1-n_0)}$. Note that, for $n \geq n_0$,

$$\overline{j_{n+1}(Z(\mathcal{P}))} \cap (X_{n+1})_G \to \overline{j_{n_0}(Z(\mathcal{P}))} \cap (X_n)_G$$

is also a trivial fibration with fiber $\mathbb{A}_k^{d(n+1-n_0)}$. By restricting the open subset $(X_n)_G$, we may suppose that $\overline{j_{n_0}(Z(\mathcal{P}))} \cap (X_n)_G$ is contained in S. It follows that the defining ideal of

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$\overline{j_{n+1}(Z(\mathcal{P}))} \cap (X_{n+1})_G$ is the radical of $\mathcal{P}_{n_0}(\mathcal{O}_{X_{n+1}})_G$. Hence the defining ideal of $Z(\mathcal{P}) \cap (X_\infty)_G$ is the radical of $\mathcal{P}_{n_0}(\mathcal{O}_{X_\infty})_G$ and (i) holds. This completes the proof. □

We will next show that the closed subsets of X_∞ satisfying Definition 3.1 in the original version which are not generically stable subsets of X_∞, are generically stable subsets of Y_∞ for some subvariety Y of *Sing* X.

Proposition C.9. *Let X be an algebraic variety over a perfect field k. Let N be an irreducible closed subset of X_∞ satisfying Definition 3.1 in the original version such that $N \subseteq (Sing\ X)_\infty$. Let $\mathcal{P} \in X_\infty$ be its generic point. Then there exists a subvariety Y of Sing X such that $N \subset Y_\infty$ and $\mathcal{P}$ is a stable point of Y_∞.*

Proof. First note that, for any subvariety Y of X, an irreducible closed subset of X_∞ satisfying Definition 3.1 in the original version for X_∞ which is contained in Y_∞, satisfies also Definition 3.1 in the original version for Y_∞. Now let N be an irreducible closed subset of X_∞ satisfying Definition 3.1 in the original version for X_∞ such that $N \subseteq (Sing\ X)_\infty$. There exists an irreducible component Y_1 of *Sing* X such that the image of the arc $h_{\mathcal{P}}$ defined by $\mathcal{P}$ is contained in Y_1, or equivalently, $N \subset (Y_1)_\infty$. Then N satisfies Definition 3.1 in the original version for $(Y_1)_\infty$. If $N \not\subseteq (Sing\ Y_1)_\infty$, then N is a generically stable subset of $(Y_1)_\infty$. Applying Proposition C.8, we conclude the result. If not, there exists an irreducible component Y_2 of *Sing* Y_1 such that $N \subseteq (Y_2)_\infty$. Arguing by induction, we conclude the result after at most $\dim X$ steps. □

Finally, we discuss why the hypothesis of a perfect base field k is needed in Theorem 4.1.

Remark C.10. Let X be a variety over a nonperfect field k, hence $\mathit{char}\, k > 0$. Let *Sing* X be its singular locus, i.e. the set of points $x \in X$ such that the local ring $\mathcal{O}_{X,x}$ is not regular, and $Z = V(J)$ the zero locus of the Jacobian ideal J of X. In general, *Sing* $X \neq Z$; the notions of regular and smooth are not equivalent over nonperfect fields. This implies that, if k is nonperfect, Theorem 4.1 does not hold in general. The reason is that we have considered *Sing* X in the corrected version of Definition 3.1 and the Jacobian ideal of X in the proof of Theorem 4.1. However, from this proof it follows that if we replace $(Sing\ X)_\infty$ by Z_∞ in the corrected version of Definition 3.1 then we obtain an analogous result. Specifically, suppose that X is a generically smooth variety over a nonperfect field k, hence $Z = V(J)$ is strictly contained in X. Let N be an irreducible closed subset of X_∞ not contained in Z_∞ and such that there exists an open affine subscheme W_0 of X_∞ such that $N \cap W_0$ is a nonempty closed subset of W_0 whose defining ideal is the radical of a finitely generated ideal. Then Theorem 4.1 holds for N.

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